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Research Article

Discussion on $\alpha - \psi$ Contractions on Generalized Metric Spaces

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We discuss the existence and uniqueness of fixed points of $\alpha - \psi$ contractive mappings in complete generalized metric spaces, introduced by Branciari. Our results generalize and improve several results in the literature.

1. Introduction and Preliminaries

Recently, Branciari [1] introduced the notion of a generalized metric, also known as rectangular metric, by replacing the triangle inequality with a more general inequality, namely, quadrilateral inequality. Since quadrilateral inequality is weaker than triangle inequality, each metric space is a generalized metric space. As it is expected, the converse of this statement is false [1]. By using this generalized metric, Branciari successfully defined an open ball and hence a topology. On the other hand, the topology of this metric fails to provide some useful topological properties:

($P1$) generalized metric needs not to be continuous;

($P2$) a convergent sequence in generalized metric space needs not to be Cauchy;

($P3$) generalized metric space needs not to be Hausdorff, and hence the uniqueness of limits cannot be guaranteed.

It is quite natural to ask whether the existing fixed point results are still valid in the setting of generalized metric spaces. The first answer to this question was given by Branciari [1] by proving an analog of the well-known Banach contraction principle. Although the statement is true, in the proof, Branciari [1] used the continuity of generalized metric which cannot be guaranteed. Samet [2] gives an example for a generalized metric which is not continuous. Later, the proof of Branciari [1] corrected by several authors; see, for example, [3–5]. The challenging nature of the topology of generalized metric has attracted attention and hence various fixed points results for different type contraction mappings on generalized metric spaces have been investigated (see, e.g., [2–13] and the references therein).

Very recently, Samet et al. [14] suggested a very interesting class of mappings, $\alpha - \psi$ contraction mappings, to investigate the existence and uniqueness of fixed point. Several well-known fixed point theorems, including the Banach mapping principle, were concluded as consequences of the main result of this interesting paper. The techniques used in this paper have been studied and improved by a number of authors; see, for example, [15–19] and the references therein.

In this paper, we investigate the existence and uniqueness of fixed point of $\alpha - \psi$ contraction mappings in the setting of generalized metric spaces by carrying the problems ($P1$)–($P3$) mentioned above. Notice that in the literature there are distinct notions that are called "generalized metric." In the sequel, when we mention "generalized metric," we correspond to the notion of "generalized metric" defined by Branciari [1].

We now recollect some fundamental definitions, notations, and basic results that will be used throughout this paper.

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^{k}(t) + v_k, \quad (1)$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In the literature such functions are called either Bianchini-Grandolfi gauge functions (see, e.g., [20–22]) or (c)-comparison functions (see, e.g., [23]).

Lemma 1 (see, e.g., [23]). *If $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$;

(ii) $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;

(iii) ψ *is continuous at* 0;

(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

In what follows, we recall the notion of generalized metric spaces.

Definition 2 (see [1]). Let X be a nonempty set and let $d : X \times X \to [0,\infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:

$$\begin{aligned} &(\text{GMS1}) \quad d(x,y) = 0 \quad \text{iff } x = y, \\ &(\text{GMS2}) \quad d(x,y) = d(y,x), \\ &(\text{GMS3}) \quad d(x,y) \leq d(x,u) + d(u,v) + d(v,y). \end{aligned} \quad (2)$$

Then the map d is called generalized metric and abbreviated as GMS. Here, the pair (X,d) is called generalized metric space.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called semimetric (see, e.g., [6]).

The concepts of convergence, Cauchy sequence, and completeness in a GMS are defined below.

Definition 3. (1) A sequence $\{x_n\}$ in a GMS (X,d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(2) A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.

(3) A GMS (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

Let X be nonempty set and let T be an endomorphism, $T : X \to X$. A point x in X is called (i) a fixed point of T if $Tx = x$, (ii) a preperiodic point of T if there are distinct natural n and m such that $T^n(x) = T^m(x)$, and (iii) a periodic point of T if there exists a natural number n so that $T^n(x) = x$, where T^n is the nth iteration of T. It is evident that all periodic points are preperiodic.

The following assumption was suggested by Wilson [6] to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points u, v there is a number $r_{u,v} > 0$ such that, for every $z \in X$,

$$r_{u,v} < d(u,z) + d(z,v). \quad (3)$$

Proposition 4 (see [4]). *In a semimetric space, assumption* (W) *is equivalent to the assertion that limits are unique.*

Proposition 5 (see [4]). *Suppose that* $\{x_n\}$ *is a Cauchy sequence in a GMS* (X,d) *with* $\lim_{n\to\infty} d(x_n, u) = 0$, *where* $u \in X$. *Then* $\lim_{n\to\infty} d(x_n, z) = d(u,z)$ *for all* $z \in X$. *In particular, the sequence* $\{x_n\}$ *does not converge to* z *if* $z \neq u$.

Definition 6 (see [14]). For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. One says that self-mapping T on X is α-admissible if, for all $x, y \in X$, one has

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \quad (4)$$

Example 7. Let $X = [0,\infty)$. We let $T : X \to X$ and $\alpha(x,y) : X \times X \to [0,\infty)$ be mappings that are defined by $Tx = \ln(x+1)$ and $\alpha(x,y) = \begin{cases} e^x & \text{if } x \geq y, \\ 0 & \text{if otherwise,} \end{cases}$ respectively. Then T is α-admissible.

Example 8. For $X = \mathbb{R}$ we define $T : X \to X$ and $\alpha(x,y) : X \times X \to [0,\infty)$ by $Tx = x/2$ and $\alpha(x,y) = \begin{cases} 2 & \text{if } x \geq y, \\ 0 & \text{if otherwise,} \end{cases}$ respectively. Then T is α-admissible.

Some interesting examples of such mappings were given in [14, 18].

The notion of $\alpha - \psi$ contractive mapping is defined in the following way.

Definition 9 (see [14]). Let (X,d) be a metric space and let $T : X \to X$ be a given mapping. One says that T is an $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)\, d(Tx, Ty) \leq \psi(d(x,y)), \quad \forall x, y \in X. \quad (5)$$

It is obvious that any contractive mapping, that is, a mapping satisfying Banach contraction, is an α–ψ contractive mapping with $\alpha(x,y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0,1)$.

2. Main Results

We present our main results in this section. First we give the analog of the notion of $\alpha - \psi$ contractive mapping, in the context of generalized metric space as follows.

Definition 10. Let (X,d) be a generalized metric space and let $T : X \to X$ be a given mapping. One says that T is an $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)\, d(Tx, Ty) \leq \psi(d(x,y)), \quad \forall x, y \in X. \quad (6)$$

Now, we state the following fixed point theorem.

Theorem 11. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be an $\alpha - \psi$ contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *T is continuous.*

Then there exists a. $u \in X$ such that $Tu = u$.

Proof. Let $x_0 \in X$ be an arbitrary point such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. Notice that the existence of such a point is guaranteed by assumption (ii) of the theorem. We construct an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If we have $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of T. Hence, for the rest of the proof, we presume that

$$x_n \neq x_{n+1} \quad \forall n. \tag{7}$$

Since T is α-admissible, we have

$$\begin{aligned} \alpha(x_0, x_1) &= \alpha(x_0, Tx_0) \\ &\geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \end{aligned} \tag{8}$$

Recursively, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{9}$$

Analogously, we derive that

$$\begin{aligned} \alpha(x_0, x_2) &= \alpha(x_0, T^2x_0) \\ &\geq 1 \Longrightarrow \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1. \end{aligned} \tag{10}$$

Iteratively, we get that

$$\alpha(x_n, x_{n+2}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{11}$$

Regarding (6) and (9), we deduce that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1}) d(Tx_n, Tx_{n-1}) \\ &\leq \psi(d(x_n, x_{n-1})), \end{aligned} \tag{12}$$

for all $n \geq 1$.

Inductively, we derive that

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0)), \quad \forall n \geq 1. \tag{13}$$

It is evident from Lemma 1 that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{14}$$

Regarding (6) and (11), we deduce that

$$\begin{aligned} d(x_{n+2}, x_n) &= d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \alpha(x_{n+1}, x_{n-1}) d(Tx_{n+1}, Tx_{n-1}) \\ &\leq \psi(d(x_{n+1}, x_{n-1})), \end{aligned} \tag{15}$$

for all $n \geq 1$.

By utilizing inequality (15), we derive that

$$d(x_{n+2}, x_n) \leq \psi^n(d(x_2, x_0)), \quad \forall n \geq 1. \tag{16}$$

Owing to Lemma 1, we find that

$$\lim_{n\to\infty} d(x_{n+2}, x_n) = 0. \tag{17}$$

Let $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Without loss of generality, assume that $m > n$. Thus, $x_m = T^{m-n}(T^nx_0) = T^nx_0 = x_n$. Consider now

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, x_n) = d(Tx_m, x_m) \\ &= d(Tx_m, Tx_{m-1}) \\ &\leq \alpha(x_m, x_{m-1}) d(Tx_m, Tx_{m-1}) \\ &\leq \psi(d(x_m, x_{m-1})) \\ &\leq \psi^{m-n}(d(x_{n+1}, x_n)). \end{aligned} \tag{18}$$

Due to (ii) of Lemma 1, inequality (18) turns into

$$d(x_{n+1}, x_n) \leq \psi^{m-n}(d(x_{n+1}, x_n)) < d(x_{n+1}, x_n), \tag{19}$$

which is a contradiction. Hence, $\{x_n\}$ has no periodic point.

In what follows, we will prove that the sequence $\{x_n\}$ is Cauchy. For this purpose, it is sufficient to examine two cases. Case (I): suppose that $k > 2$ and k is odd. Let $k = 2m+1, k \geq 1$. Then, by using the quadrilateral inequality together with (16), we find

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + d(x_{n+2m}, x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^n(d(x_1, x_0)) \\ &\leq \sum_{p=n}^{+\infty} \psi^n(d(x_1, x_0)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{20}$$

Case (II): let $k > 2$ and let k be even. Let $k = 2m, k \geq 1$. Then, by applying the quadrilateral inequality together with (16) and (17), we find

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \\ &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + d(x_{n+2m-1}, x_{n+2m}) \\ &\leq \sum_{p=n}^{n+k-1} \psi^n(d(x_2, x_0)) \\ &\leq \sum_{p=n}^{+\infty} \psi^n(d(x_2, x_0)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{21}$$

By combining the expressions (20) and (21) we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{22}$$

Since T is continuous, we obtain from (22) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0. \tag{23}$$

From (22) and (23) we get immediately that $\lim_{n\to\infty} Tx_n x_0 = \lim_{n\to\infty} Tx_n = Tu$. Taking Proposition 5 into account, we conclude that u is a fixed point of T; that is, $Tu = u$. □

Theorem 12. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be an $\alpha - \psi$ contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Following the lines in the proof of Theorem 11, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ converges for some $u \in X$. From (9) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (6), for all k, we get that

$$\begin{aligned} d(x_{n(k)+1}, Tu) &= d(Tx_{n(k)}, Tu) \\ &\leq \alpha(x_{n(k)}, u)\, d(Tx_{n(k)}, Tu) \\ &\leq \psi(d(x_{n(k)}, u)). \end{aligned} \tag{24}$$

Letting $k \to \infty$ in the above equality, we find that

$$\lim_{k\to\infty} d(x_{n(k)+1}, Tu) = 0. \tag{25}$$

By Proposition 5, we obtain that u is a fixed point of T; that is, $Tu = u$. □

For the uniqueness, we need an additional condition.

(U) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y) \geq 1$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 13. *Adding condition (U) to the hypotheses of Theorem 11 (resp., Theorem 12), one obtains that u is the unique fixed point of T.*

Proof. In what follows we will show that u is a unique fixed point of T. We will use the *reductio ad absurdum*. Let v be another fixed point of T with $v \neq u$. It is evident that $\alpha(u, v) = \alpha(Tu, Tv)$.

Now, due to (6), we have

$$\begin{aligned} d(u, v) &\leq \alpha(u, v)\, d(u, v) \\ &= \alpha(Tu, Tv)\, d(Tu, Tv) \\ &\leq \psi(d(u, v)) < d(u, v) \end{aligned} \tag{26}$$

which is a contradiction. Hence, $u = v$. □

As an alternative condition for the uniqueness of a fixed point of a $\alpha - \psi$ contractive mapping, we will consider the following hypothesis.

(H) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 14. *Adding conditions (H) and (W) to the hypotheses of Theorem 11 (resp., Theorem 12), one obtains that u is the unique fixed point of T.*

Proof. Suppose that v is another fixed point of T. From (H), there exists $z \in X$ such that

$$\alpha(u, z) \geq 1, \qquad \alpha(v, z) \geq 1. \tag{27}$$

Since T is α-admissible, from (27), we have

$$\alpha(u, T^n z) \geq 1, \quad \alpha(v, T^n z) \geq 1, \quad \forall n. \tag{28}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \geq 0$ and $z_0 = z$. From (28), for all n, we have

$$\begin{aligned} d(u, z_{n+1}) &= d(Tu, Tz_n) \\ &\leq \alpha(u, z_n)\, d(Tu, Tz_n) \\ &\leq \psi(d(u, z_n)). \end{aligned} \tag{29}$$

Iteratively, by using inequality (29), we get that

$$d(u, z_{n+1}) \leq \psi^n(d(u, z_0)), \tag{30}$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{31}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{32}$$

Regarding (W) together with (31) and (32), it follows that $u = v$. Thus we proved that u is the unique fixed point of T. □

Corollary 15. *Adding condition (H) to the hypotheses of Theorem 11 (resp., Theorem 12) and assuming that (X, d) is Hausdorff, one obtains that u is the unique fixed point of T.*

The proof is clear and hence it is omitted. Indeed, Hausdorffness implies the uniqueness of the limit. Thus, the theorem above yields the conclusions.

Example 16. Let $X = A \cup B \cup C$ where $A = (-\infty, 0)$, $B = \{1/2, 1/3, 1/4, 1/5\}$, and $C = [1, 2]$. Define the generalized metric d on X as follows:

$$\begin{aligned} &d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0.3, \\ &d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = 0.2, \\ &d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{3}\right) = 0.6, \\ &d\left(\frac{1}{2},\frac{1}{2}\right) = d\left(\frac{1}{3},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{5}\right) = 0, \\ &d(x,y) = |x-y| \quad \text{otherwise.} \end{aligned} \tag{33}$$

It is clear that d does not satisfy triangle inequality on A. Indeed,

$$0.6 = d\left(\frac{1}{2},\frac{1}{4}\right) \geq d\left(\frac{1}{2},\frac{1}{3}\right) + d\left(\frac{1}{3},\frac{1}{4}\right) = 0.5. \tag{34}$$

Notice that (GMS3) holds, so d is a generalized metric.

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x \in C, \\ \frac{1}{4} & \text{if } x \in B, \\ 0 & \text{if } x \in A. \end{cases} \tag{35}$$

Define

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in B \cup C, \\ 0 & \text{if otherwise.} \end{cases} \tag{36}$$

It is clear that T is an $\alpha - \psi$ contractive mapping with $\psi(t) = t/2$ for all $t \in [0, \infty)$. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x = 1/2$, we have

$$\alpha\left(\frac{1}{2}, T\frac{1}{2}\right) = \alpha\left(\frac{1}{2},\frac{1}{4}\right) = 1. \tag{37}$$

Notice also that T is α-admissible mapping. To show this assume that $x, y \in X$ with $\alpha(x, y) \geq 1$. It yields that $x, y \in B \cup C$. Owing to the definition of the mapping T, we have

$$Tx, Ty \in B \cup C, \quad \text{and hence } \alpha(Tx, Ty) \geq 1. \tag{38}$$

Thus, the mapping T is α-admissible. It is clear that the mapping T is not continuous. On the other hand, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\{x_n\} \in B \cup C$. Recall that the sequence $\{x_n\}$ is defined iteratively, by $x_n = Tx_{n-1}$ for each integer $n \geq 1$, with an initial (arbitrary) point x_0. In this case, the initial point x_0 lies in either B or C. If $x_0 \in B$, then the sequence $\{x_n\}$ is a constant sequence and hence tends to $1/4 \in B$. Thus, $\alpha(x_n, x_{n+1}) \geq 1$ for all n implies that $\alpha(x_n, 1/4) = \alpha(x_n, 1/4) \geq 1$. If $x_0 \in C$, then the sequence $\{x_n\}$ is a constant sequence and hence tends to $1/2 \in B$. Thus, $\alpha(x_n, x_{n+1}) \geq 1$ for all n implies that $\alpha(x_n, 1/2) = \alpha(x_n, 1/2) \geq 1$.

Then T satisfies the conditions of Theorem 12 and has a (unique) fixed point on X; that is, $x = 1/4$.

3. Consequences

Now, we will show that many existing results in the literature can be deduced easily from our Theorems 11 and 12.

Corollary 17. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)), \tag{39}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Let $\alpha : X \times X \to [0, \infty)$ be the mapping defined by $\alpha(x, y) = 1$, for all $x, y \in X$. Then T is an $\alpha - \psi$-contraction mapping. It is evident that all conditions of Theorem 11 are satisfied. Hence, T has a unique fixed point. □

The following fixed point theorems follow immediately from Corollary 17 by taking $\psi(t) = \lambda t$, where $\lambda \in (0, 1)$.

Corollary 18 (Branciari [1]). *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y), \tag{40}$$

for all $x, y \in X$. Then T has a unique fixed point.

Remark 19. Our results improve and correct the results of Branciari [1] in which the analog of Banach fixed point theorem was proved. In the literature, to correct the proof of Branciari [1], some authors assume some superfluous conditions such as Hausdorffness of the induced topology of generalized metric space and continuity of generalized metric function. Inspired by the interesting papers of [3, 4] we prove the analog of Banach fixed point theorem in the context of generalized metric space without any further condition.

The notion of transitivity of mapping $\alpha : X \times X \to [0, +\infty)$ was introduced in [24, 25] as follows.

Definition 20 (see [24, 25]). Let $N \in \mathbb{N}$. One says that α is N-transitive (on X) if

$$x_0, x_1, \ldots, x_{N+1} \in X : \alpha(x_i, x_{i+1}) \geq 1, \tag{41}$$

for all $i \in \{0, 1, \ldots, N\} \Rightarrow \alpha(x_0, x_{N+1}) \geq 1$.

In particular, we say that α is transitive if it is 1-transitive; that is,

$$\begin{aligned} &x, y, z \in X : \alpha(x, y) \geq 1, \\ &\alpha(y, z) \geq 1 \Longrightarrow \alpha(x, z) \geq 1. \end{aligned} \tag{42}$$

As consequences of Definition 20, we obtain the following remarks.

Remark 21 (see [24, 25]). (1) Any function $\alpha : X \times X \to [0, +\infty)$ is 0-transitive.

(2) If α is N-transitive, then it is kN-transitive for all $k \in \mathbb{N}$.

(3) If α is transitive, then it is N-transitive for all $N \in \mathbb{N}$.

(4) If α is N-transitive, then it is not necessarily transitive for all $N \in \mathbb{N}$.

Corollary 22. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be an $\alpha - \psi$ contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and α is transitive;*

(iii) *either*

(a) *T is continuous,or*

(b) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. Regarding assumption (ii) of the theorem, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Consequently, we have $\alpha(Tx_0, T^2 x_0) \geq 1$, by (i). Since α is transitive, we derive that $\alpha(x_0, T^2 x_0) \geq 1$. Hence all conditions of Theorem 11 (and, resp., Theorem 12) are satisfied. □

The following are evident from Theorems 13 and 14.

Corollary 23. *Adding condition (U) to the hypotheses of Corollary 22, one concludes that u is the unique fixed point of T.*

Corollary 24. *Adding conditions (H) and (W) to the hypotheses of Corollary 22, one obtains that u is the unique fixed point of T.*

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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