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*Research Article*

# Practical Stability of Impulsive Discrete Systems with Time Delays

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The purpose of this paper is to investigate the practical stability problem for impulsive discrete systems with time delays. By using Lyapunov functions and the Razumikhin-type technique, some criteria which guarantee the practical stability and uniformly asymptotically practical stability of the addressed systems are provided. Finally, two examples are presented to illustrate the criteria.

## 1. Introduction

As we all know, in many applications, we use discrete systems rather than continuous ones as the mathematical modeling, for example, numerical analysis, control theory, population models, and computer science [1–3]. Therefore, more and more attention has been paid to the theory of discrete systems, and some results for the stability of discrete systems have been obtained over the past few years [4–8].

The theory of practical stability has developed into a branch of the theory of motion stability [9]. Its notion is very useful, since it only needs to stabilize a system into a region of phase space. Based on this method, the desired state of a system can be unstable only if it oscillates sufficiently near this state. Recently, there has been a significant development in the theory of practical stability [10–15]. Moreover, impulses and time delays exist in many processes of dynamic systems, for example, physics, chemical technology, population dynamics, and neural networks, and they may impact systems seriously [16–30]. Therefore, it is necessary and important to analyze the practical stability of impulsive discrete systems with time delays.

In [7, 8], authors have obtained some results for asymptotic stability and exponential stability of impulsive discrete systems with time delays. Unfortunately, there is almost no result concerning uniformly asymptotically practical stability of impulsive discrete systems with time delays. The purpose of this paper is to establish some criteria which guarantee uniformly asymptotically practical stability of the addressed systems by using Lyapunov functions and the Razumikhin-type technique. This work is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, the main results are presented. In Section 4, two examples are discussed to illustrate the results.

## 2. Preliminaries

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers, $\mathbb{R}^m$ the $m$-dimensional real space equipped with the Euclidean norm $\|\cdot\|$, $\mathbb{Z}$ the set of integers, and $\mathbb{Z}_+$ the set of positive integers. For any $r > 0$, $r \in \mathbb{Z}_+$, $J \triangleq \{-r, -r+1, -r+2, \ldots, -1, 0\}$, and set $C(\mathbb{R}_+, \mathbb{R}_+) \triangleq \{\phi : \mathbb{R}_+ \to \mathbb{R}_+ \mid \phi \text{ is continuous}\}$. Let $S \triangleq \{\varphi : J \to \mathbb{R}^m\}$. Let $S_\rho \triangleq \{\varphi \in S : \|\varphi\| < \rho\}$. The norm of $\varphi$ is defined by $\|\varphi\|_J = \max_{s\in J}|\varphi(s)|$. The impulse times $n_k$ satisfy $0 < n_1 < n_2 < \cdots < n_k < \cdots, n_k, k \in \mathbb{Z}_+$, and $\lim_{k\to+\infty} n_k = +\infty$.

Consider the following impulsive discrete systems with time delays:

$$x(n+1) = f(n, \overline{x}_n), \quad n \ge n_0, \ n \in \mathbb{Z}_+,$$
$$\overline{x}(n) = \begin{cases} x(n), & n \ne n_k, \ k \in \mathbb{Z}_+, \\ x(n_k) + I_k(n_k, x(n_k)), & n = n_k, \ k \in \mathbb{Z}_+, \end{cases} \tag{1}$$
$$x_{n_0}(s) = \varphi(s), \quad s \in J,$$

where $0 \le n_0 < n_1$, $\varphi \in S$, $f \in \mathbb{Z}_+ \times S_\rho \to \mathbb{R}^m$, $f(n,0) = 0$. For each $n \ge n_0$, $x_n \in S_\rho$ is defined by $x_n(s) = x(n+s)$, $s \in J$. For each $k \in \mathbb{Z}_+$, $I_k \in \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $I_k(n,0) = 0$, and, for any $\rho > 0$, there exists a $\rho_1 \in (0,\rho)$ such that $x \in S(\rho_1)$ implies that $x + I_k \in S(\rho)$.

In this paper, we assume that $f$ and $I_k$ satisfy certain conditions such that the solution of system (1) exists on $[n_0 - r, +\infty) \cap \mathbb{Z}_+$ and is unique [4]. We denote by $\overline{x}(n) = \overline{x}(n, n_0, \varphi)$ the solution of system (1) with initial value $\varphi$.

For convenience, we define the following classes of functions:

$K = \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : w$ is strictly increasing and $w(0) = 0\}$;

$K_1 = \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : w(0) = 0$ and $w(s) > 0$ for $s > 0\}$;

$K_2 = \{\psi \in C(\mathbb{R}_+, \mathbb{R}_+) : \psi$ is increasing and $\psi(s) < s$ for $s > 0\}$.

In addition, we introduce some definitions as follows.

*Definition 1* (see [9]). Given two constants $\lambda$ and $A$, $0 < \lambda < A$. Then, the impulsive discrete system (1) with respect to $(\lambda, A)$ is said to be

($S_1$) practically stable, if $\|\varphi\|_J < \lambda$ implies $\|x(n)\| < A$, $n \ge n_0$, $n \in \mathbb{Z}_+$,

($S_2$) uniformly practically stable if ($S_1$) holds, for every $n_0 \in \mathbb{Z}_+$,

($S_3$) asymptotically practically stable, if ($S_1$) holds and, for any $\epsilon > 0$, there exists $T = T(n_0, \epsilon) > 0$, $T \in \mathbb{Z}_+$, such that $\|\varphi\|_J < \lambda$ implies $\|x(n)\| < \epsilon$, $n \ge n_0 + T$, $n \in \mathbb{Z}_+$,

($S_4$) uniformly asymptotically practically stable if ($S_2$) holds and the latter part of ($S_3$) holds for a constant $T = T(\epsilon) > 0$, $T \in \mathbb{Z}_+$, only dependent on $\epsilon$.

## 3. Main Results

**Theorem 2.** *Assume that there exist functions $a, b \in K$, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi \in K_2$, $V : \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}_+$, such that*

(i) *$0 < \lambda < A$ are given,*

(ii) *$a(\|x\|) \le V(n,x) \le b(\|x\|)$ for $(n,x) \in \mathbb{Z}_+ \times \mathbb{R}^m$,*

(iii) *$V(n_k, \overline{x}(n_k)) = V(n_k, x(n_k) + I_k(n_k, x(n_k))) \le \psi(V(n_k, x(n_k)))$;*

(iv) *there is a function $P(s)$ continuous and nondecreasing for $s \ge 0$ and satisfying $P(s) > \psi^{-1}(s)$, $s > 0$, such that, for any solution $x(n)$ of system (1), $P(V(n, x(n))) \ge V(n+s, x(n+s))$, $s \in J$, implies that*

$$\Delta V(n, x(n)) = V(n+1, x(n+1)) - V(n, \overline{x}(n)) \le \omega(V(n, \overline{x}(n))), \tag{2}$$

*where $\tau \triangleq \max_{k \in \mathbb{Z}_+}\{n_{k+1} - n_k\}$ and*

$$\tau \sup_{s>0} \frac{\omega(s)}{s} < 1 - 2\left(\inf_{s>0} \frac{\psi^{-1}(s)}{s}\right)^{-1}, \tag{3}$$

(v) *$b(\lambda) < \psi(a(A))$.*

*Then, the system* (1) *with respect to* $(\lambda, A)$ *is uniformly asymptotic practically stable.*

*Proof.* Let

$$q \triangleq \sup_{s>0} \frac{\omega(s)}{s}, \qquad p \triangleq \left(\inf_{s>0} \frac{\psi^{-1}(s)}{s}\right)^{-1} < 1. \tag{4}$$

For any $n_0 \ge 0$, let $\overline{x}(n) \doteq \overline{x}(n, n_0, \varphi)$ be the solution of system (1) through $(n_0, \varphi)$, where $(n_0, \varphi) \in \mathbb{Z}_+ \times S$, and $\|\varphi\|_J < \lambda$. It suffices to show that

$$\|x\| < A, \qquad n \ge n_0, \quad n \in \mathbb{Z}_+. \tag{5}$$

Now, we show that

$$V(n, x(n)) \le \psi^{-1}(b(\lambda)), \quad n \in [n_0, n_1] \cap \mathbb{Z}_+. \tag{6}$$

If it does not hold, then there exists a $r \in [n_0, n_1] \cap \mathbb{Z}_+$, such that $V(r, x(r)) > \psi^{-1}(b(\lambda))$. Let $r_2 = \min\{n : V(n, x(n)) > \psi^{-1}(b(\lambda)), n \in [n_0, n_1] \cap \mathbb{Z}_+\}$. Since $V(n_0, x(n_0)) \le b(\lambda) \le \psi^{-1}(b(\lambda))$, it is clear that $r_2 > n_0$. Let $r_1 = \max\{n : V(n, x(n)) \le b(\lambda), n \in [n_0, r_2) \cap \mathbb{Z}_+\}$. Thus,

$$V(r_2, x(r_2)) > \psi^{-1}(b(\lambda)), \qquad V(r_1, x(r_1)) \le b(\lambda), \\ b(\lambda) < V(t, x(t)) \le \psi^{-1}(b(\lambda)), \quad n \in (r_1, r_2) \cap \mathbb{Z}_+. \tag{7}$$

Hence, we obtain

$$\begin{aligned} V(r_2, x(r_2)) - V(r_1, x(r_1)) &> \psi^{-1}(b(\lambda)) - b(\lambda) \\ &= \psi^{-1}(b(\lambda))\left(1 - \frac{b(\lambda)}{\psi^{-1}(b(\lambda))}\right) \\ &\ge \psi^{-1}(b(\lambda))(1-p). \end{aligned} \tag{8}$$

By (7), we obtain that, for any $n \in [r_1, r_2] \cap \mathbb{Z}_+$,

$$\begin{aligned} P(V(n, x(n))) &> \psi^{-1}(V(n, x(n))) \ge \psi^{-1}(b(\lambda)) \\ &\ge V(n+s, x(n+s)), \quad s \in J. \end{aligned} \tag{9}$$

Using condition (iv), the inequality $\Delta V(n, x(n)) \leq \omega(V(n, \overline{x}(n))) = \omega(V(n, x(n)))$ holds for all $n \in [r_1, r_2] \cap \mathbb{Z}_+$. Thus,

$$\begin{aligned} V(r_2, x(r_2)) - V(r_1, x(r_1)) &= V(r_2, x(r_2)) - V(r_2 - 1, x(r_2 - 1)) \\ &\quad + V(r_2 - 1, x(r_2 - 1)) - V(r_2 - 2, x(r_2 - 2)) \\ &\quad + \cdots + V(r_1 + 1, x(r_1 + 1)) - V(r_1, x(r_1)) \\ &= \Delta V(r_2 - 1, x(r_2 - 1)) + \Delta V(r_2 - 2, x(r_2 - 2)) \\ &\quad + \cdots + \Delta V(r_1, x(r_1)) \\ &\leq \omega(V(r_2 - 1, x(r_2 - 1))) + \omega(V(r_2 - 2, x(r_2 - 2))) \\ &\quad + \cdots + \omega(V(r_1, x(r_1))) \\ &\leq V(r_2 - 1, x(r_2 - 1)) \frac{\omega(V(r_2 - 1, x(r_2 - 1)))}{V(r_2 - 1, x(r_2 - 1))} \\ &\quad + \cdots + V(r_1, x(r_1)) \frac{\omega(V(r_1, x(r_1)))}{V(r_1, x(r_1))} \\ &\leq \psi^{-1}(b(\lambda)) q\tau. \end{aligned} \tag{10}$$

From (8) and (10), it can be deduced that $1 - p < q\tau$, which is a contradiction with the condition (iv) and, thus, (6) holds.

Then, it follows from condition (iii) that

$$\begin{aligned} V(n_1, \overline{x}(n_1)) &= V(n_1, x(n_1) + I_1(n_1, x(n_1))) \\ &\leq \psi(V(n_1, x(n_1))) \leq b(\lambda). \end{aligned} \tag{11}$$

Next, we claim that

$$V(n, x(n)) \leq \psi^{-1}(b(\lambda)), \quad n \in [n_1, n_2] \cap \mathbb{Z}_+. \tag{12}$$

If this assertion is not true, then there exists a $r \in [n_1, n_2] \cap \mathbb{Z}_+$, such that $V(r, x(r)) > \psi^{-1}(b(\lambda))$. Let $r_4 = \min\{n : V(n, x(n)) > \psi^{-1}(b(\lambda)), n \in [n_1, n_2] \cap \mathbb{Z}_+\}$. Since $V(n_1, x(n_1)) \leq b(\lambda) \leq \psi^{-1}(b(\lambda))$, we have

$$r_4 > n_1, \quad V(r_4, x(r_4)) > \psi^{-1}(b(\lambda)). \tag{13}$$

Let $r_3 = \max\{n : V(n, x(n)) \leq b(\lambda), n \in [n_1, r_4) \cap \mathbb{Z}_+\}$. Thus,

$$V(r_3, x(r_3)) \leq b(\lambda), \tag{14}$$

$$b(\lambda) < V(n, x(n)) \leq \psi^{-1}(b(\lambda)), \quad n \in (r_3, r_4) \cap \mathbb{Z}_+. \tag{15}$$

Hence, we obtain

$$\begin{aligned} V(r_4, x(r_4)) - V(r_3, x(r_3)) &> \psi^{-1}(b(\lambda)) - b(\lambda) \\ &= \psi^{-1}(b(\lambda)) \left(1 - \frac{b(\lambda)}{\psi^{-1}(b(\lambda))}\right) \\ &\geq \psi^{-1}(b(\lambda))(1 - p). \end{aligned} \tag{16}$$

Considering (15), we obtain, for any $n \in [r_3, r_4] \cap \mathbb{Z}_+$,

$$\begin{aligned} P(V(n, x(n))) &> \psi^{-1}(V(n, x(n))) \geq \psi^{-1}(b(\lambda)) \\ &\geq V(n + s, x(n + s)), \quad s \in J. \end{aligned} \tag{17}$$

Using condition (iv), the inequality $\Delta V(n, x(n)) \leq \omega(V(n, \overline{x}(n))) = \omega(V(n, x(n)))$ holds for all $n \in [r_3, r_4] \cap \mathbb{Z}_+$. Thus,

$$\begin{aligned} V(r_4, x(r_4)) - V(r_3, x(r_3)) &= V(r_4, x(r_4)) - V(r_4 - 1, x(r_4 - 1)) \\ &\quad + V(r_4 - 1, x(r_4 - 1)) - V(r_4 - 2, x(r_4 - 2)) \\ &\quad + \cdots + V(r_3 + 1, x(r_3 + 1)) - V(r_3, x(r_3)) \\ &= \Delta V(r_4 - 1, x(r_4 - 1)) + \Delta V(r_4 - 2, x(r_4 - 2)) \\ &\quad + \cdots + \Delta V(r_3, x(r_3)) \\ &\leq \omega(V(r_4 - 1, x(r_4 - 1))) + \omega(V(r_4 - 2, x(r_4 - 2))) \\ &\quad + \cdots + \omega(V(r_3, x(r_3))) \\ &\leq V(r_4 - 1, x(r_4 - 1)) \frac{\omega(V(r_4 - 1, x(r_4 - 1)))}{V(r_4 - 1, x(r_4 - 1))} \\ &\quad + \cdots + V(r_3, x(r_3)) \frac{\omega(V(r_3, x(r_3)))}{V(r_3, x(r_3))} \\ &\leq \psi^{-1}(b(\lambda)) q\tau. \end{aligned} \tag{18}$$

From (16) and (18), it can be deduced that $1 - p < q\tau$, which is a contradiction with the condition (iv) and, thus, (12) holds.

Then, it follows from condition (iii) that

$$\begin{aligned} V(n_2, \overline{x}(n_2)) &= V(n_2, x(n_2) + I_2(n_2, x(n_2))) \\ &\leq \psi(V(n_2, x(n_2))) \leq b(\lambda). \end{aligned} \tag{19}$$

Similarly, it can be deduced that

$$V(n, x(n)) \leq \psi^{-1}(b(\lambda)), \quad n \in [n_2, n_3] \cap \mathbb{Z}_+. \tag{20}$$

By simple induction, we can prove that

$$\begin{aligned} V(n, x(n)) &\leq \psi^{-1}(b(\lambda)), \quad n \in [n_k, n_{k+1}] \cap \mathbb{Z}_+, \ k \in \mathbb{Z}_+, \\ V(n_{k+1}, \overline{x}(n_{k+1})) &= V(n_{k+1}, x(n_{k+1}) + I_{k+1}(n_{k+1}, x(n_{k+1}))) \\ &\leq \psi(V(n_{k+1}, x(n_{k+1}))) \\ &\leq b(\lambda) \\ &< \psi^{-1}(b(\lambda)). \end{aligned} \tag{21}$$

It follows from conditions (ii) and (v) that

$$\begin{aligned} V(n, x(n)) &\leq \psi^{-1}(b(\lambda)) < a(A), \\ \|x(n)\| &\leq a^{-1}(V(n, x(n))) \\ &< a^{-1}(a(A)) = A, \quad n \geq n_0, \ n \in \mathbb{Z}_+. \end{aligned} \tag{22}$$

This inequality implies that the system (1) with respect to $(\lambda, A)$ is uniformly practically stable.

Next, we show that the system (1) with respect to $(\lambda, A)$ is uniformly asymptotically practically stable. For any $\epsilon$, $0 < \epsilon < A$, there exist numbers $a = a(\epsilon) > 0$, $0 < d < a$, such that

$$\begin{gathered} P(s) > \psi^{-1}(s) + a, \\ \psi^{-1}(s) + a > \psi^{-1}(s+d), \\ s \in \left[a(\epsilon), \psi^{-1}(b(\lambda))\right]. \end{gathered} \tag{23}$$

Let $N = N(\epsilon) \in \mathbb{Z}_+$ satisfy $a(\epsilon) + (N-1)d \le \psi^{-1}(b(\lambda)) \le a(\epsilon) + Nd$, and $T = (N-1)\nu r$, $T \in \mathbb{Z}_+$, where $\nu \ge 1$. We will prove that

$$V(n, x(n)) \le \psi^{-1}(a(\epsilon)), \quad n \ge n_0 + T, \ n \in \mathbb{Z}_+. \tag{24}$$

In order to do this, we first prove that there exists a $T_1 \ge n_0$, $T_1 \in \mathbb{Z}_+$, such that

$$V(T_1, x(T_1)) \le a(\epsilon) + (N-1)d. \tag{25}$$

If (25) does not hold, then, for any $n \ge n_0$, $n \in \mathbb{Z}_+$, $V(n, x(n)) > a(\epsilon) + (N-1)d$.

Note that, for $s \in J$,

$$\begin{aligned} P(V(n, x(n))) &> \psi^{-1}(V(n, x(n))) + a \\ &\ge \psi^{-1}(a(\epsilon) + (N-1)d) + a \\ &> \psi^{-1}(a(\epsilon) + Nd) \ge \psi^{-1}(b(\lambda)) \\ &\ge V(n+s, x(n+s)). \end{aligned} \tag{26}$$

Thus,

$$\Delta V(n, x(n)) \le \omega(V(n, \overline{x}(n))), \quad n \ge n_0. \tag{27}$$

Hence, we obtain

$$\begin{aligned} &V(n_2, \overline{x}(n_2)) - V(n_1 - 1, \overline{x}(n_1 - 1)) \\ &\quad = V(n_2, \overline{x}(n_2)) - V(n_2, x(n_2)) + V(n_2, x(n_2)) \\ &\qquad - V(n_2 - 1, x(n_2 - 1)) + V(n_2 - 1, x(n_2 - 1)) \\ &\qquad - \cdots + V(n_1 + 1, x(n_1 + 1)) - V(n_1, \overline{x}(n_1)) \\ &\qquad + V(n_1, \overline{x}(n_1)) - V(n_1, x(n_1)) \\ &\qquad + V(n_1, x(n_1)) - V(n_1 - 1, x(n_1 - 1)) \\ &\quad \le \psi(V(n_2, x(n_2))) - V(n_2, x(n_2)) \\ &\qquad + \psi(V(n_1, x(n_1))) - V(n_1, x(n_1)) \\ &\qquad + \Delta V(n_2 - 1, x(n_2 - 1)) + \Delta V(n_2 - 2, x(n_2 - 2)) \\ &\qquad + \cdots + \Delta V(n_1 - 1, x(n_1 - 1)) \\ &\quad \le \psi(V(n_2, x(n_2))) - V(n_2, x(n_2)) \\ &\qquad + \psi(V(n_1, x(n_1))) - V(n_1, x(n_1)) \\ &\qquad + \omega(V(n_2 - 1, x(n_2 - 1))) \\ &\qquad + \omega(V(n_2 - 2, x(n_2 - 2))) \\ &\qquad + \cdots + \omega(V(n_1 - 1, x(n_1 - 1))) \\ &\quad \le V(n_2, x(n_2)) \left( \frac{\psi(V(n_2, x(n_2)))}{V(n_2, x(n_2))} - 1 \right) \\ &\qquad + V(n_1, x(n_1)) \left( \frac{\psi(V(n_1, x(n_1)))}{V(n_1, x(n_1))} - 1 \right) \\ &\qquad + V(n_2 - 1, x(n_2 - 1)) \frac{\omega(V(n_2 - 1, x(n_2 - 1)))}{V(n_2 - 1, x(n_2 - 1))} \\ &\qquad + V(n_2 - 2, x(n_2 - 2)) \frac{\omega(V(n_2 - 2, x(n_2 - 2)))}{V(n_2 - 2, x(n_2 - 2))} \\ &\qquad + \cdots + V(n_1 - 1, x(n_1 - 1)) \frac{\omega(V(n_1 - 1, x(n_1 - 1)))}{V(n_1 - 1, x(n_1 - 1))} \\ &\quad \le \psi^{-1}(b(\lambda))(p-1) + \psi^{-1}(b(\lambda)) q\tau \\ &\qquad + \psi^{-1}(b(\lambda))(p-1) \\ &\quad \le \psi^{-1}(b(\lambda))(2p - 2 + q\tau). \end{aligned} \tag{28}$$

Thus,

$$V(n_2, \overline{x}(n_2)) \le \psi^{-1}(b(\lambda))(2p - 1 + q\tau) < 0, \tag{29}$$

which is a contradiction. Thus, there exists a $T_1 \ge n_0$, $T_1 \in \mathbb{Z}_+$, such that (25) holds.

Next, we prove that

$$V(n, x(n)) \le \psi^{-1}(a(\epsilon) + (N-1)d), \quad n \ge T_1, \ n \in \mathbb{Z}_+. \tag{30}$$

Let $m = \min\{m \in \mathbb{Z}_+ : n_m \ge T_1\}$, and we show that

$$V(n, x(n)) \le \psi^{-1}(a(\epsilon) + (N-1)d), \quad n \in [T_1, n_m] \cap \mathbb{Z}_+. \tag{31}$$

If (31) does not hold, then there is a $r \in [T_1, n_m] \cap \mathbb{Z}_+$ such that

$$V(r, x(r)) > \psi^{-1}(a(\epsilon) + (N-1)d). \tag{32}$$

Let $r^\star = \min\{n : V(n, x(n)) > \psi^{-1}(a(\epsilon) + (N-1)d), n \in [T_1, t_m]\}$. Since

$$V(T_1, x(T_1)) \le a(\epsilon) + (N-1)d \le \psi^{-1}(a(\epsilon) + (N-1)d), \tag{33}$$

we have

$$r^\star > T_1, \quad V(r^\star, x(r^\star)) > \psi^{-1}(a(\epsilon) + (N-1)d). \tag{34}$$

Let $\hat{r} = \max\{n : V(n, x(n)) \le a(\epsilon) + (N-1)d, n \in [T_1, r^\star)\}$. Note

$$V(r^\star, x(r^\star)) > \psi^{-1}(a(\epsilon) + (N-1)d) > a(\epsilon) + (N-1)d. \tag{35}$$

Thus,

$$\begin{gathered}\hat{r} < r^\star, \quad V(\hat{r}, x(\hat{r})) \le a(\epsilon) + (N-1)d, \\ a(\epsilon) + (N-1)d < V(n, x(n)) \le \psi^{-1}(a(\epsilon) + (N-1)d), \\ n \in (\hat{r}, r^\star) \cap \mathbb{Z}_+.\end{gathered} \tag{36}$$

Hence, we obtain

$$\begin{aligned}&V(r^\star, x(r^\star)) - V(\hat{r}, x(\hat{r})) \\ &\quad > \psi^{-1}(a(\epsilon) + (N-1)d) - (a(\epsilon) + (N-1)d) \\ &\quad = \psi^{-1}(a(\epsilon) + (N-1)d)\left(1 - \frac{a(\epsilon) + (N-1)d}{\psi^{-1}(a(\epsilon) + (N-1)d)}\right) \\ &\quad \ge \psi^{-1}(a(\epsilon) + (N-1)d)(1-p).\end{aligned} \tag{37}$$

On the other hand, note that, for any $n \in [\hat{r}, r^\star] \cap \mathbb{Z}_+$,

$$\begin{aligned}P(V(n, x(n))) &> \psi^{-1}(V(n, x(n))) \\ &\quad + a \ge \psi^{-1}(a(\epsilon) + (N-1)d) + a \\ &> \psi^{-1}(a(\epsilon) + Nd) \ge \psi^{-1}(b(\lambda)) \\ &\ge V(n+s, x(n+s)), \quad s \in J.\end{aligned} \tag{38}$$

Using condition (iv), the inequality $\Delta V(n, x(n)) \le \omega(V(n, \overline{x}(n))) = \omega(V(n, x(n)))$ holds for all $n \in [\hat{r}, r^\star] \cap \mathbb{Z}_+$. Thus,

$$\begin{aligned}&V(r^\star, x(r^\star)) - V(\hat{r}, x(\hat{r})) \\ &\quad = V(r^\star, x(r^\star)) - V(r^\star - 1, x(r^\star - 1)) \\ &\qquad + V(r^\star - 1, x(r^\star - 1)) - V(r^\star - 2, x(r^\star - 2)) \\ &\qquad + \cdots + V(\hat{r} + 1, x(\hat{r} + 1)) - V(\hat{r}, x(\hat{r})) \\ &\quad = \Delta V(r^\star - 1, x(r^\star - 1)) + \Delta V(r^\star - 2, x(r^\star - 2)) \\ &\qquad + \cdots + \Delta V(\hat{r}, x(\hat{r})) \\ &\quad \le \omega(V(r^\star - 1, x(r^\star - 1))) + \omega(V(r^\star - 2, x(r^\star - 2))) \\ &\qquad + \cdots + \omega(V(\hat{r}, x(\hat{r}))) \\ &\quad \le V(r^\star - 1, x(r^\star - 1))\frac{\omega(V(r^\star - 1, x(r^\star - 1)))}{V(r^\star - 1, x(r^\star - 1))} \\ &\qquad + \cdots + V(\hat{r}, x(\hat{r}))\frac{\omega(V(\hat{r}, x(\hat{r})))}{V(\hat{r}, x(\hat{r}))} \\ &\quad \le \psi^{-1}(a(\epsilon) + (N-1)d)q\tau.\end{aligned} \tag{39}$$

From (37) and (39), it can be deduced that $1 - p < q\tau$, which is a contradiction. Thus (31) holds.

Then, from condition (iv), we get

$$\begin{aligned}V(n_m, \overline{x}(n_m)) &= V(n_m, x(n_m) + I_m(n_m, x(n_m))) \\ &\le \psi(V(n_m, x(n_m))) \le a(\epsilon) + (N-1)d.\end{aligned} \tag{40}$$

Similarly, it can be deduced that

$$\begin{gathered}V(n, x(n)) \le \psi^{-1}(a(\epsilon) + (N-1)d), \\ n \in [n_m, n_{m+1}] \cap \mathbb{Z}_+.\end{gathered} \tag{41}$$

By simple induction, one may derive that

$$\begin{aligned}&V(n, x(n)) \le \psi^{-1}(a(\epsilon) + (N-1)d), \\ &\qquad n \in [n_k, n_{k+1}] \cap \mathbb{Z}_+, \quad k \ge m, \\ &V(n_{k+1}, \overline{x}(n_{k+1})) \\ &\quad = V(n_{k+1}, x(n_{k+1}) + I_{k+1}(n_{k+1}, x(n_{k+1}))) \\ &\quad \le \psi(V(n_{k+1}, x(n_{k+1}))) \\ &\quad \le a(\epsilon) + (N-1)d.\end{aligned} \tag{42}$$

Thus, (30) holds.

Similarly, we can prove that there exists a $T_2 \ge T_1 + \nu r$, $\nu \ge 1$ such that

$$V(T_2, x(T_2)) \le a(\epsilon) + (N-2)d. \tag{43}$$

By simple induction, we can prove, in general, that

$$V\left(T_j, x\left(T_j\right)\right) \leq a(\epsilon) + (N-j)d,$$
$$V(n, x(n)) \leq \psi^{-1}\left(a(\epsilon) + (N-j)d\right), \quad (44)$$
$$n \geq T_j, \quad j = 1, 2, \ldots, N.$$

Therefore, when choosing $j = N$, we obtain

$$V(n, x(n)) \leq \psi^{-1}(a(\epsilon)), \quad n \geq T_N, \quad (45)$$

where $T_N \geq n_0 + (N-1)\nu\tau$. Therefore,

$$\|x\| \leq a^{-1}\left(\psi^{-1}(a(\epsilon))\right), \quad n \geq n_0 + T, \quad (46)$$

where $T = (N-1)\nu r$. The proof is complete. □

*Remark 3.* It can be found from Theorem 2 that it requires that the distance between two adjacent impulse times cannot be too long, and meanwhile the function $V$ should decrease at impulse times. We can see that impulses do contribute to the system's practical stability behavior. In the following, another result will be presented from the impulsive perturbation point of view, which is different from Theorem 2.

**Theorem 4.** *Assume that there exist functions $a, b \in K$, $V: \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}_+$ such that*

(i) $0 < \lambda < A$ *are given,*

(ii) $a(\|x\|) \leq V(n, x) \leq b(\|x\|)$ *for* $(n, x) \in \mathbb{Z}_+ \times \mathbb{R}^m$,

(iii) $V(n_k, \overline{x}(n_k)) = V(n_k, x(n_k) + I_k(n_k, x(n_k))) \leq (1 + \beta_k)V(n_k, x(n_k))$, *where* $\beta_k \geq 0$, $\sum_{k=1}^{\infty} \beta_k < \infty$,

(iv) $V(n, x(n)) \geq V(n+s, x(n+s))$, $s \in J$, *implies that*

$$\Delta V(n, x(n)) = V(n, x(n)) - V(n-1, \overline{x}(n-1)) \leq 0, \quad (47)$$

*where $x(n)$ is a solution of system* (1),

(v) $Mb(\lambda) < a(A)$, $M = \prod_{k=1}^{\infty}(1 + \beta_k) < \infty$.

*Then, the system* (1) *with respect to* $(\lambda, A)$ *is uniformly practically stable.*

*Proof.* For any $n_0 \geq 0$, let $\overline{x}(n) \doteq \overline{x}(n, n_0, \varphi)$ be the solution of system (1) through $(n_0, \varphi)$, where $(n_0, \varphi) \in \mathbb{Z}_+ \times S$ and $\|\varphi\|_J < \lambda$. It suffices to show that

$$\|x\| < A, \qquad n \geq n_0, \quad n \in \mathbb{Z}_+. \quad (48)$$

Next, we prove that

$$V(n, x(n)) \leq Mb(\lambda), \quad n \in [n_0, +\infty) \cap \mathbb{Z}_+. \quad (49)$$

First, we show that

$$V(n, x(n)) \leq b(\lambda), \quad n \in [n_0, n_1] \cap \mathbb{Z}_+. \quad (50)$$

If it does not hold, then there exists a $r \in [n_0, n_1] \cap \mathbb{Z}_+$ such that

$$V(r, x(r)) > b(\lambda). \quad (51)$$

Let $r_1 = \min\{n : V(n, x(n)) > b(\lambda), n \in [n_0, n_1] \cap \mathbb{Z}_+\}$. Since $V(n_0, x(n_0)) \leq b(\lambda)$, it is clear that

$$V(r_1, x(r_1)) > b(\lambda),$$
$$V(r_1 - 1, x(r_1 - 1)) \leq b(\lambda),$$
$$\Delta V(r_1, x(r_1)) = V(r_1, x(r_1)) - V(r_1 - 1, \overline{x}(r_1 - 1)) > 0,$$
$$r_1 > n_0. \quad (52)$$

Thus, for $s \in J$,

$$V(r_1, x(r_1)) > b(\lambda) \geq V(r_1 + s, x(r_1 + s)). \quad (53)$$

By condition (iv), we have that

$$\Delta V(r_1, x(r_1)) \leq 0, \quad (54)$$

which is a contradiction. Thus, (50) holds.

From (50) and condition (iii), we obtain

$$V(n_1, \overline{x}(n_1)) \leq (1 + \beta_1)V(n_1, x(n_1)) \leq (1 + \beta_1)b(\lambda). \quad (55)$$

Next, we show that

$$V(n, x(n)) \leq (1 + \beta_1)b(\lambda), \quad n \in [n_1, n_2] \cap \mathbb{Z}_+. \quad (56)$$

If this assertion is not true, then there exists a $r \in [n_1, n_2] \cap \mathbb{Z}_+$ such that

$$V(r, x(r)) > (1 + \beta_1)b(\lambda). \quad (57)$$

Let $r_2 = \min\{n : V(n, x(n)) > (1 + \beta_1)b(\lambda), n \in [n_1, n_2] \cap \mathbb{Z}_+\}$. Since $V(n_1, x(n_1)) \leq (1 + \beta_1)b(\lambda)$, we get

$$V(r_2, x(r_2)) > (1 + \beta_1)b(\lambda),$$
$$V(r_2 - 1, x(r_2 - 1)) \leq (1 + \beta_1)b(\lambda),$$
$$\Delta V(r_2, x(r_2)) = V(r_2, x(r_2)) - V(r_2 - 1, \overline{x}(r_2 - 1)) > 0,$$
$$r_2 > n_1. \quad (58)$$

Thus, for $s \in J$,

$$V(r_2, x(r_2)) > (1 + \beta_2)b(\lambda) \geq V(r_2 + s, x(r_2 + s)). \quad (59)$$

By condition (iv), we have

$$\Delta V(r_2, x(r_2)) \leq 0, \quad (60)$$

which is a contradiction. Thus, (56) holds.

Considering (30) and condition (iii), it can be deduced that

$$V(n_2, \overline{x}(n_2)) \leq (1 + \beta_2)V(n_2, x(n_2))$$
$$\leq (1 + \beta_1)(1 + \beta_2)b(\lambda). \quad (61)$$

By simple induction, we have

$$V(n,x(n)) \le (1+\beta_1)(1+\beta_2)\cdots(1+\beta_k)\,b(\lambda), \quad n\in[n_k,n_{k+1}]\cap\mathbb{Z}_+, \quad k\in\mathbb{Z}_+, \tag{62}$$

which, together with (50) and condition (v), yields that

$$V(n,x(n)) \le Mb(\lambda) < a(A), \quad n\ge n_0,\ n\in\mathbb{Z}_+. \tag{63}$$

Therefore, from condition (ii), we have

$$\|x\| \le A, \qquad n\ge n_0, \quad n\in\mathbb{Z}_+. \tag{64}$$

Thus, system (1) with respect to $(\lambda, A)$ is uniformly practically stable.

The proof is complete. □

**Theorem 5.** *Assume that there exist functions $a, b \in K$, $P, \omega \in K_1$, $V: \mathbb{Z}_+\times\mathbb{R}^m \to \mathbb{R}_+$, such that*

(i) $0<\lambda<A$ *are given,*

(ii) $a(\|x\|)\le V(n,x)\le b(\|x\|)$ *for* $(n,x)\in\mathbb{Z}_+\times\mathbb{R}^m$,

(iii) $V(n_k,\overline{x}(n_k)) = V(n_k, x(n_k) + I_k(n_k,x(n_k))) \le (1+\beta_k)V(n_k,x(n_k))$, *where* $\beta_k\ge 0$, $\sum_{k=1}^{\infty}\beta_k<\infty$,

(iv) $P(V(n,x(n)))\ge V(n+s,x(n+s))$, $s\in J$, *implies that*

$$\Delta V(n,x(n)) = V(n,x(n)) - V(n-1,\overline{x}(n-1)) \le -\omega(V(n,x(n))), \tag{65}$$

*with* $P(s)>Ms$, $s>0$, $M=\prod_{k=1}^{\infty}(1+\beta_k)<\infty$, *where* $x(n)$ *is a solution of system* (1);

(v) $Mb(\lambda)<a(A)$.

*Then, the system* (1) *with respect to* $(\lambda, A)$ *is uniformly asymptotically practically stable.*

*Proof.* For any $n_0\ge 0$, let $\overline{x}(n) \doteq \overline{x}(n,n_0,\varphi)$ be the solution of system (1) through $(n_0,\varphi)$, where $(n_0,\varphi)\in\mathbb{Z}_+\times S$, and $\|\varphi\|_J<\lambda$. From Theorem 4, it is easy to see that the system (1) with respect to $(\lambda, A)$ is uniformly practically stable. Now, we show that the system (1) with respect to $(\lambda, A)$ is uniformly asymptotically practically stable.

For any $\epsilon\in(0,A)$, there exists number $d=d(\epsilon)>0$ such that

$$P(s) > Ms + d, \quad s\in\left[\frac{a(\epsilon)}{M}, Mb(\lambda)\right]. \tag{66}$$

Let $N=N(\epsilon)$ be the smallest positive integer such that

$$\frac{a(\epsilon)+Nd}{M} \ge Mb(\lambda),$$
$$\gamma = \inf_{a(\epsilon)/M\le s\le Mb(\lambda)} \omega(s) > \left(1+M\overline{M}\right)b(\lambda), \quad \overline{M} = \sum_{k=1}^{\infty}\beta_k. \tag{67}$$

We will prove that there exists $T\in\mathbb{Z}_+$ such that

$$V(n,x(n)) \le a(\epsilon), \quad n\ge n_0+T,\ n\in\mathbb{Z}_+. \tag{68}$$

To this end, we first prove that there exists $T_1\in(n_k,n_{k+1})\cap\mathbb{Z}_+$, $k\in\mathbb{Z}_+$, such that

$$V(n,x(n)) \le a(\epsilon) + (N-1)d, \quad n\ge T_1,\ n\in\mathbb{Z}_+. \tag{69}$$

In fact, when $n\in[n_0,T_1]$, there exists a $N_1\in[n_m,n_{m+1})\cap\mathbb{Z}_+\subseteq[n_0,T_1]\cap\mathbb{Z}_+$, $m\in\mathbb{Z}_+$, such that

$$V(N_1,x(N_1)) \le \frac{a(\epsilon)+(N-1)d}{M}. \tag{70}$$

If (70) does not hold, it is clear that, for any $n\in[n_0,T_1]\cap\mathbb{Z}_+$,

$$V(n,x(n)) > \frac{a(\epsilon)+(N-1)d}{M}, \qquad \frac{a(\epsilon)}{M} \le V(n,x(n)) \le Mb(\lambda). \tag{71}$$

Thus, for $s\in J$,

$$\begin{aligned} P(V(n,x(n))) &\ge MV(n,x(n)) + d \ge a(\epsilon) \\ &\quad + (N-1)d + d = a(\epsilon) + Nd \\ &\ge Mb(\lambda) \ge V(n+s,x(n+s)). \end{aligned} \tag{72}$$

It follows from condition (iv) that

$$\Delta V(n,x(n)) \le -\omega(V(n,x(n))) \le -\gamma. \tag{73}$$

On the other hand, since there at least exists one point $T_1$ which is not an impulsive point, we obtain

$$\begin{aligned} &V(T_1,x(T_1)) - V(n_0,x(n_0)) \\ &= V(T_1,x(T_1)) - V(T_1-1,x(T_1-1)) \\ &\quad + V(T_1-1,x(T_1-1)) - V(T_1-2,x(T_1-2)) \\ &\quad + \cdots + V(n_0+1,x(n_0+1)) - V(n_0,x(n_0)) \\ &= \Delta V(T_1,x(T_1)) + \Delta V(T_1-1,x(T_1-1)) \\ &\quad + \cdots + \Delta V(n_0+1,x(n_0+1)) \\ &\le -\omega(V(T_1,x(T_1))) + \sum_{j=1}^{k} V\left(n_{j-1},x\left(n_{j-1}\right)\right)\beta_j \\ &\le -\gamma + M\overline{M}b(\lambda). \end{aligned} \tag{74}$$

Thus,

$$V(T_1,x(T_1)) \le Mb(\lambda)\left(1+\overline{M}\right) - \gamma < 0, \tag{75}$$

which is a contradiction. Hence, when $n\in[n_0,T_1]\cap\mathbb{Z}_+$, there exists a $N_1\in[n_m,n_{m+1})\cap\mathbb{Z}_+\subseteq[n_0,T_1]\cap\mathbb{Z}_+$, $m\in\mathbb{Z}_+$, such that

$$V(N_1,x(N_1)) \le \frac{a(\epsilon)+(N-1)d}{M}. \tag{76}$$

Then, we claim that

$$V(n,x(n)) \le \frac{a(\epsilon)+(N-1)d}{M}, \quad n\in[N_1,n_{m+1}]\cap\mathbb{Z}_+. \tag{77}$$

If (77) does not hold, there exists a $\widehat{N} \in [N_1, n_{m+1}] \cap \mathbb{Z}_+$ such that

$$V\left(\widehat{N}, x\left(\widehat{N}\right)\right) > \frac{a(\epsilon) + (N-1)d}{M}. \quad (78)$$

Let $\widetilde{N} = \min\{n : V(n, x(n)) > (a(\epsilon) + (N-1)d)/M, n \in [N_1, n_{m+1}] \cap \mathbb{Z}_+\}$. Since

$$V(N_1, x(N_1)) \le \frac{a(\epsilon) + (N-1)d}{M}, \quad (79)$$

we have that

$$V\left(\widetilde{N}, x\left(\widetilde{N}\right)\right) > \frac{a(\epsilon) + (N-1)d}{M}, \quad \Delta V\left(\widetilde{N}, x\left(\widetilde{N}\right)\right) > 0, \quad \widetilde{N} > N_1. \quad (80)$$

Note that $a(\epsilon)/M \le V(\widetilde{N}, x(\widetilde{N})) \le Mb(\lambda)$; thus for $s \in J$,

$$P\left(V\left(\widetilde{N}, x\left(\widetilde{N}\right)\right)\right) \ge MV\left(\widetilde{N}, x\left(\widetilde{N}\right)\right) + d \ge a(\epsilon) + (N-1)d + d = a(\epsilon) + Nd \ge Mb(\lambda) \ge V\left(\widetilde{N}+s, x\left(\widetilde{N}+s\right)\right). \quad (81)$$

It follows from condition (iv) that

$$\Delta V\left(\widetilde{N}, x\left(\widetilde{N}\right)\right) \le -\omega\left(V\left(\widetilde{N}, x\left(\widetilde{N}\right)\right)\right) < 0, \quad (82)$$

which is a contradiction. Thus, (77) holds.

Considering (77) and condition (iii), it can be deduced that

$$V(n_{m+1}, \overline{x}(n_{m+1})) \le (1+\beta_{m+1}) V(n_{m+1}, x(n_{m+1})) \le (1+\beta_{m+1}) \frac{a(\epsilon) + (N-1)d}{M}. \quad (83)$$

Similarly, we may show

$$V(n, x(n)) \le (1+\beta_{m+1}) \frac{a(\epsilon) + (N-1)d}{M}, \quad n \in [n_{m+1}, n_{m+2}],$$
$$V(n_{m+2}, \overline{x}(n_{m+2})) \le (1+\beta_{m+2}) V(n_{m+2}, x(n_{m+2})) \le (1+\beta_{m+2})(1+\beta_{m+1}) \times \frac{a(\epsilon) + (N-1)d}{M}. \quad (84)$$

By simple induction, we can prove in general that

$$V(n, x(n)) \le \prod_{j=1}^{i}(1+\beta_{m+j}) \frac{a(\epsilon) + (N-1)d}{M} \le a(\epsilon) + (N-1)d, \quad n \in [n_{m+i}, n_{m+i+1}], \ i \in \mathbb{Z}_+. \quad (85)$$

Thus, (69) holds.

Next, we prove that there exists $T_2 \in (n_l, n_{l+1}) \cap \mathbb{Z}_+, l \in \mathbb{Z}_+$, $T_2 > T_1 + qr, q > 1, q \in \mathbb{Z}_+$, such that

$$V(n, x(n)) \le a(\epsilon) + (N-2)d, \quad n \ge T_2, \ n \in \mathbb{Z}_+. \quad (86)$$

In fact, when $n \in [T_1 + r, T_2]$, there exists a $N_2 \in [n_{\overline{m}}, n_{\overline{m}+1}] \cap \mathbb{Z}_+ \subseteq [T_1 + r, T_2] \cap \mathbb{Z}_+, \overline{m} \in \mathbb{Z}_+$, such that

$$V(N_2, x(N_2)) \le \frac{a(\epsilon) + (N-1)d}{M}. \quad (87)$$

If (87) does not hold, it is clear that, for any $n \in [T_1 + r, T_2] \cap \mathbb{Z}_+$,

$$V(n, x(n)) > \frac{a(\epsilon) + (N-2)d}{M}, \quad \frac{a(\epsilon)}{M} \le V(n, x(n)) \le Mb(\lambda). \quad (88)$$

Thus, for $s \in J$,

$$P(V(n, x(n))) \ge MV(n, x(n)) + d \ge a(\epsilon) + (N-2)d + d = a(\epsilon) + (N-1)d \ge Mb(\lambda) \ge V(n+s, x(n+s)). \quad (89)$$

It follows from condition (iv) that

$$\Delta V(n, x(n)) \le -\omega(V(n, x(n))) \le -\gamma. \quad (90)$$

On the other hand, since there at least exists one point $T_2$ which is not an impulsive point, we obtain

$$\begin{aligned} V(T_2, x(T_2)) - V(T_1 + r, x(T_1 + r)) &= V(T_2, x(T_2)) - V(T_2 - 1, x(T_2 - 1)) \\ &\quad + V(T_2 - 1, x(T_2 - 1)) - V(T_2 - 2, x(T_2 - 2)) \\ &\quad + \cdots + V(T_1 + r + 1, x(T_1 + r + 1)) \\ &\quad - V(T_1 + r, x(T_1 + r)) \\ &= \Delta V(T_2, x(T_2)) + \Delta V(T_2 - 1, x(T_2 - 1)) \\ &\quad + \cdots + \Delta V(T_1 + r + 1, x(T_1 + r + 1)) \\ &\le -\omega(V(T_2, x(T_2))) + \sum_{j=T_1+r}^{n_{\overline{m}}} V(n_{j-1}, x(n_{j-1}))\beta_j \\ &\le -\gamma + M\overline{M}b(\lambda). \end{aligned} \quad (91)$$

Thus,

$$V(T_2, x(T_2)) \le Mb(\lambda)\left(1 + \overline{M}\right) - \gamma \le 0, \quad (92)$$

which is a contradiction. Hence, when $n \in [T_1 + r, T_2] \cap \mathbb{Z}_+$, there exists a $N_2 \in [n_{\overline{m}}, n_{\overline{m}+1}) \subseteq [T_1 + r, T_2] \cap \mathbb{Z}_+, \overline{m} \in \mathbb{Z}_+$, such that

$$V(N_2, x(N_2)) \le \frac{a(\epsilon) + (N-2)d}{M}. \quad (93)$$

Similarly, we can prove that (86) holds.

By simple induction, we have that

$$V(n, x(n)) \leq a(\epsilon) + (N - i) d, \quad n \geq n_0 + T_i, \quad i = 1, 2, \ldots, N, \quad n \in \mathbb{Z}_+. \tag{94}$$

Therefore, when choosing $i = N$, we obtain

$$V(n, x(n)) \leq a(\epsilon), \quad n \geq n_0 + T_N, \ n \in \mathbb{Z}_+. \tag{95}$$

From condition (ii), we have that

$$\|x(n)\| \leq \epsilon, \quad n \geq n_0 + T_N, \ n \in \mathbb{Z}_+. \tag{96}$$

The proof is complete. □

## 4. Applications

The following illustrative examples will demonstrate the effectiveness of our results.

*Example 6.* Consider the following impulsive discrete system:

$$\begin{gathered} x(n+1) = cx(n) + dx(n-1), \quad n \geq n_0, \ n \in \mathbb{Z}_+, \\ \overline{x}(n) = \begin{cases} x(n), & n \neq n_k, \ k \in \mathbb{Z}_+, \\ \eta x(n_k), & n = n_k, \ k \in \mathbb{Z}_+, \end{cases} \\ x_{n_0}(s) = \varphi(s), \quad s \in J, \end{gathered} \tag{97}$$

where $c, d, \eta$ are any three constants and $0 < \eta < 1$.

*Property 1.* Given constants $\lambda$, $A$ satisfy $\lambda < \eta A$, and there is a constant $\zeta > 1/\eta$.

Then, the system (97) with respect to $(\lambda, A)$ is uniformly asymptotically practically stable if

$$\tau(|c-1| + \zeta|d|) < 1 - 2\eta, \tag{98}$$

where $\tau \triangleq \max_{k \in \mathbb{Z}_+}\{n_{k+1} - n_k\}$.

*Proof.* Choose $V(n, x(n)) = |x(n)|$, where $x(n)$ is a solution of system (97). Let $b(s) = a(s) = s$, $P(s) = \zeta s$, $\omega(s) = (|c-1| + \zeta|d|)s$, $\psi(s) = \eta s$, $s > 0$, and then

$$\begin{aligned} &\Delta V(n, x(n)) \\ &= V(n+1, x(n+1)) - V(n, \overline{x}(n)) \\ &= |x(n+1)| - |x(n)| \\ &= |cx(n) + dx(n-1)| - |x(n)| \\ &\leq |(c-1)x(n) + dx(n-1)| \\ &\leq |c-1| \cdot |x(n)| + |d| \cdot |x(n-1)| \\ &= |c-1| V(n, x(n)) + |d| V(n-1, x(n-1)) \\ &\leq |c-1| V(n, x(n)) + |d| \zeta V(n, x(n)) \\ &= (|c-1| + \zeta|d|) V(n, x(n)). \end{aligned} \tag{99}$$

By Theorem 2, the above property can be easily derived. □

*Example 7.* Consider the following impulsive discrete system:

$$\begin{gathered} x(n+1) = \frac{x(n)[(1+d)x(n-1) + cd]}{x(n-1) + c}, \quad n \geq n_0, \ n \in \mathbb{Z}_+, \\ \overline{x}(n) = \begin{cases} x(n), & n \neq n_k, \ k \in \mathbb{Z}_+, \\ \eta x(n_k), & n = n_k, \ k \in \mathbb{Z}_+, \end{cases} \\ x_{n_0}(s) = \varphi(s), \quad s \in J, \end{gathered} \tag{100}$$

where $c, d, \eta$ are three constants and $c > 0$, $0 < \eta < 1$.

*Property 2.* Given constants $\lambda$, $A$ satisfy $\lambda < \eta A$, and there is a constant $\zeta > 1/\eta$.

Then, the system (100) with respect to $(\lambda, A)$ is uniformly asymptotically practically stable if

$$\tau|d| < 1 - 2\eta, \tag{101}$$

where $\tau \triangleq \max_{k \in \mathbb{Z}_+}\{n_{k+1} - n_k\}$.

*Proof.* Choose $V(n, x(n)) = |x(n)|$, where $x(n)$ is a solution of system (100). Let $b(s) = a(s) = s$, $P(s) = \zeta s$, $\omega(s) = s \cdot (|d| + 1/(1 + c/\zeta s) - 1)$, $\psi(s) = \eta s$, $s > 0$, and then

$$\begin{aligned} &\Delta V(n, x(n)) \\ &= V(n+1, x(n+1)) - V(n, \overline{x}(n)) \\ &= |x(n+1)| - |x(n)| \\ &= \left|\frac{x(n)[(1+d)x(n-1) + cd]}{x(n-1) + c}\right| - |x(n)| \\ &= |x(n)| \cdot \left(\left|\frac{dx(n-1) + cd + x(n-1)}{x(n-1) + c}\right| - 1\right) \\ &= |x(n)| \cdot \left(\left|d + \frac{1}{1 + c/x(n-1)}\right| - 1\right) \\ &\leq |x(n)| \cdot \left(|d| + \frac{1}{1 + c/V(n-1, x(n-1))} - 1\right) \\ &\leq V(n, x(n)) \cdot \left(|d| + \frac{1}{1 + c/\zeta V(n, x(n))} - 1\right) \\ &= \omega(V(n)). \end{aligned} \tag{102}$$

By Theorem 2, the above property can be easily derived. □

*Remark 8.* According to Theorem 2, we obtained the sufficient conditions guaranteeing uniformly asymptotically practical stability of the two impulsive discrete systems, respectively. In other words, the results we presented are effective for both linear and nonlinear impulsive discrete systems.

## 5. Conclusion

In this paper, we considered the impulsive discrete systems with time delays. Based on Lyapunov functions and the

Razumikhin-type technique, the practical stability and uniformly asymptotically practical stability have been presented, which is dependent on both the impulses and the time delays. To our knowledge, there is almost no result concerning the problem of practical stability for impulsive discrete systems with delays. According to the analysis, we can see that impulses do contribute to the system's practical stability behavior. Two examples have been illustrated to demonstrate the usefulness of the proposed method.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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