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Research Article

Landau-Type Theorems for Certain Biharmonic Mappings

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Let $F(z) = |z|^2 g(z) + h(z)$ $(|z| < 1)$ be a biharmonic mapping of the unit disk $\mathbb{D}$, where g and h are harmonic in $\mathbb{D}$. In this paper, the Landau-type theorems for biharmonic mappings of the form $L(F)$ are provided. Here L represents the linear complex operator $L = (z\partial/\partial z) - \overline{(z\partial/\partial z)}$ defined on the class of complex-valued C^1 functions in the plane. The results, presented in this paper, improve the related results of earlier authors.

1. Introduction

Suppose that $f(z) = u(x, y) + iv(x, y), z = x + iy$ is a four times continuously differentiable complex-valued function in a domain $D \subset \mathbb{C}$. If f satisfies the biharmonic equation $\Delta(\Delta f) = 0$, then we call that f is biharmonic, where Δ represents the Laplacian operator:

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1}$$

Biharmonic functions arise in many physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering (see [1] for details). It is known that a mapping F is biharmonic in a simply connected domain D if and only if F has the following representation:

$$F(z) = |z|^2 g(z) + h(z), \tag{2}$$

where $g(z)$ and $h(z)$ are complex-valued harmonic functions in D [1]. Also, it is known that $g(z)$ and $h(z)$ can be expressed as

$$\begin{aligned} g(z) &= g_1(z) + \overline{g_2}(z), \quad z \in D, \\ h(z) &= h_1(z) + \overline{h_2}(z), \quad z \in D, \end{aligned} \tag{3}$$

where g_1, g_2, k_1, and k_2 are analytic in D [2, 3].

For a continuously differentiable mapping f in D, we define

$$\begin{aligned} \Lambda_f(z) &= \max_{0\leqslant\theta\leqslant 2\pi} \left| f_z + e^{-2i\theta} f_{\overline{z}} \right| = \left| f_z \right| + \left| f_{\overline{z}} \right|, \\ \lambda_f(z) &= \min_{0\leqslant\theta\leqslant 2\pi} \left| f_z + e^{-2i\theta} f_{\overline{z}} \right| = \left| \left| f_z \right| - \left| f_{\overline{z}} \right| \right|. \end{aligned} \tag{4}$$

We use J_f to denote the Jacobian of f

$$J_f(z) = \left| f_z(z) \right|^2 - \left| f_{\overline{z}}(z) \right|^2. \tag{5}$$

Then $J_f = \lambda_f \Lambda_f$ if $J_f \geqslant 0$.

In [4], the authors considered the following differential operator L defined on the class of complex-valued C^1 functions:

$$L = z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}. \tag{6}$$

Evidently, L is a complex linear operator and satisfies the usual product rule:

$$\begin{aligned} L(af + bg) &= aL(f) + bL(g), \\ L(fg) &= fL(g) + gL(f), \end{aligned} \tag{7}$$

where a and b are complex constants; f and g are C^1 functions. In addition, the operator L possesses a number of interesting properties. For instance, it is easy to see that the operator L preserves both harmonicity and biharmonicity. Many other basic properties are stated in [4].

Landau's theorem states that if f is an analytic function on the unit disk $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{D}$, then f is univalent in the disk $\mathbb{D}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$ with $r_0 = 1/(M + \sqrt{M^2 - 1})$, and $f(\mathbb{D}_{r_0})$ contains a disk $\mathbb{D}_{R_0}$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f(z) = Mz((1 - Mz)/(M - z))$. Recently, many authors considered the Landau-type theorems for harmonic mappings [5–9] and biharmonic mappings [1, 4, 10–13]. Chen et al. [10] obtained the Landau-type theorems for biharmonic mappings of the form $L(F)$ as follows.

Theorem A (see [10]). *Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $F(0) = h(0) = 0$ and $J_h(0) = 1$, where $g(z)$ and $h(z)$ are harmonic in $\mathbb{D}$. Assume that both $|g(z)|$ and $|h(z)|$ are bounded by M. Then there is a constant ρ_1 $(0 < \rho_1 < 1)$ such that $L(F)$ is univalent in $\mathbb{D}_{\rho_1}$, where ρ_1 satisfies the following equation:*

$$\frac{\pi}{4M} - \frac{6M\rho_1^2}{(1-\rho_1)^2} - \frac{4M\rho_1^3}{(1-\rho_1)^3} - \frac{16M}{\pi^2} m_1 \arctan \rho_1 - \frac{4M\rho_1}{(1-\rho_1)^3} = 0, \tag{8}$$

where $m_1 \approx 6.059$ is the minimum value of the function

$$\frac{2 - x^2 + (4/\pi)\arctan x}{x(1-x^2)}, \tag{9}$$

for $0 < x < 1$. The minimum is attained at $x \approx 0.588$. Moreover, the range $L(F)(\mathbb{D}_{\rho_1})$ contains a schlicht disk $\mathbb{D}_{R_1}$, where

$$R_1 = \rho_1 \left[\frac{\pi}{4M} - \frac{2M\rho_1^2}{(1-\rho_1)^2} - \frac{16M}{\pi^2} m_1 \arctan \rho_1 \right]. \tag{10}$$

Theorem B (see [10]). *Let $F(z) = |z|^2 g(z)$ be a biharmonic mapping in $\mathbb{D}$ such that $g(0) = 0$, $J_g(0) = 1$, and $|g(z)| < M$, where $g(z)$ is harmonic in $\mathbb{D}$. Then there is a constant ρ_2 $(0 < \rho_2 < 1)$ such that $L(F)$ is univalent in $\mathbb{D}_{\rho_2}$, where ρ_2 satisfies the following equation:*

$$\frac{\pi}{4M} - \frac{48M}{\pi^2} m_1 \arctan \rho_2 - \frac{2M\rho_2}{(1-\rho_2)^3} = 0, \tag{11}$$

where m_1 is defined as in Theorem A. Moreover, $L(F)(\mathbb{D}_{\rho_2})$ contains a disk $\mathbb{D}_{R_2}$ with

$$R_2 = \rho_2^3 \left[\frac{\pi}{4M} - \frac{16M}{\pi^2} m_1 \arctan \rho_2 \right]. \tag{12}$$

However, these results are not sharp. The main object of this paper is to improve Theorems A and B. We get three versions of Landau-type theorems for biharmonic mappings of the form $L(F)$, where F belongs to the class of biharmonic mappings, and Theorems 11 and 14 improve Theorems A and B. In order to establish our main results, we need to recall the following lemmas.

Lemma 1 (see [6, 14]). *Suppose that $f(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ such that $|f(z)| \leqslant M$ for all $\mathbb{D}$. Then*

$$\Lambda_f(z) \leqslant \frac{4M}{\pi\left(1-|z|^2\right)}, \quad z \in \mathbb{D}. \tag{13}$$

The inequality is sharp.

Lemma 2 (see [9, 12, 15]). *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{D}$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Then $|a_0| \leqslant M$ and for any $n \geqslant 1$*

$$|a_n| + |b_n| \leqslant \frac{4M}{\pi}. \tag{14}$$

These estimates are sharp.

Lemma 3 (see [8, 11]). *Suppose that f is a harmonic mapping of $\mathbb{D}$ with $f(0) = \lambda_f(0) - 1 = 0$. If $\Lambda_f \leqslant \Lambda$ for $z \in \mathbb{D}$; then*

$$|a_n| + |b_n| \leqslant \frac{\Lambda^2 - 1}{n\Lambda}, \quad n = 2, 3, \ldots. \tag{15}$$

These estimates are sharp.

Lemma 4 (see [11]). *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{D}$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. If $|J_f(0)| = 1$; then $\lambda_f(0) \geqslant \lambda_0(M)$, where $M_0 = \pi/2\sqrt[4]{2\pi^2 - 16} \approx 1.1296$ and*

$$\lambda_0(M) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{M^2-1} + \sqrt{M^2+1}}, & 1 \leqslant M \leqslant M_0, \\ \dfrac{\pi}{4M}, & M > M_0. \end{cases} \tag{16}$$

Lemma 5 (see [13]). *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. If f satisfies $|f(z)| \leqslant M$ for all $z \in \mathbb{D}$ and $|J_f(0)| = 1$, then*

$$\left(\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 \right)^{1/2} \leqslant \sqrt{M^4 - 1} \cdot \lambda_f(0). \tag{17}$$

Lemma 6. *Suppose that $M > 0$, $\Lambda \geqslant 1$. Then the equation*

$$\varphi(r) = 1 - \frac{12Mr^2}{\pi(1-r^2)} - \frac{8Mr^3}{\pi(1-r)^3} - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{2r - r^2}{(1-r)^2} = 0 \tag{18}$$

has a unique root in $(0, 1)$.

Proof. It is easy to prove that the function φ is continuous and strictly decreasing on $[0, 1)$, $\varphi(0) = 1 > 0$, and $\lim_{r \to 1^-} \varphi(r) = -\infty$. Hence, the assertion follows from the mean value theorem. This completes the proof. □

Lemma 7. *Suppose that* $M_1 > 0$, $M_2 \geqslant 1$, *and* $\lambda_0(M_2)$ *is defined by* (16). *Then the equation*

$$\lambda_0(M_2) - \frac{12M_1r^2}{\pi(1-r^2)} - \frac{8M_1r^3}{\pi(1-r)^3} - \lambda_0(M_2)\sqrt{M_2^4-1} \cdot \left[\frac{2r\sqrt{4r^2+r^4+1}}{(1-r^2)^{5/2}} + \frac{r\sqrt{r^4-3r^2+4}}{(1-r^2)^{3/2}}\right] = 0 \quad (19)$$

has a unique root in $(0,1)$.

Lemma 8. *Let* $M \geqslant 1$. *Then the equation*

$$1 - \sqrt{M^4-1} \cdot \left[\frac{3r\sqrt{r^4-3r^2+4}}{(1-r^2)^{3/2}} + \frac{2r\sqrt{4r^2+r^4+1}}{(1-r^2)^{5/2}}\right] = 0 \quad (20)$$

has a unique root in $(0,1)$.

Lemma 9. *For any* $z_1 \neq z_2$ *in* $\mathbb{D}_r$ $(0<r<1)$, *we have*

$$\int_0^1 \left|tz_1 + (1-t)z_2\right|^2 dt \geqslant \frac{|z_1|^3 + |z_2|^3}{3(|z_1|+|z_2|)} > 0. \quad (21)$$

2. Main Results

We first establish a new version of the Landau-type theorem for biharmonic mappings on the unit disk $\mathbb{D}$ as follows.

Theorem 10. *Let* $F(z) = |z|^2g(z) + h(z)$ *be a biharmonic mapping of the unit disk* $\mathbb{D}$, *with* $F(0) = h(0) = \lambda_F(0) - 1 = 0$, $|g(z)| \leqslant M$, *and* $\Lambda_h(z) \leqslant \Lambda$ *for* $z \in \mathbb{D}$, *where* $M > 0, \Lambda \geqslant 1$. *Then* $L(F)$ *is univalent in the disk* $\mathbb{D}_{r_0}$, *where* r_0 *is the unique root in* $(0,1)$ *of the equation*

$$1 - \frac{12Mr^2}{\pi(1-r^2)} - \frac{8Mr^3}{\pi(1-r)^3} - \frac{\Lambda^2-1}{\Lambda} \cdot \frac{2r-r^2}{(1-r)^2} = 0, \quad (22)$$

and $L(F)(\mathbb{D}_{r_0})$ *contains a schlicht disk* $\mathbb{D}_{\sigma_0}$, *where*

$$\sigma_0 = r_0\left[1 - \frac{\Lambda^2-1}{\Lambda} \cdot \frac{r_0}{1-r_0} - \frac{4Mr_0^2}{\pi(1-r_0)^2}\right]. \quad (23)$$

Proof. Let $F(z) = |z|^2g(z) + h(z)$ satisfy the hypothesis of Theorem 10, where

$$g(z) = g_1(z) + \overline{g_2(z)} = \sum_{n=0}^{\infty} a_nz^n + \sum_{n=0}^{\infty} \overline{b_n}\overline{z}^n,$$
$$h(z) = h_1(z) + \overline{h_2(z)} = \sum_{n=1}^{\infty} c_nz^n + \sum_{n=1}^{\infty} \overline{d_n}\overline{z}^n \quad (24)$$

are harmonic in $\mathbb{D}$. As L is linear and $L(|z|^2) = 0$, we may set

$$H := L(F) = |z|^2L(g) + L(h). \quad (25)$$

Then we have

$$H_z = 2|z|^2g_z + |z|^2zg_{zz} - \overline{z}^2g_{\overline{z}} + h_z + zh_{zz},$$
$$H_{\overline{z}} = -2|z|^2g_{\overline{z}} - |z|^2\overline{z}g_{\overline{z}\overline{z}} + z^2g_z - h_{\overline{z}} - \overline{z}h_{\overline{z}\overline{z}}. \quad (26)$$

Note that $\lambda_F(0) = ||c_1| - |d_1|| = \lambda_h(0) = 1$; by Lemma 3, we have

$$|c_n| + |d_n| \leqslant \frac{\Lambda^2-1}{n\Lambda}, \quad n = 2,3,\ldots. \quad (27)$$

Thus, for $z_1 \neq z_2$ in $\mathbb{D}_r$ $(0<r<r_0)$, we have

$$\begin{aligned}\left|H(z_1) - H(z_2)\right| &= \left|\int_{[z_1,z_2]} H_z(z)\,dz + H_{\overline{z}}(z)\,d\overline{z}\right| \\ &\geqslant \left|\int_{[z_1,z_2]} h_z(0)\,dz - h_{\overline{z}}(0)\,d\overline{z}\right| \\ &\quad - 2\left|\int_{[z_1,z_2]} |z|^2\left(g_zdz - g_{\overline{z}}d\overline{z}\right)\right| \\ &\quad - \left|\int_{[z_1,z_2]} |z|^2\left(zg_{zz}dz - \overline{z}g_{\overline{z}\overline{z}}d\overline{z}\right)\right| \\ &\quad - \left|\int_{[z_1,z_2]} zh_{zz}dz - \overline{z}h_{\overline{z}\overline{z}}d\overline{z}\right| \\ &\quad - \left|\int_{[z_1,z_2]} z^2g_zd\overline{z} - \overline{z}^2g_{\overline{z}}dz\right| \\ &\quad - \left|\int_{[z_1,z_2]} \left(h_z - h_z(0)\right)dz - \left(h_{\overline{z}} - h_{\overline{z}}(0)\right)d\overline{z}\right|.\end{aligned} \quad (28)$$

Let

$$\begin{aligned} I_1 &= \left|\int_{[z_1,z_2]} h_z(0)\,dz - h_{\overline{z}}(0)\,d\overline{z}\right|, \\ I_2 &= \left|\int_{[z_1,z_2]} |z|^2\left(g_zdz - g_{\overline{z}}d\overline{z}\right)\right|, \\ I_3 &= \left|\int_{[z_1,z_2]} |z|^2\left(zg_{zz}dz - \overline{z}g_{\overline{z}\overline{z}}d\overline{z}\right)\right|, \\ I_4 &= \left|\int_{[z_1,z_2]} zh_{zz}dz - \overline{z}h_{\overline{z}\overline{z}}d\overline{z}\right|, \\ I_5 &= \left|\int_{[z_1,z_2]} z^2g_zd\overline{z} - \overline{z}^2g_{\overline{z}}dz\right|, \\ I_6 &= \left|\int_{[z_1,z_2]} \left(h_z - h_z(0)\right)dz - \left(h_{\overline{z}} - h_{\overline{z}}(0)\right)d\overline{z}\right|. \end{aligned} \quad (29)$$

By Lemmas 1, 2, and 3, elementary calculations yield that

$$\begin{aligned}
I_1 &\geqslant \int_{[z_1,z_2]} \lambda_h(0)\,|dz| = \lambda_h(0)\,|z_1-z_2| = |z_1-z_2|,\\
I_2 &\leqslant \int_{[z_1,z_2]} |z|^2 \left(|g_z|\,|dz| + |g_{\bar z}|\,|d\bar z|\right)\\
&\leqslant r^2 |z_1-z_2|\,\Lambda_g(z) \leqslant |z_1-z_2| \frac{4Mr^2}{\pi(1-r^2)},\\
I_3 &\leqslant |z_1-z_2| \sum_{n=2}^{\infty} n(n-1)\left(|a_n|+|b_n|\right) r^{n+1}\\
&\leqslant |z_1-z_2| \frac{8Mr^3}{\pi(1-r)^3},\\
I_4 &\leqslant |z_1-z_2| \sum_{n=2}^{\infty} n(n-1)\left(|c_n|+|d_n|\right) r^{n-1}\\
&\leqslant |z_1-z_2| \frac{\Lambda^2-1}{\Lambda}\cdot\frac{r}{(1-r)^2},\\
I_5 &\leqslant \int_{[z_1,z_2]} \left(|z|^2 |g_z|\,|d\bar z| + |\bar z|^2 |g_{\bar z}|\,|dz|\right)\\
&\leqslant r^2 |z_1-z_2|\,\Lambda_g(z) \leqslant |z_1-z_2| \frac{4Mr^2}{\pi(1-r^2)},\\
I_6 &\leqslant |z_1-z_2| \sum_{n=2}^{\infty} n\left(|c_n|+|d_n|\right) r^{n-1}\\
&\leqslant |z_1-z_2| \frac{\Lambda^2-1}{\Lambda}\cdot\frac{r}{1-r}.
\end{aligned} \tag{30}$$

Using these estimates and Lemma 6, we obtain

$$\begin{aligned}
&|H(z_1)-H(z_2)|\\
&\quad\geqslant I_1 - 2I_2 - I_3 - I_4 - I_5 - I_6\\
&\quad\geqslant |z_1-z_2| \left[1 - \frac{12Mr^2}{\pi(1-r^2)} - \frac{8Mr^3}{\pi(1-r)^3} - \frac{\Lambda^2-1}{\Lambda}\cdot\frac{2r-r^2}{(1-r)^2}\right] > 0,
\end{aligned} \tag{31}$$

which implies $H(z_1) \neq H(z_2)$.

For any z such that $z \in \partial\mathbb{D}_{r_0}$, by Lemmas 2, 4, and 5, we obtain

$$\begin{aligned}
|H(z)| &= \left| |z|^2 (zg_z - \bar z g_{\bar z}) + (zh_z - \bar z h_{\bar z}) \right|\\
&\geqslant |zh_z(0) - \bar z h_{\bar z}(0)|\\
&\quad - |z(h_z - h_z(0)) - \bar z (h_{\bar z} - h_{\bar z}(0))|\\
&\quad - \left| |z|^2 (zg_z - \bar z g_{\bar z}) \right|\\
&\geqslant r_0 \left[1 - \sum_{n=2}^{\infty} \left(|c_n|+|d_n|\right) n r_0^{n-1} - \sum_{n=1}^{\infty} \left(|a_n|+|b_n|\right) n r_0^{n+1}\right]\\
&\geqslant r_0 \left[1 - \frac{\Lambda^2-1}{\Lambda}\cdot\frac{r_0}{1-r_0} - \frac{4Mr_0^2}{\pi(1-r_0)^2}\right] = \sigma_0.
\end{aligned} \tag{32}$$

This completes the proof. □

Next we improve Theorem A as follows.

Theorem 11. *Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $\mathbb{D}$, with $F(0) = h(0) = J_F(0) - 1 = 0$, $|g(z)| \leqslant M_1$, and $|h(z)| \leqslant M_2$ for $z \in \mathbb{D}$, where $M_1 > 0$, $M_2 \geqslant 1$. Then $L(F)$ is univalent in the disk $\mathbb{D}_{r_3}$, where r_3 is the unique root in $(0,1)$ of the equation*

$$\begin{aligned}
&\lambda_0(M_2) - \frac{12M_1r^2}{\pi(1-r^2)} - \frac{8M_1r^3}{\pi(1-r)^3} - \lambda_0(M_2)\sqrt{M_2^4-1}\\
&\quad\cdot\left[\frac{2r\sqrt{4r^2+r^4+1}}{(1-r^2)^{5/2}} + \frac{r\sqrt{r^4-3r^2+4}}{(1-r^2)^{3/2}}\right] = 0,
\end{aligned} \tag{33}$$

and $L(F)(\mathbb{D}_{r_3})$ contains a schlicht disk $\mathbb{D}_{\sigma_3}$, where $\lambda_0(M_2)$ is defined by (16) and

$$\begin{aligned}
\sigma_3 = r_3 \Bigg[&\lambda_0(M_2) - \lambda_0(M_2)\sqrt{M_2^4-1}\\
&\cdot\frac{r_3\sqrt{r_3^4-3r_3^2+4}}{(1-r_3^2)^{3/2}} - \frac{4M_1r_3^2}{\pi(1-r_3)^2}\Bigg].
\end{aligned} \tag{34}$$

Proof. Note that $J_F(0) = |c_1|^2 - |d_1|^2 = J_h(0) = 1$; by Lemma 4, we have

$$\lambda_h(0) \geqslant \lambda_0(M_2). \tag{35}$$

We adopt the same method in Theorem 10, for $z_1 \neq z_2$ in $\mathbb{D}_r (0 < r < r_3)$; by Lemmas 1, 2, and 5, we get

$$I_1 \geqslant \int_{[z_1,z_2]} \lambda_h(0)\,|dz| = \lambda_h(0)\,|z_1 - z_2|,$$

$$\begin{aligned}
I_2 &\leqslant \int_{[z_1,z_2]} |z|^2 \left(|g_z|\,|dz| + |g_{\bar z}|\,|d\bar z|\right) \\
&\leqslant r^2 |z_1 - z_2|\, \Lambda_g(z) \leqslant |z_1 - z_2| \frac{4M_1 r^2}{\pi(1-r^2)}, \\
I_3 &\leqslant |z_1 - z_2| \sum_{n=2}^{\infty} n(n-1)\left(|a_n| + |b_n|\right) r^{n+1} \\
&\leqslant |z_1 - z_2| \frac{8M_1 r^3}{\pi(1-r)^3}, \\
I_4 &\leqslant |z_1 - z_2| \sum_{n=2}^{\infty} n(n-1)\left(|c_n| + |d_n|\right) r^{n-1} \\
&\leqslant |z_1 - z_2| \left(\sum_{n=2}^{\infty} \left(|c_n| + |d_n|\right)^2\right)^{1/2} \\
&\quad \cdot \left(\sum_{n=2}^{\infty} n^2 (n-1)^2 r^{2(n-1)}\right)^{1/2} \\
&\leqslant |z_1 - z_2|\, \lambda_h(0) \sqrt{M_2^4 - 1} \cdot \frac{2r\sqrt{4r^2 + r^4 + 1}}{(1-r^2)^{5/2}}, \\
I_5 &\leqslant \int_{[z_1,z_2]} \left(|z|^2 |g_z|\,|d\bar z| + |\bar z|^2 |g_{\bar z}|\,|dz|\right) \\
&\leqslant r^2 |z_1 - z_2|\, \Lambda_g(z) \leqslant |z_1 - z_2| \frac{4M_1 r^2}{\pi(1-r^2)}, \\
I_6 &\leqslant |z_1 - z_2| \sum_{n=2}^{\infty} n\left(|c_n| + |d_n|\right) r^{n-1} \\
&\leqslant |z_1 - z_2| \left(\sum_{n=2}^{\infty} \left(|c_n| + |d_n|\right)^2\right)^{1/2} \\
&\quad \cdot \left(\sum_{n=2}^{\infty} n^2 r^{2(n-1)}\right)^{1/2} \\
&\leqslant |z_1 - z_2|\, \lambda_h(0) \sqrt{M_2^4 - 1} \cdot \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1-r^2)^{3/2}}.
\end{aligned} \tag{36}$$

Using these estimates and Lemma 7, by (35), we obtain

$$\begin{aligned}
&|H(z_1) - H(z_2)| \\
&\geqslant I_1 - 2I_2 - I_3 - I_4 - I_5 - I_6 \\
&\geqslant |z_1 - z_2| \Bigg(\lambda_h(0) - \frac{12M_1 r^2}{\pi(1-r^2)} \\
&\quad - \frac{8M_1 r^3}{\pi(1-r)^3} - \lambda_h(0)\sqrt{M_2^4 - 1} \\
&\quad \cdot \left[\frac{2r\sqrt{4r^2 + r^4 + 1}}{(1-r^2)^{5/2}} + \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1-r^2)^{3/2}}\right]\Bigg) \\
&\geqslant |z_1 - z_2| \Bigg(\lambda_0(M_2) - \frac{12M_1 r^2}{\pi(1-r^2)} \\
&\quad - \frac{8M_1 r^3}{\pi(1-r)^3} - \lambda_0(M_2)\sqrt{M_2^4 - 1} \\
&\quad \cdot \left[\frac{2r\sqrt{4r^2 + r^4 + 1}}{(1-r^2)^{5/2}} + \frac{r\sqrt{r^4 - 3r^2 + 4}}{(1-r^2)^{3/2}}\right]\Bigg) > 0,
\end{aligned} \tag{37}$$

which implies $H(z_1) \neq H(z_2)$.

For any z such that $z \in \partial\mathbb{D}_{r_3}$, by (35) and Lemmas 2 and 5, we obtain

$$\begin{aligned}
|H(z)| &\geqslant r_3 \Bigg[\lambda_h(0) - \sum_{n=2}^{\infty} \left(|c_n| + |d_n|\right) n r_3^{n-1} \\
&\quad - \sum_{n=1}^{\infty} \left(|a_n| + |b_n|\right) n r_3^{n+1}\Bigg] \\
&\geqslant r_3 \Bigg[\lambda_h(0) - \lambda_h(0)\sqrt{M_2^4 - 1} \\
&\quad \cdot \frac{r_3\sqrt{r_3^4 - 3r_3^2 + 4}}{(1-r_3^2)^{3/2}} - \frac{4M_1 r_3^2}{\pi(1-r_3)^2}\Bigg] \\
&\geqslant r_3 \Bigg[\lambda_0(M_2) - \lambda_0(M_2)\sqrt{M_2^4 - 1} \\
&\quad \cdot \frac{r_3\sqrt{r_3^4 - 3r_3^2 + 4}}{(1-r_3^2)^{3/2}} - \frac{4M_1 r_3^2}{\pi(1-r_3)^2}\Bigg] = \sigma_3.
\end{aligned} \tag{38}$$

This completes the proof. □

Setting $M_1 = M_2 = M$ in Theorem 11, we have the following corollary.

Corollary 12. *Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk $\mathbb{D}$, with $F(0) = h(0) = J_F(0) - 1 = 0$, and both $g(z)$ and $h(z)$ are bounded by M. Then $L(F)$ is* *univalent in the disk* $\mathbb{D}_{r_1}$*, where* r_1 *is the minimum root of the equation*

$$\lambda_0(M) - \frac{12Mr^2}{\pi(1-r^2)} - \frac{8Mr^3}{\pi(1-r)^3} - \lambda_0(M)\sqrt{M^4-1} \cdot \left[\frac{2r\sqrt{4r^2+r^4+1}}{(1-r^2)^{5/2}} + \frac{r\sqrt{r^4-3r^2+4}}{(1-r^2)^{3/2}}\right] = 0, \quad (39)$$

and $L(F)(\mathbb{D}_{r_1})$ *contains a schlicht disk* $\mathbb{D}_{\sigma_1}$*, where*

$$\sigma_1 = r_1 \left[\lambda_0(M) - \lambda_0(M)\sqrt{M^4-1} \cdot \frac{r_1\sqrt{r_1^4-3r_1^2+4}}{(1-r_1^2)^{3/2}} - \frac{4Mr_1^2}{\pi(1-r_1)^2}\right]. \quad (40)$$

In order to show Corollary 12 improves Theorem A, we use Mathematica to compute the approximate values for various choices of M as in Table 1.

Remark 13. From Table 1 we can see, for the same M,

$$r_1 > \rho_1, \qquad \sigma_1 > R_1. \quad (41)$$

Finally we improve Theorems B as follows.

Theorem 14. *Let* $F(z) = |z|^2 g(z)$ *be a biharmonic mapping in* $\mathbb{D}$ *such that* $g(0) = 0$*,* $J_g(0) = 1$ *and* $|g(z)| < M$*, where* $M \geqslant 1$ *and* $g(z)$ *is harmonic in* $\mathbb{D}$*. Then* $L(F)$ *is univalent in the disk* $\mathbb{D}_{r_2}$*, where* r_2 *is the minimum positive root in* $(0,1)$ *of the following equation:*

$$1 - \sqrt{M^4-1} \cdot \left[\frac{3r\sqrt{r^4-3r^2+4}}{(1-r^2)^{3/2}} + \frac{2r\sqrt{4r^2+r^4+1}}{(1-r^2)^{5/2}}\right] = 0, \quad (42)$$

and $L(F)(\mathbb{D}_{r_2})$ *contains a schlicht disk* $\mathbb{D}_{\sigma_2}$ *with*

$$\sigma_2 = r_2^3 \lambda_0(M)\left[1 - \sqrt{M^4-1} \cdot \frac{r_2\sqrt{r_2^4-3r_2^2+4}}{(1-r_2^2)^{3/2}}\right], \quad (43)$$

where $\lambda_0(M)$ *is defined by* (16).

Proof. Let

$$g(z) = g_1(z) + \overline{g_2(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n \quad (44)$$

Let $H(z) := L(F) = |z|^2 L(g)$; then we have

$$H_z = 2|z|^2 g_z - \overline{z}^2 g_{\overline{z}} + z|z|^2 g_{zz},$$
$$H_{\overline{z}} = -2|z|^2 g_{\overline{z}} + z^2 g_z - \overline{z}|z|^2 g_{\overline{z}\overline{z}}. \quad (45)$$

For $z_1 \neq z_2$ in $\mathbb{D}_r$ $(0 < r < r_2)$, by Lemmas 4, 5, 8, and 9, we get

$$\begin{aligned}
&|H(z_1) - H(z_2)| \\
&\geqslant |z_1 - z_2| \left(\int_0^1 |tz_1 + (1-t)z_2|^2 dt\right) \\
&\quad \times \left[\lambda_g(0) - 3\sum_{n=2}^{\infty} n(|a_n| + |b_n|) r^{n-1} - \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|) r^{n-1}\right] \\
&\geqslant |z_1 - z_2| \frac{|z_1|^3 + |z_2|^3}{3(|z_1| + |z_2|)} \\
&\quad \times \left[\lambda_g(0) - 3\lambda_g(0)\sqrt{M^4-1} \cdot \frac{r\sqrt{r^4-3r^2+4}}{(1-r^2)^{3/2}} - \sqrt{M^4-1} \cdot \lambda_g(0) \cdot \frac{2r\sqrt{4r^2+r^4+1}}{(1-r^2)^{5/2}}\right] \\
&\geqslant |z_1 - z_2| \frac{|z_1|^3 + |z_2|^3}{3(|z_1| + |z_2|)} \lambda_0(M) \\
&\quad \times \left[1 - 3\sqrt{M^4-1} \cdot \frac{r\sqrt{r^4-3r^2+4}}{(1-r^2)^{3/2}} - \sqrt{M^4-1} \cdot \frac{2r\sqrt{4r^2+r^4+1}}{(1-r^2)^{5/2}}\right] > 0,
\end{aligned} \quad (46)$$

which implies $H(z_1) \neq H(z_2)$.

For any z such that $z \in \partial\mathbb{D}_{r_2}$, by Lemmas 4 and 5, we obtain

$$\begin{aligned}
|H(z)| &= \left|L(|z|^2 g)\right| \\
&\geqslant \left||z|^2 (z g_z(0) - \overline{z} g_{\overline{z}}(0))\right| - \left||z|^2 (z(g_z - g_z(0)) - \overline{z}(g_{\overline{z}} - g_{\overline{z}}(0)))\right| \\
&\geqslant r_2^3 \left[\lambda_g(0) - \sum_{n=2}^{\infty} n(|a_n| + |b_n|) r_2^{n-1}\right] \\
&\geqslant r_2^3 \lambda_0(M) \left[1 - \sqrt{M^4-1} \cdot \frac{r_2\sqrt{r_2^4-3r_2^2+4}}{(1-r_2^2)^{3/2}}\right] = \sigma_2.
\end{aligned} \quad (47)$$

This completes the proof of Theorem 14. □

In order to show Theorem 14 improves Theorem B, we use Mathematica to compute the approximate values for various choices of M as in Table 2.

Remark 15. From Table 2 we can see, for the same M,

$$r_2 > \rho_2, \qquad \sigma_2 > R_2. \quad (48)$$

Table 1: The values of r_1, σ_1 are in Corollary 12. The values of ρ_1, R_1 are in Theorem A.

	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$
ρ_1	0.0527621	0.0139445	0.00626165	0.00353488	0.0022661
r_1	0.357671	0.0593158	0.0269865	0.015355	0.00988556
R_1	0.013793	0.00164514	0.00048245	0.00020277	0.00010364
σ_1	0.216467	0.0119479	0.00357231	0.00151701	0.00077955

Table 2: The values of r_2, σ_2 are in Theorem 14. The values of ρ_2, R_2 are in Theorem B.

	$M = 2$	$M = 3$	$M = 4$	$M = 5$
ρ_2	0.00623234	0.00277176	0.00155948	0.00099817
r_2	0.032209	0.0139701	0.00782686	0.00500376
R_2	6.54254×10^{-8}	3.83564×10^{-9}	5.12297×10^{-10}	1.07466×10^{-10}
σ_2	9.84416×10^{-6}	5.35363×10^{-7}	7.06092×10^{-8}	1.47596×10^{-8}

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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