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Research Article

Multiplicity of Periodic Solutions for a Higher Order Difference Equation

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We study a higher order difference equation. By Lyapunov-Schmidt reduction methods and computations of critical groups, we prove that the equation has four M-periodic solutions.

1. Introduction

Considering the following higher order difference equation

$$\sum_{i=0}^{k} a_i \left(x_{n-i} + x_{n+i}\right) + f\left(n, x_n\right) = 0, \quad n \in \mathbf{Z}, \tag{1}$$

where $k \in \mathbf{N}$, $\mathbf{N}$ and $\mathbf{Z}$ are the sets of all positive integers and integers, respectively, $f \in C^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$, $\mathbf{R}$ is the set of all real numbers, and there exists a positive integer M such that, for any $(t, z) \in (\mathbf{R} \times \mathbf{R})$, $f(t + M, Z) = f(t, Z)$, $F(t, z) = \int_0^z f(t, s)\mathrm{d}s$.

Throughout this paper, for $a, b \in \mathbf{Z}$, we define $\mathbf{Z}(a) := \{a, a+1, \ldots\}$, $\mathbf{Z}(a, b) := \{a, a+1, \ldots, b\}$ when $a \leq b$.

When $k = 1$, $a_0 = -1$, $a_1 = 1$, (1) can be reduced to the following second order difference equation:

$$\Delta^2 x_{n-1} + f\left(n, x_n\right) = 0, \quad n \in \mathbf{Z}. \tag{2}$$

Equation (2) can be seen as an analogue discrete form of the following second order differential equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + f(t, x) = 0. \tag{3}$$

In recent years, much attention has been given to second order Hamiltonian systems and elliptic boundary value problems by a number of authors; see [1–3] and references therein. On one hand, there have been many approaches to study periodic solutions of differential equations or difference equations, such as critical point theory (which includes the minimax theory, the Kaplan-Yorke method, and Morse theory), fixed point theory, and coincidence theory; see, for example, [4–20].

Among these approaches, Morse theory is an important tool to deal with such problems. However, there are, at present, only a few papers dealing with higher order difference equation except [21–23]. On the other hand, under some assumptions, the functional f may not satisfy the Palasis-Smale condition. Thus, we cannot apply the Morse theory to f directly. To go around this difficulty, Tang and Wu [24] and Liu [25] obtain many interesting results of elliptic boundary value problems by combining Morse theory with Lyapunov-Schmidt reduction method or minimax principle. Inspired by this, we study the existence of periodic solutions of a higher order difference equation (1) by combining computations of critical groups with Lyapunov-Schmidt reduction method, and an existence theorem on multiple periodic solutions for such an equation is obtained.

For a given integer $M > 0$, let

$$\lambda_j = -2\sum_{s=0}^{k} a_s \cos\frac{2s\pi}{M} j, \quad j = 1, \ldots, M. \tag{4}$$

We denote $p_1 = M/2$ when M is even, or $p_1 = (M+1)/2$ when M is odd. Because of $\lambda_{M-j} = \lambda_j$, $j \in \mathbf{Z(1, M)}$, then, λ_j, $j \in \mathbf{Z(1, M)}$ has p_1 different values. Therefore, we can write these numbers in such a way:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{p_1}. \tag{5}$$

Assume $\lambda_{\min} = \min\{\lambda_j, \lambda_j \neq 0, j = 1, \ldots, p_1\}$, $\lambda_{\max} = \max\{\lambda_j, \lambda_j \neq 0, j = 1, \ldots, p_1\}$.

Combing Morse theory with Lyapunov-Schmidt reduction method, we have the following results.

Theorem 1. *Suppose that* $M \geq 2k + 1$, $a_0 + \sum_{s=1}^{k} |a_s| < 0$, *and* $f(t, z) = f(z)$; *we assume that*

(f_1) $f(z) \in C^1(\mathbf{R}, \mathbf{R})$, $f(0) = 0$, $f'(0) < \lambda_{\min} < f_\infty = \lambda_m \leq \lambda_{\max}$, $m \in \mathbf{N}(1, p_1)$, *where* $f_\infty = \lim_{|z|\to\infty} f(z)/z$;

(f_2) *there exists a constant* $\gamma \geq \lambda_1$ *such that* $f'(z) \leq \gamma < \lambda_{m+1}$;

(f_3) *for any* $t \in \mathbf{Z}$,

$$F(z) - \frac{1}{2}\lambda_m |z|^2 \longrightarrow +\infty, \quad \text{as } |z| \longrightarrow \infty. \tag{6}$$

Then (1) *possesses at least four nontrivial* M*-periodic solutions.*

This paper is divided into four parts. Section 2 presents variational structure. In Section 3, we present some propositions. The proof of Theorem 1 is given in Section 4.

2. Preliminaries

To apply Morse theory to study the existence of periodic solutions of (1), we will construct suitable variational structure.

Let $\mathbf{S}$ be the set of sequences $x = \{x_n\}_{n=-\infty}^{+\infty}$, where $x_n \in \mathbf{R}$. For any $x, y \in \mathbf{S}$ and $a, b \in \mathbf{R}$, $ax + by$ is defined by

$$ax + bx := \{ax_n + bx_n\}. \tag{7}$$

Then $\mathbf{S}$ is a vector space.

For any given positive integer M, E_M is defined as a subspace of $\mathbf{S}$ by

$$E_M = \{x = \{x_n\} \in \mathbf{S} \mid x_{n+M} = x_n, n \in \mathbf{Z}\}. \tag{8}$$

E_M can be equipped with inner product $\langle \cdot, \cdot \rangle_{E_M}$ and norm $\|\cdot\|_{E_M}$ as follows:

$$\langle x, y \rangle_M = \sum_{j=1}^{M} x_j \cdot y_j, \quad \forall x, y \in E_M,$$
$$\|x\|_{E_M} = \left(\sum_{j=1}^{M} x_j^2\right)^{1/2}, \quad \forall x \in E_M, \tag{9}$$

where $|\cdot|$ denotes the Euclidean Norm in $\mathbf{R}^M$, and $x_n \cdot y_n$ denotes the usual scalar product in $\mathbf{R}$.

Define a linear map $L : E_M \to \mathbf{R}^M$ by

$$Lx = (x_1, \ldots, x_M)^T. \tag{10}$$

It is easy to see that the map L defined in (10) is a linear homeomorphism with $\|x\|_{E_M} = |Lx|$ and $(E_M, \langle \ldots, \ldots \rangle)_{E_M}$ is a finite dimensional Hilbert space, which can be identified with $\mathbf{R}^M$.

For (1), we consider the functional I defined on E_M by

$$I(x) = -\frac{1}{2}\sum_{n=1}^{M}\sum_{i=0}^{k} a_i (x_{n-i} + x_{n+i}) x_n - \sum_{n=1}^{M} F(n, x_n), \quad \forall x \in E_M, \tag{11}$$

where $x_{n+M} = x_n$, $\forall x \in E_M$, $F(t, z) = \int_0^z f(t, s)ds$.

Since E_M is linearly homeomorphic to $\mathbf{R}^M$, by the continuity of $f(t, z)$, I can be viewed as continuously differentiable functional defined on a finite dimensional Hilbert space. That is, $I \in C^1(E_M, \mathbf{R})$. If we define $x_0 := x_M$, then

$$\frac{\partial I(x)}{\partial x_n} = -\left[\sum_{i=0}^{k} a_i (x_{n-i} + x_{n+i}) + f(n, x_n)\right], \tag{12}$$

where $n \in \mathbf{Z}(1, M)$. Therefore, $x \in E_M$ is a critical point of I; that is, $I'(x) = 0$ if and only if

$$\sum_{i=0}^{k} a_i (x_{n-i} + x_{n+i}) + f(n, x_n) = 0, \quad n \in \mathbf{Z}(1, M). \tag{13}$$

On the other hand, $\{x_n\} \in E_M$ is M-periodic in n, and $f(t, z)$ is M-periodic in t; hence, $x \in E_M$ is a critical point of I if and only if $\sum_{i=0}^{k} a_i(x_{n-i} + x_{n+i}) + f(n, x_n) = 0$ for any $n \in \mathbf{Z}$, and $x = \{x_n\}$ is a M-periodic solution of (1). Thus, we reduce the problem of finding M-periodic solutions of (1) to that of seeking critical points of the functional I in E_M.

Apparently, $I(x) \in C^2(E_M, \mathbf{R})$. Consider

$$\begin{aligned}
\left(I'(x), v\right) &= -\frac{1}{2}\sum_{n=1}^{M}\sum_{i=1}^{k} a_i \left[(x_{n-i} + x_{n+i}) v_n + (v_{n-i} + v_{n+i}) x_n\right] - \sum_{n=1}^{M} f(n, x_n) v_n, \\
\left(I''(x) v, w\right) &= -\frac{1}{2}\sum_{n=1}^{M}\sum_{i=1}^{k} a_i \left[(w_{n-i} + w_{n+i}) v_n + (v_{n-i} + v_{n+i}) w_n\right] - \sum_{n=1}^{M} f'(n, x_n) v_n w_n,
\end{aligned} \tag{14}$$

for all $x, v, w \in E_M$. For convenience, we write $x \in E_M$ as $x = (x_1, x_2, \ldots, x_M)^T$.

In view of $x_{n+M} = x_n$, $\forall x = (x_1, x_2, \ldots, x_M)^T \in E_M$, $n \in \mathbf{Z}$, when $M \geq 2k + 1$, I can be rewritten as

$$I(x) = \frac{1}{2} x^T A x - \sum_{n=1}^{M} F(n, x_n), \tag{15}$$

where

$$-A=\begin{pmatrix}
2a_0 & a_1 & a_2 & \cdots & a_{k-1} & a_k & 0 & 0 & \cdots & 0 & a_k & a_{k-1} & \cdots & a_2 & a_1\\
a_1 & 2a_0 & a_1 & \cdots & a_{k-2} & a_{k-1} & a_k & 0 & \cdots & 0 & 0 & a_k & \cdots & a_3 & a_2\\
a_2 & a_1 & 2a_0 & \cdots & a_{k-3} & a_{k-2} & a_{k-1} & a_k & \cdots & 0 & 0 & 0 & \cdots & a_4 & a_3\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
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a_2 & a_3 & a_4 & \cdots & 0 & 0 & 0 & 0 & \cdots & a_k & a_{k-1} & a_{k-2} & \cdots & 2a_0 & a_1\\
a_1 & a_2 & a_3 & \cdots & a_k & 0 & 0 & 0 & \cdots & a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 & 2a_0
\end{pmatrix}_{M\times M}. \tag{16}$$

Let the eigenvalues of A be $\lambda_1', \lambda_1', \ldots, \lambda_M'$, and let A be a circulant matrix [18] denoted by

$$A \overset{\text{def}}{=} \operatorname{Circ}\{-2a_0, -a_1, -a_2, \ldots, -a_k, 0, \ldots, \\ 0, -a_k, -a_{k-1}, \ldots, -a_2, -a_1\}. \tag{17}$$

By [18], the eigenvalues of A are

$$\lambda_j' = -2a_0 - \sum_{s=1}^{k} a_s \left\{\exp i \frac{2j\pi}{M}\right\}^{s} - \sum_{s=1}^{k} a_s \left\{\exp i \frac{2j\pi}{M}\right\}^{M-s} \\ = -2\sum_{s=0}^{k} a_s \cos\left(\frac{2js\pi}{M}\right), \tag{18}$$

where $j = 1, \ldots, M$.

According to (18), for any positive integer M with $M \geq 2k+1$, we know that.

If $a_0 + \sum_{s=1}^{k} |a_s| < 0$, then $\lambda_j' > 0$ $(j = 1, 2, \ldots, M)$. That is, the matrix A is positive definite.

Comparing (18) with (4), we know that $\lambda_j' = \lambda_j$ $(j = 1, \ldots, M)$, then, the matrix A has p_1 different eigenvalues denoted in such a way:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{p_1}. \tag{19}$$

3. Main Propositions

In order to prove our main results, we will give several propositions and notations as follows.

Definition 2 (see [4]). Let X be a Banach space, let $J \in C^1(X, \mathbf{R})$, and let $H_q(A, B)$ be the qth singular relative homology group of the topological pair (A, B) with coefficients in an Abelian group G. $\beta_q = \operatorname{rank} H_q(A, B)$ is called the q-dimension Betti number. Let u be an isolated critical point of J with $J(u) = c$, $c \in \mathbf{R}$, and let U be a neighborhood of u_0 in which J has no critical points except u_0. Then the group

$$C_q(J, u_0) := H_q\left(J_c \bigcap U, J_c \bigcap U \setminus \{u_0\}\right), \quad q = 0, 1, 2, \ldots \tag{20}$$

is called the qth critical group of J at u, here $J_c = J^{-1}(-\infty, c]$. Assume that J satisfies PS condition; J has no critical value less than $\alpha \in \mathbf{R}$; then the qth critical group at infinity of J is defined as

$$C_q(J, \infty) := H_q(X, J_a), \quad q = 0, 1, 2, \ldots. \tag{21}$$

If $J''(u_0) = 0$, then the Morse index of J at u_0 is defined as the dimension of the maximal subspace of X on which the quadratic form $(J''(u_0)v, v)$ is negative definite. Define $K_c = \{u \in X : J'(u) = 0, J(u) = c\}$. We need the following condition.

(A) Suppose that $a < b$ are two regular values of J; J has at most finitely many critical points on $J^{-1}[a, b]$ and the rank of the critical group for every critical point is finite.

Definition 3 (see [4]). Assume that J satisfies condition (A); $c_1 < c_2 < \cdots < c_m$ are all critical values of J in $[a, b]$ and

$K_{c_i} = \{z_1^i, z_2^i, \ldots, z_{n_i}^i\}$, $i = 1, 2, \ldots, m$. Choose $0 < \epsilon < \min\{c_1 - a, c_2 - c_1, \ldots, c_m - c_{m-1}, b - c_m\}$. Define

$$\begin{aligned} M_q &= M_q(a,b) \\ &= \sum_{i=1}^{m} \operatorname{rank} H_q\left(J_{c_i+\epsilon}, J_{c_i-\epsilon}\right) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n_i} \operatorname{rank} C_q\left(J, z_j^i\right), \quad q = 0, 1, \ldots \end{aligned} \tag{22}$$

Then M_q is called the qth Morse-type number of J about the interval $[a, b]$.

Here the critical groups of J at an isolated critical point u describe the local behavior of J near u, while the critical groups of J at infinity describe the global property of J. The Morse inequality gives the relation between them.

Proposition 4 (see [4]). *Suppose that $J \in C^1(X, \mathbf{R})$ satisfies the PS condition and has only isolated critical points, and the critical values of f are bounded below. Then we have*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1+t) Q(t), \tag{23}$$

where $M_q = \sum_{J'(u)=0} \operatorname{rank} C_q(J, u)$, $\beta_q = \operatorname{rank} C_q(J, \infty)$; Q is a formal series with nonnegative integer coefficients.

Now we recall the Lyapunov-Schmidt reduction method.

Proposition 5 (see [5]). *Let X be a separable Hilbert space with inner product $\langle u, v\rangle$ and norm $\|u\|$ and let X^- and X^+ be closed subspaces of X such that $X = X^- \oplus X^+$. Let $J \in C^1(X, \mathbf{R})$. If there is a real number $\beta > 0$ such that, for all $v \in X^-$, $w_1, w_2 \in X^+$, there holds*

$$\left\langle \nabla f\left(v + w_1\right) - \nabla J\left(v + w_2\right), w_1 - w_2 \right\rangle \geq \beta \left\| w_1 - w_2 \right\|^2, \tag{24}$$

then we have the following:

(i) *there exists a continuous function $\psi : X^- \to X^+$ such that*

$$J\left(v + \psi(v)\right) = \min_{w \in X^+} J(v + w), \tag{25}$$

and $\psi(v)$ is the unique member of X^+ such that

$$\left\langle \nabla J\left(v + \psi(v)\right), w \right\rangle = 0, \quad \forall w \in X^+; \tag{26}$$

(ii) *the functional $\varphi \in C^1(X^-, \mathbf{R})$ defined by $\varphi(v) = J(v + \psi(v))$ and*

$$\left\langle \nabla \varphi(v), v_1 \right\rangle = \left\langle \nabla J\left(v + \psi(v)\right), v \right\rangle, \quad \forall v, v_1 \in X^-; \tag{27}$$

(iii) *an element $v \in X^-$ is a critical point of φ if and only if $v + \psi(v)$ is a critical point of J.*

Proposition 6 (see [25]). *Assume that the assumptions of Proposition 5 hold, then at any isolated critical point v of φ we have*

$$C_q(\varphi, v) \cong C_q\left(f, \psi(v)\right), \quad q = 0, 1, 2, \ldots. \tag{28}$$

Proposition 7 (see [25]). *Assume that the assumptions of Proposition 5 hold, if there exists a compact mapping $T : X \to X$ such that, for any $u \in X$, we have $\nabla J(u) = u - T(u)$, then we have φ:*

$$ind\left(\nabla \varphi, v\right) = ind\left(\nabla J, v + \psi(v)\right) \tag{29}$$

at any isolated critical point v of φ.

4. Proof of Theorem

Consider the following C^1 functional:

$$I(x) = \frac{1}{2} x^T A x - \sum_{n=1}^{M} F\left(n, x_n\right). \tag{30}$$

As we know, the PS condition is an important part of critical point theory. However, under our assumptions (f_1)–(f_3), the functional I may not satisfy PS condition. Thus, we cannot apply the Morse theory directly. But the truncated functional $I_\pm$ does satisfy the PS condition. So we can obtain two critical points of I via mountain pass lemma; then we can obtain other critical points by combing Morse theory with Lyapunov-Schmidt reduction method.

At first, we consider the truncated problem

$$\sum_{i=1}^{k} a_i\left(x_{n-i} + x_{n+i}\right) + f_+\left(x_n\right) = 0, \quad n \in \mathbf{Z},\ \mathbf{k} \in \mathbf{N}, \tag{31}$$

where

$$f_+(z) = \begin{cases} f(n, z), & z \geq 0, \\ 0, & z < 0, \end{cases} \tag{32}$$

$$\sum_{i=1}^{k} a_i\left(x_{n-i} + x_{n+i}\right) + f_-\left(x_n\right) = 0, \quad n \in \mathbf{Z},\ \mathbf{k} \in \mathbf{N}, \tag{31$'$}$$

where

$$f_-(z) = \begin{cases} f(n, z), & z \leq 0, \\ 0, & z > 0. \end{cases} \tag{33}$$

Then the functional $I_+ : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ corresponding to (31) can be written as

$$I_+(x) = \frac{1}{2} x^T A x - \sum_{n=1}^{M} F_+\left(x_n\right), \tag{34}$$

where $F_+(n, z) = \int_0^z f_+(n, s) ds$. Apparently, $I_+ \in C^1$.

The functional $I_- : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ corresponding to (31)$'$ can be written as

$$I_-(x) = \frac{1}{2} x^T A x - \sum_{n=1}^{M} F_-\left(x_n\right), \tag{35}$$

where $F_-(n, z) = \int_0^z f_-(n, s) ds$. Apparently, $I_- \in C^1$.

We only consider the case of I_+; the case of I_- is similar and omitted.

By (f_1), we know that

$$\lim_{z\to-\infty}\frac{f_+(z)}{z}=0,\qquad \lim_{z\to+\infty}\frac{f_+(z)}{z}=f_\infty. \tag{36}$$

Then there exist real number $\epsilon>0$ (small enough) and $C_\epsilon>0$ such that

$$f_+(z)=f_\infty z+C_\epsilon,\quad \text{if } z\longrightarrow+\infty. \tag{37}$$

Lemma 8. *Under the conditions of Theorem 1, the functional $I_+(x)$ satisfies the PS condition.*

Proof. Let $\{x^q\}\in E_M$ be such a sequence; that is, there exists a positive constant M_1 such that $|I_+(x^q)|\le M_1$, $\forall q\in\mathbf{N}$, and that $|(I_+'(x^q),v)|\to 0$ as $q\to+\infty$, $\forall v\in E_M$.

Therefore,

$$\begin{aligned}2M_1&\ge 2I_+(x^q)-\left(I_+'(x^q),x^q\right)\\&=\sum_{n=1}^{M}\left[f_+(x_n^q)x_n^q-F_+(x_n^q)\right]\\&=\sum_{n=1}^{M}\left[\left(f_\infty(x_n^q)^2+C_\epsilon x_n^q\right)-\left(\frac12 f_\infty(x_n^q)^2+C_\epsilon x_n^q+C_\epsilon\right)\right]\\&=\frac12 f_\infty\|x^q\|^2-C_\epsilon M.\end{aligned} \tag{38}$$

That is, $\{x^q\}\in E_M$ is a bounded sequence in the finite dimensional space E_M. Consequently, it has a convergent subsequence. Thus, we obtain Lemma 8.

Let $z^+=\max(z,0)$, $z^-=\max(-z,0)$, and $z=z^+-z^-$. □

Lemma 9. *If $x\in E_M$ is a local minimizer of I_+, then x must be a local minimizer of I.*

Proof. Let $x>0$ be a local minimizer of I_+; then for any sequence $\{x^q\}\subset E_M$, $x^q\to x$ $(q\to\infty)$, for big enough q, we have $I(x^q)\ge I(x)$.

In fact,

$$\begin{aligned}I(x^q)-I(x)&=I(x^q)-I_+(x)\\&\ge I(x^q)-I_+(x^q)\\&=\sum_{n=1}^{M}\left[F_+(x_n^q)-F(x_n^q)\right]\\&=-\sum_{n\in\mathbf{Z}(1,M),x_n^q<0}F(x_n^q).\end{aligned} \tag{39}$$

Because $x^q\to x$, $x_n^q=(x_n^q)^+-(x_n^q)^-$, and $x_n=(x_n)^+-(x_n)^-$, so $(x_n^q)^+\to(x_n)^+=x_n$, $-(x_n^q)^-\to 0^-$.

For any $n\in\mathbf{Z}(1,M)$, if $(x_n^q)^-=0$, then $I(x^q)=I(x)$.

If $-(x_n^q)^-\to 0^-$, by (f_1), $f(0)=0$, and $0<f'(z)<\gamma$, then $f(z)<0$ for $z\to 0^-$. Therefore, $-\sum_{n\in\mathbf{Z}(1,M),x_n^q<0}F(x_n^q)>0$; that is, $I(x^q)>I(x)$.

The proof of Lemma 9 is complete. □

It is easy to see that the zero function 0 is a local minimizer of I_+, and $I_+(s\phi_1)\to-\infty$ as $s\to+\infty$, where ϕ_1 is a first eigenfunction corresponding to the first nonzero eigenvalue of A. Thus, by the mountain pass lemma we obtain a critical point x_+ of I_+. However, it is true that x_+ is a critical point of I if x_+ is a critical point of I_+; then we deduce that x_+ is a critical point of I with

$$C_q(I,x_+)\cong\delta_{q,1}G,\quad x_+>0 \text{ in } E_M. \tag{40}$$

Similarly, we obtain another critical point x_- of I and

$$C_q(I,x_-)\cong\delta_{q,1}G,\quad x_-<0 \text{ in } E_M. \tag{41}$$

Next we will prove that I has two more nonzero critical points. We decompose $E_M=X^-\oplus X^+$ according to $f_\infty=\lambda_m$. We set

$$\begin{aligned}X^-&=\bigoplus_{i=1}^{m}\operatorname{Ker}(A-\lambda_iI),\\X^+&=\bigoplus_{i=m+1}^{M}\operatorname{Ker}(A-\lambda_iI),\\E_M&=X^-\bigoplus X^+.\end{aligned} \tag{42}$$

Since $f'(z)\le\gamma<\lambda_{m+1}$, for any $v\in X^-$ and $w_1,w_2\in X^+$, we have

$$\begin{aligned}&\left\langle\nabla I(n,v+w_1)-\nabla I(n,v+w_2),w_1-w_2\right\rangle\\&\quad\ge\beta\|w_1-w_2\|^2,\end{aligned} \tag{43}$$

where $\beta=1-\gamma\lambda_{m+1}^{-1}$. Then, by Proposition 5, there exist a continuous map $\psi: X^-\to X^+$ and a C^1-functional $\varphi: X^-\to\mathbf{R}$ such that

$$\varphi(v)=I(v+\psi(v))=\min_{w\in X^+}. \tag{44}$$

We need to show that φ has at least five critical points. Hence, we assume that φ has no critical value less than some $\alpha\in\mathbf{R}$.

Lemma 10. *Suppose that $f\in C^1(\mathbf{R},\mathbf{R})$ satisfies (f_1)–(f_3), then the functional φ is anticoercive.*

Proof. According to (f_3), there exists $R>0$ such that

$$\frac12\lambda_m z^2-F(z)\le 0,\quad |z|\ge R. \tag{45}$$

Then, for any $z\in\mathbf{R}$, we have

$$\frac12\lambda_m z^2-F(z)\le T=\max_{|z|\le R}\left|\frac12\lambda_m z^2-F(z)\right|. \tag{46}$$

Assume that $\{v^t\}_{t=1}^{\infty}$ is a sequence in X^- such that $\|v^t\|\to\infty$. Let $\xi^t=v^t/\|v^t\|$, then $\|\xi^t\|=1$. Because of $\dim X^-<\infty$, there exist some $\xi\in X^-$ such that, up to subsequence $\|\xi^t-\xi\|\to 0$, $\|\xi\|=1$.

In particular, $\xi \neq 0$, meas Θ = meas$\{n \in \mathbf{Z}[1, M] : \xi_n \neq 0\} > 0$. For $n \in \Theta$, $|v_n^t| \rightarrow \infty$. Hence, by (f_3),

$$\sum_{n\in\Theta} \left(\frac{1}{2}\lambda_m \left\|v^t\right\|^2 - F\left(v_n^t\right)\right) \longrightarrow -\infty, \quad \text{as } t \longrightarrow \infty. \tag{47}$$

By the above discussion, we have

$$\begin{aligned}
\varphi\left(v^t\right) &\leq I\left(v^t\right) \\
&= -\frac{1}{2}\sum_{n=1}^{M}\sum_{i=1}^{k} a_i \left(v_{n-i}^t + v_{n+i}^t\right) v_n^t - \sum_{n=1}^{M} F\left(v_n^t\right) \\
&\leq \frac{1}{2}\lambda_m \left\|v_t\right\|^2 - \sum_{n=1}^{M} F\left(v_n^t\right) \\
&= \sum_{n\in\Theta} \left[\frac{1}{2}\lambda_m \left\|v_t\right\|^2 - F\left(v_n^t\right)\right] \\
&\quad + \sum_{n\in[1,M]\setminus\Theta} \left[\frac{1}{2}\lambda_m \left\|v_t\right\|^2 - F\left(v_n^t\right)\right] \\
&\leq \sum_{n\in\Theta} \left[\frac{1}{2}\lambda_m \left\|v^t\right\|^2 - F\left(v_n^t\right)\right] + MT \\
&\longrightarrow -\infty.
\end{aligned} \tag{48}$$

This concludes the proof. □

Because φ is anticoercive, we choose $a < b < \alpha$ and $\rho > r > 0$ such that

$$A_\rho \subset \varphi_a \subset A_r \subset \varphi_b, \tag{49}$$

where $A_\rho = \{v \in X^- : \|v\| \geq \rho\}$. Since φ has no critical value in $[a, b]$, $H_*(\varphi_b, \varphi_a) = 0$.

Thus, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 = H_q(\varphi_b, \varphi_a) & \longrightarrow & H_q(X^-, \varphi_a) & \xrightarrow{i_*} & H_q(X^-, \varphi_b) & \xrightarrow{\partial_*} & H_{q-1}(\varphi_b, \varphi_a) = 0 \\
\uparrow & & \uparrow & \searrow^{l_*} & \uparrow & & \uparrow \\
0 = H_q(A_r, A_\rho) & \longrightarrow & H_q(X^-, A_\rho) & \xrightarrow{k_*} & H_q(X^-, A_r) & \xrightarrow{\partial_*} & H_{q-1}(A_r, A_\rho) = 0
\end{array} \tag{50}$$

where all the homomorphisms except ∂_* are induced by inclusions. The exactness of rows implies that i_*, k_* are isomorphisms. Hence $l_* : H_q(X^-, \varphi_a) \rightarrow H_q(X^-, A_r)$ is also an isomorphism, and we get

$$C_q(\varphi, \infty) = H_q\left(X^-, \varphi_a\right) \cong H_q\left(X^-, A_r\right) = \delta_{q,m} G. \tag{51}$$

Because the anticoercive functional φ is defined on the m-dimensional X^-, it has a critical point v, with

$$C_q(\varphi, v) \cong \delta_{q,m} G. \tag{52}$$

Let 0, v_+, v_- be the projection of 0, x_+, x_- in X^-, respectively. Then they are all critical points of φ. By (11), (14), and Proposition 6, and 0 is a local minimizer of I, we have

$$\begin{aligned}
C_q\left(\varphi, v_\pm\right) &\cong C_q\left(I, x_\pm\right) \cong \delta_{q,1} Q, \\
C_q(\varphi, 0) &\cong C_q(I, 0) \cong \delta_{q,0} Q.
\end{aligned} \tag{53}$$

If 0, v_+, v_-, v are the only critical points of φ, then by Proposition 4 with $t = -1$,

$$(-1)^0 + 2 \times (-1)^1 + (-1)^m = (-1)^m. \tag{54}$$

This is impossible. Thus φ has at least five critical points. So I also has five critical points, four of which are nonzero. Therefore, (1) has at least four nontrivial solutions. This completes the proof of Theorem 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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