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Research Article

Integral φ_0-Stability in terms of Two Measures for Impulsive Differential Equations with "Supremum"

Peiguang Wang,[1] Xiaojing Liu,[2] and Qing Xu[2]

[1] *College of Electronic and Information Engineering, Hebei University, Baoding 071002, China*
[2] *College of Mathematics and Computer Science, Hebei University, Baoding 071002, China*

Correspondence should be addressed to Peiguang Wang; pgwang@hbu.edu.cn

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This paper establishes a criterion on integral φ_0-stability in terms of two measures for impulsive differential equations with "supremum" by using the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method. Meantime, an example is given to illustrate our result.

1. Introduction

In this paper, we discuss the integral φ_0-stability in terms of two measures for impulsive differential equations with "supremum":

$$\begin{aligned} x' &= F\left(t, x(t), \sup_{s\in[t-r,t]} x(s)\right) \quad \text{for } t \geq 0,\ t \neq \tau_k, \\ x(\tau_k + 0) &= I_k(x(\tau_k - 0)) \quad \text{for } k = 1, 2, \ldots, \\ x(t) &= \phi(t), \quad t \in [t_0 - r, t_0], \end{aligned} \tag{1}$$

and its perturbed impulsive differential equations with "supremum"

$$\begin{aligned} x' &= F\left(t, x(t), \sup_{s\in[t-r,t]} x(s)\right) + G\left(t, x(t), \sup_{s\in[t-r,t]} x(s)\right) \\ &\qquad \text{for } t \geq 0, \quad t \neq \tau_k, \\ x(\tau_k + 0) &= I_k(x(\tau_k - 0)) + J_k(x(\tau_k - 0)) \\ &\qquad \text{for } k = 1, 2, \ldots, \\ x(t) &= \phi(t), \quad t \in [t_0 - r, t_0], \end{aligned} \tag{2}$$

where $x \in R^n$, $F, G : R^+ \times R^n \times R^n \rightarrow R^n$, $F(t, 0, 0) = G(t, 0, 0) \equiv 0$, $I_k, J_k : R^n \rightarrow R^n$, $I_k(0) = J_k(0) \equiv 0$, $k = 1, 2, \ldots$, $r > 0$, $t_0 \in R^+$, and $\phi \in (PC[t_0 - r, t_0], R^n)$. Let R^n be n-dimensional Euclidean space with norm $\|x\|$, $R^+ = [0, \infty)$, and $\{\tau_k\}_1^\infty$ a sequence of fixed points in R^+ such that $\tau_{k+1} > \tau_k$ and $\lim_{k\to\infty}\tau_k = \infty$. We denote by $x(t; t_0, \phi)$ the solution of (1). In our further investigation we will assume that solution $x(t; t_0, \phi)$ is defined on $[t_0 - r, \infty)$ for any initial function $\phi \in PC([t_0 - r, t_0], R^n)$.

The research on impulsive differential equations with "supremum" problem, Bainov et al. [1] justified the partial averaging for impulsive differential equations, He et al. [2] discussed the periodic boundary value problem for first order impulsive differential equations, Agarwal and Hristova [3] studied the strict stability in terms of two measures for impulsive differential equations, Stamova and Stamov [4] investigated the global stability of models based on impulsive differential equations and variable impulsive perturbations, and Hristova [5, 6] obtained the φ_0-stability in terms of two measures for impulsive differential equations.

In recent years, the integral stability theory has been rapid development (see [7–12]). For example, Soliman and Abdalla [10] introduced integral φ_0-stability of perturbed system of ordinary differential equations. Hristova [12] studied the integral stability in terms of two measures for impulsive differential equations with "supremum." However, the corresponding theory of impulsive differential equations with "supremum" is still at an initial stage of its development, especially for integral φ_0-stability in terms of two measures. Motivated by the idea of [5, 6, 10, 12], in this work, by employing the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method, we extend the notions of φ_0-stability in terms of two measures to integral φ_0-stability in terms of two measures for impulsive differential equations with "supremum."

2. Preliminaries

Denote by $PC(X,Y)$ $(X \subset R,\ Y \subset R^n)$ the set of all functions $u : X \rightarrow Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $\tau_k \in X$ and which are continuous from the left at the points $\tau_k \in X$, $u(\tau_k) = u(\tau_k - 0)$.

We denote by $PC^1(X,Y)$ the set of all function $u \in PC(X,Y)$ which are continuously differentiable for $t \in X$, $t \neq \tau_k$.

Let $x, y \in R^n$. Denote by $(x \cdot y)$ the dot product of both vectors x and y.

Let $\mathscr{K} \subset R^n$ be a cone, and $\mathscr{K}^* = \{\varphi \in R^n : (\varphi \cdot x) \geq 0$ for any $x \in \mathscr{K}\}$ is adjoint cone.

We give the following notations for convenience:

$$\begin{aligned} K &= \{a \in C(R^+, R^+) : \\ &\qquad a(s) \text{ is strictly increasing, } a(0) = 0\}; \\ CK &= \{b \in C[R^+ \times R^+, R^+] : \\ &\qquad b(t, \cdot) \in K \text{ for any fixed } t \in [0,\infty)\}; \\ \Gamma &= \Big\{h \in C[[-r,\infty) \times R^n, \mathscr{K}] : \\ &\qquad \inf_{x \in R^n} h(t,x) = 0 \text{ for each } t \in [-r,\infty)\Big\}. \end{aligned} \tag{3}$$

Let $h_0, h \in \Gamma$, $\varphi_0 \in \mathscr{K}^*$, $t \in R^+$, and $\phi \in PC([t_0 - r, t_0], R^n)$. Define

$$H_0(t,\phi,\varphi_0) = \sup\{(\varphi_0 \cdot h_0(t+s, \phi(t+s))) : s \in [-r,0]\}, \tag{4}$$

$$H(t,\phi,\varphi_0) = \sup\{(\varphi_0 \cdot h_0(t+s, \phi(t+s))) : s \in [-r,0]\}. \tag{5}$$

Let ρ, t and $T > 0$ be constants, $\varphi_0 \in \mathscr{K}^*$, $h \in \Gamma$. Define sets:

$$\begin{aligned} S(h,\rho,\varphi_0) &= \{(t,x) \in R^+ \times R^n : (\varphi_0 \cdot h(t,x)) < \rho\}; \\ S^c(h,\rho,\varphi_0) &= \{(t,x) \in R^+ \times R^n : (\varphi_0 \cdot h(t,x)) \geq \rho\}; \\ \Omega(t,T,\rho) &= \{(x,y) \in R^n \times R^n : \\ &\qquad (\varphi_0 \cdot h(t,x)) < \rho \text{ for } s \in [t, t+T], \\ &\qquad (\varphi_0 \cdot h(t,y)) < \rho \text{ for } s \in [t-r, t+T]\}. \end{aligned} \tag{6}$$

In our further investigations we use the following comparison scalar impulsive ordinary differential equation:

$$\begin{aligned} u' &= g_1(t,u), \quad t \neq \tau_k, \\ u(\tau_k + 0) &= \xi_k(u(\tau_k)), \\ u(t_0) &= u_0, \\ &\qquad k = 1, 2, \ldots, \end{aligned} \tag{7}$$

the scalar impulsive ordinary differential equation:

$$\begin{aligned} w' &= g_2(t,w), \quad t \neq \tau_k, \\ w(\tau_k + 0) &= \eta_k(w(\tau_k)), \\ w(t_0) &= w_0, \\ &\qquad k = 1, 2, \ldots, \end{aligned} \tag{8}$$

and its perturbed scalar impulsive ordinary differential equation:

$$\begin{aligned} w' &= g_2(t,w) + q(t), \quad t \neq \tau_k, \\ w(\tau_k + 0) &= \eta_k(w(\tau_k)) + \gamma_k(w(\tau_k)), \\ w(t_0) &= w_0, \\ &\qquad k = 1, 2, \ldots, \end{aligned} \tag{9}$$

where $u, w \in R$, $g_1(t,0) = g_2(t,0) \equiv 0$, $\xi_k(0) = 0$, $\eta_k(0) = 0$, $k = 1, 2, \ldots$.

Assume that solutions of the scalar impulsive equations (7), (8), and (9) exist on $[t_0,\infty)$ for any initial values. Meanwhile, we give some definitions and lemmas. The details can be found in [5].

Definition 1 (see [5]). We say that function $V(t,x) : [-r,\infty) \times R^n \rightarrow \mathscr{K}$, $V = (V_1, V_2, \ldots, V_n)$, belongs to the class Λ if

(A1) $V(t,x) \in PC^1([-r,\infty) \times R^n, \mathscr{K})$;

(A2) for each $k = 1, 2, \ldots$ and $x \in R^n$ there exist the finite limits

$$V(\tau_k - 0, x) = \lim_{t \uparrow \tau_k} V(t,x), \qquad V(\tau_k + 0, x) = \lim_{t \downarrow \tau_k} V(t,x); \tag{10}$$

(A3) there exist constants $M_i > 0$, $i = 1, 2, \ldots, n$, such that $|V_i(t,x) - V_i(t,y)| \leq M_i \|x - y\|$ for any $t \in R^+$, $x, y \in R^n$.

Definition 2 (see [5]). Let $\varphi_0 \in \mathscr{K}^*$, $h \in \Gamma$ be given. The function $V(t,x) \in \Lambda$ is said to be φ_0-strongly h-decrescent if there exist a constant $\delta > 0$ and a function $a \in K$ such that $(t,x) \in [-r,\infty) \times R^n : (\varphi_0 \cdot h(t,x)) < \delta$ implies that $(\varphi_0 \cdot V(t,x)) \leq a(\varphi_0 \cdot h(t,x))$.

Let $V(t,x) \in \Lambda$, $t \in \Omega$, $t \neq \tau_k$, $x \in R^n$, and $\phi \in PC([t-r,t],R^n)$. We define a derivative of the function $V(t,x)$ along the trajectory of solution of (1) as follows:

$$D_{(1)}V(t,\phi(t)) = \lim_{\epsilon\to 0}\sup\frac{1}{\epsilon}\left\{V\left(t+\epsilon,\phi(t) + \epsilon F\left(t,\phi(t),\sup_{s\in[-r,0]}\phi(t+s)\right)\right) - V(t,\phi(t))\right\}. \tag{11}$$

Similarly we define a derivative of the function $V(t,x) \in \Lambda$ along the trajectory of solution of the perturbed system (2) for $t \in \Omega$, $t \neq \tau_k$, $x \in R^n$, and $\phi \in PC([t-r,t],R^n)$ as follows:

$$D_{(2)}V(t,\phi(t)) = \lim_{\epsilon\to 0}\sup\frac{1}{\epsilon}\left\{V\left(t+\epsilon,\phi(t) + \epsilon\left(F\left(t,\phi(t),\sup_{s\in[-r,0]}\phi(t+s)\right) + G\left(t,\phi(t),\sup_{s\in[-r,0]}\phi(t+s)\right)\right)\right) - V(t,\phi(t))\right\}. \tag{12}$$

Definition 3 (see [5]). Let $\varphi_0 \in \mathscr{K}^*$, $h, h_0 \in \Gamma$ be given. The function h_0 is φ_0-uniformly finer than h if there exist a constant $\delta > 0$ and a function $a \in K$, such that for any point $(t,x) \in [0,\infty)\times R^n : (\varphi_0 \cdot h_0(t,x)) < \delta$ the inequality $(\varphi_0 \cdot h(t,x)) \leq a(\varphi_0 \cdot h_0(t,x))$ holds.

Lemma 4 (see [5]). *Let $h, h_0 \in \Gamma$, $\varphi_0 \in \mathscr{K}^*$ be given, and $h_0(t,x)$ is φ_0-uniformly finer than $h(t,x)$ with a constant δ and a function $a \in K$. Then for any $t \in R^+$ and $\phi \in PC([t-r,t],R^n)$ inequality $H_0(t,\phi,\varphi_0) < \delta$ implies $H(t,\phi,\varphi_0) \leq a(H_0(t,\phi,\varphi_0))$, where functions H and H_0 are defined by* (4), (5).

In our further investigations we use the following comparison result.

Lemma 5 (see [5]). *Let the following conditions be fulfilled.*

($B1$) *The vector $\varphi_0 \in \mathscr{K}^*$ and function $V \in \Lambda$ are such that*

(i) *for any number $t \geq 0 : t \neq \tau_k$ and any function $\psi \in PC([t-r,t],R^n)$ such that $(\varphi_0 \cdot V(t,\psi(t))) \geq (\varphi_0 \cdot V(t+s,\psi(t+s)))$ for $s \in [-r,0)$ the inequality*

$$(\varphi_0 \cdot D_{(1)}V(t,\psi(t))) \leq g_1(t,(\varphi_0 \cdot V(t,\psi(t)))) \tag{13}$$

holds, where $g_1 \in PC(R^+ \times R^+, R^+)$.

(ii) $(\varphi_0 \cdot V(\tau_k+0,I_k(x))) \leq \xi_k(\varphi_0 \cdot V(\tau_k,x))$, $k = 1,2,\ldots$, $x \in R^n$, *and $\tau_k \in [t_0,T]$, where functions $\xi_k \in K$.*

($B2$) *Function $x(t;t_0,\phi)$ is a solution of* (1) *that is defined for $t \in [t_0-r,T]$, where $\phi \in PC([t_0-r,t_0],R^n)$.*

($B3$) *Function $u^*(t) = u^*(t;t_0,u_0)$ is the maximal solution of* (7) *with initial condition $u^*(t_0) = u_0$ that is defined for $t \in [t_0,T]$.*

Then the inequality $\sup_{s\in[-r,0]}(\varphi_0 \cdot V(t_0+s,\phi(t_0+s))) \leq u_0$ implies the validity of the inequality $(\varphi_0 \cdot V(t,x(t))) \leq u^*(t)$ for $t \in [t_0,T]$.

Definition 6. Let $h_0, h \in \Gamma$. System of impulsive differential equations with "supremum" (1) is said to be

($S1$) (H_0,h)-equi-integral φ_0-stable if for every $\alpha \geq 0$ and for any $t_0 \geq 0$ there exists a positive function $\beta = \beta(t_0,\alpha) \in CK$ which is continuous in t_0 for each α and such that for maximal solution $y^*(t;t_0,\phi)$ of the perturbed system of impulsive differential equations with "supremum" (2) the inequality

$$(\varphi_0 \cdot h(t,y^*(t;t_0,\phi))) < \beta, \quad t \geq t_0 \tag{14}$$

holds, provided that

$$H_0(t_0,\phi,\varphi_0) \leq \alpha, \tag{15}$$

and for every $T > 0$,

$$\int_{t_0}^{t_0+T}\sup_{(x,y)\in\Omega(t_0,T,\beta)}\|G(s,x,y^*)\|\,ds + \sum_{t_0\leq\tau_k\leq t_0+T}\sup_{x:h(\tau_k,x)<\beta}\|J_k(x)\| \leq \alpha, \tag{16}$$

where $H_0(t_0,\phi,\varphi_0)$ is defined by (4) and $\phi \in PC([t_0-r,t_0],R^n)$;

($S2$) (H_0,h)-uniform-integrally φ_0-stable if ($S1$) is satisfied, where δ is independent on t_0.

Remark 7. We note that in the case when $h_0(t,x) \equiv \|x\|$ and $h(t,x) \equiv \|x\|$ the (H_0,h)-equi-integral (uniform-integral) φ_0-stability reduces to equi-integral (uniform-integral) φ_0-stability.

3. Main Result

Theorem 8. *Let the following conditions be fulfilled.*

($H1$) *Functions $h_0, h \in \Gamma$; h_0 is φ_0-uniformly finer than h.*

($H2$) *There exists a function $V_1 \in \Lambda$ that is φ_0-strongly h_0-decrescent and*

(i) *for any number $t \geq 0$, $t \neq \tau_k$, and any function $\psi \in PC([t-r,t],R^n)$, such that $(\varphi_0 \cdot V_1(t,\psi(t))) > (\varphi_0 \cdot V_1(t+s,\psi(t+s)))$ for $s \in [-r,0)$ and $(t,\psi(t)) \in S(h,\rho,\varphi_0)$ the inequality*

$$(\varphi_0 \cdot D_{(1)} V_1(t, \psi(t))) \le g_1(t, (\varphi_0 \cdot V_1(t, \psi(t)))) \quad (17)$$

holds, where $\rho > 0$ is a constant.

(ii) $(\varphi_0 \cdot V_1(\tau_k + 0, I_k(x))) \le \xi_k(\varphi_0 \cdot V_1(\tau_k, x))$, *for* $(\tau_k, x) \in S(h, \rho, \varphi_0)$, $k = 1, 2, \ldots$.

$(H3)$ *For any number $\mu > 0$ there exists a function $V_2^{(\mu)} \in \Lambda$ such that*

(iii) $b(\varphi_0 \cdot h(t, x)) \le (\varphi_0 \cdot V_2^{(\mu)}(t, x)) \le a(\varphi_0 \cdot h_0(t, x))$ *for* $(t, x) \in [-r, \infty) \times R^n$, *where* $a, b \in K$ *and* $\lim_{u \to \infty} b(u) = \infty$.

(iv) *For any number* $t \ge 0$, $t \ne \tau_k$, *and any function* $\psi \in PC([t - r, t], R^n)$, *such that* $(t, \psi(t)) \in S(h, \rho, \varphi_0) \cap S^c(h_0, \mu, \varphi_0)$ *and* $(\varphi_0 \cdot (V_1(t, \psi(t)) + V_2^{(\mu)}(t, \psi(t)))) > (\varphi_0 \cdot (V_1(t+s, \psi(t+s)) + V_2^{(\mu)}(t+s, \psi(t+s))))$ *for* $s \in [-r, 0)$ *the inequality*

$$\begin{aligned}&\left(\varphi_0 \cdot \left(D_{(1)} V_1(t, \psi(t)) + D_{(2)} V_2^{(\mu)}(t, \psi(t))\right)\right)\\&\quad \le g_2\left(t, \varphi_0 \cdot \left(V_1(t, \psi(t)) + V_2^{(\mu)}(t, \psi(t))\right)\right)\end{aligned} \quad (18)$$

holds.

(v) $(\varphi_0 \cdot (V_1(\tau_k + 0, I_k(x)) + V_2^{(\mu)}(\tau_k + 0, I_k(x)))) \le \eta_k(\varphi_0 \cdot (V_1(\tau_k, x) + V_2^{(\mu)}(\tau_k, x)))$ *for* $(\tau_k, x) \in S(h, \rho, \varphi_0) \cap S^c(h_0, \mu, \varphi_0)$, $k = 1, 2, \ldots$.

$(H4)$ *Zero solution of the scalar impulsive differential equation* (7) *is equi-stable.*

$(H5)$ *Zero solution of the scalar impulsive differential equation* (8) *is uniform-integrally stable.*

Then system of impulsive differential equations with "supremum" (1) *is (H_0, h)-uniform-integrally φ_0-stable.*

Proof. Since function $V_1(t, x)$ is φ_0-strongly h_0-decrescent, there exist a constant $\rho_1 \in (0, \rho)$ and a function $\psi_1 \in K$ such that $(\varphi_0 \cdot h_0(t, x)) < \rho_1$ implies that

$$(\varphi_0 \cdot V_1(t, x)) \le \psi_1((\varphi_0 \cdot h_0(t, x))). \quad (19)$$

Since $h_0(t, x)$ is φ_0-uniformly finer than $h(t, x)$, there exist a constant $\rho_0 \in (0, \rho_1)$ and a function $\psi_2 \in K$ such that $(\varphi_0 \cdot h_0(t, x)) < \rho_0$ implies that

$$(\varphi_0 \cdot h(t, x)) \le \psi_2(\varphi_0 \cdot h_0(t, x)), \quad (20)$$

where $\psi_2(\rho_0) < \rho_1$.

According to Lemma 4, the inequality $H_0(t, \phi, \varphi_0) < \rho_0$ implies

$$H(t, \phi, \varphi_0) \le \psi_2(H_0(t, \phi, \varphi_0)), \quad \phi \in PC([t - r, t], R^n). \quad (21)$$

Let $t_0 \ge 0$ be a fixed point. Choose a number $\alpha > 0$ such that $\alpha < \rho_0$.

According to condition $(H3)$ of Theorem 8, there exists a function $V_2^{(\alpha)}(t, x)$ that is Lipshitz with a constant M_2. Let M_1 be the Lipshitz constant of function $V(t, x)$.

Denote $(M_1 + M_2)\alpha = \alpha_1$. Without loss of generality we assume $\alpha_1 < b(\rho)$.

Since the zero solution of the scalar impulsive differential equation (7) is equi-stable, there exists a function $\delta_1 = \delta_1(t_0, \alpha_1) > 0$ such that the inequality $|u_0| < \delta_1$ implies

$$|u(t; t_0, u_0)| < \frac{\alpha_1}{2}, \quad t \ge t_0, \quad (22)$$

where $u(t; t_0, u_0)$ is a solution of (7).

Since the function $\psi_1 \in K$ there exists a $\delta_2 = \delta_2(\delta_1) > 0$, $\delta_2 < \rho_1$, such that for $|u| < \delta_2$ the inequality

$$\psi_1(u) < \delta_1 \quad (23)$$

holds.

Since the zero solution of the scalar impulsive differential equation (8) is uniform-integrally stable, there exists a function $\beta_1 = \beta_1(\alpha_1) \in CK$, $b(\rho) > \beta_1 \ge \alpha_1$, such that for every solution of the perturbed impulsive equation (9) the inequality

$$|w(t; t_0, w_0)| < \beta_1, \quad t \ge t_0, \quad (24)$$

holds, provided that

$$|w_0| < \alpha_1 \quad (25)$$

and for every $T > 0$,

$$\int_{t_0}^{t_0+T} |q(s)|\, ds + \sum_{t_0 \le \tau_k \le t_0+T} |\gamma_k| \le \alpha_1. \quad (26)$$

Since the function $b \in K$, $\lim_{s \to \infty} b(s) = \infty$, and $\psi_2(\alpha) < \psi_2(\rho_0) < \rho_1 < \rho$, we could choose a constant $\beta = \beta(\beta_1) > 0$, $\rho > \beta > \alpha$, $\beta > \psi_2(\alpha)$, such that

$$b(\beta) \ge \beta_1. \quad (27)$$

Since the function $a, \psi_2 \in K$, and $\beta > \psi_2(\alpha)$, we can find a $\delta_3 = \delta_3(\alpha_1, \beta) > 0$, $\alpha < \delta_3 < \min(\delta_2, \rho_0)$, such that the inequalities

$$a(\delta_3) < \frac{\alpha_1}{2}, \qquad \psi_2(\delta_3) < \beta \quad (28)$$

hold.

From (21) and (28) it follows that $H_0(t_0, \phi, \varphi_0) < \alpha$ implies

$$H(t_0, \phi, \varphi_0) \le \psi_2(H_0(t_0, \phi, \varphi_0)) < \psi_2(\alpha) < \psi_2(\delta_3) < \beta; \quad (29)$$

that is, $h(t, \phi, \varphi_0) < \beta$ for $t \in [t_0 - r, t_0]$.

Now let the initial functions $\phi \in PC([t_0 - r, t_0], R^n)$ be such that

$$H_0(t_0, \phi, \varphi_0) < \alpha \quad (30)$$

and let the perturbed functions in impulsive equation with "supremum" (2) be such that

$$\int_{t_0}^{t_0+T} \sup_{x,y\in\Omega(t_0,T,\beta)} \|G(s,x,y^*)\| ds + \sum_{t_0\le\tau_k\le t_0+T} \|J_k(x)\| \le \alpha \tag{31}$$

for every $T > 0$.

Let $y^*(t) = y^*(t;t_0,\phi)$ be a solution of (2), where the initial function and the perturbed functions satisfy (30) and (31); then

$$(\varphi_0 \cdot h(t, y^*(t;t_0,\phi))) < \beta, \quad t \ge t_0. \tag{32}$$

Suppose it is not true. There exists a point $t^* > t_0$ such that

$$\begin{aligned}(\varphi_0 \cdot h(t^*, y^*(t^*;t_0,\phi))) &= \beta,\\ (\varphi_0 \cdot h(t, y^*(t;t_0,\phi))) &< \beta,\\ t &\in [t_0, t^*).\end{aligned} \tag{33}$$

Case 1. Let $t^* \ne \tau_k$, $k = 1, 2, \ldots$. Then from the continuity of the maximal solution $y^*(t;t_0,\phi)$ at point t^* follows that $(\varphi_0 \cdot h(t^*, y^*(t^*;t_0,\phi))) = \beta$.

If we assume that $(\varphi_0 \cdot h_0(t^*, y^*(t^*))) \le \delta_3$ then from the choice of δ_3 and inequality (28) it follows $(\varphi_0 \cdot h(t^*, y^*(t^*;t_0,\phi))) \le (\varphi_0 \cdot h_0(t^*, y^*(t^*;t_0,\phi))) \le \psi_2(\delta_3) < \beta$ that contradicts (33).

Therefore

$$(\varphi_0 \cdot h_0(t^*, y^*(t^*))) \le \delta_3, \qquad H_0(t_0,\phi,\varphi_0) < \alpha < \delta_1. \tag{34}$$

Case 1.1. Let there exist a point $t_0^* \in (t_0, t^*)$, $t_0^* \ne \tau_k$, $k = 1, 2, \ldots$, such that $\delta_3 = (\varphi_0 \cdot h_0(t^*, y^*(t^*)))$ and $(t, y^*(t)) \in S(h,\beta,\varphi_0) \cap S^c(h_0,\delta_3,\varphi_0)$. Since $\beta < \rho$ and $\delta_3 > \alpha$ it follows that

$$(t, y^*(t)) \in S(h,\rho,\varphi_0) \cap S^c(h_0,\alpha,\varphi_0), \quad t \in [t_0^*, t^*). \tag{35}$$

Define a function $\phi^*(t) = y^*(t)$ for $t \in [t_0^* - r, t_0^*]$ and let $r_1(t;t_0^*,u_0)$ be the maximal solution of impulsive scalar differential equation (7) where $u_0 = \sup_{s\in[-r,0]}(\varphi_0 \cdot V_1(t_0^*, \phi^*(t_0^*)))$. Let $x^*(t) \equiv x^*(t;t_0^*,\phi^*)$ be the solution of the impulsive equations (1), $t \in [t_0^* - r, t_0^*]$. From conditions (i), (ii) of Theorem 8, according to Lemma 5, it follows that

$$(\varphi_0 \cdot V_1(t, x^*(t))) \le r_1(t;t_0^*,u_0), \quad t \in [t_0^*, t^*]. \tag{36}$$

From the choice of the point t_0^* it follows that $(\varphi_0 \cdot h_0(t_0^*, \phi^*(t_0^*))) = (\varphi_0 \cdot h_0(t_0^*, y^*(t_0^*))) = \delta_3 < \delta_2$.

According to inequalities (19) and (23) we obtain

$$\begin{aligned}u_0 &= (\varphi_0 \cdot V_1(t_0^*, \phi^*(t_0^*)))\\ &\le \psi_1(\varphi_0 \cdot h_0(t_0^*, \phi^*(t_0^*))) < \delta_1.\end{aligned} \tag{37}$$

From inequalities (22) and (36) it follows that $(\varphi_0 \cdot V_1(t, x^*(t))) \le r_1(t;t_0^*,u_0) < \alpha_1/2$ for $t \in [t_0^*, t^*]$, or

$$(\varphi_0 \cdot V_1(t_0^*, y^*(t_0^*))) < (\varphi_0 \cdot V_1(t_0^*, x(t_0^*))) < \frac{\alpha_1}{2}. \tag{38}$$

From inequality (28) and condition (iii) of Theorem 8, it follows that

$$\begin{aligned}\left(\varphi_0 \cdot V_2^{(\alpha)}(t_0^*, y^*(t_0^*))\right) &< a(\varphi_0 \cdot h_0(t_0^* + s, y^*(t_0^* + s)))\\ &= a(\delta_3) < \frac{\alpha_1}{2}.\end{aligned} \tag{39}$$

Consider function $V_2^{(\alpha)}(t,x)$ that is defined in condition $(H7)$ of Theorem 8, and define the function

$$V(t,x) = V_1(t,x) + V_2^{(\alpha)}(t,x), \tag{40}$$

the function $V(t,x)$ satisfies the conditions of Lemma 5. Let point $t \in [t_0^*, t^*]$, $t \ne t_k$, and function $\psi \in PC([t-r,t], R^n)$ be such that $(t,\psi(t)) \in S(h,\beta,\varphi_0) \cap S^c(h_0,\alpha,\varphi_0)$, $(\psi(t), \sup_{s\in[-r,0]}\psi(t+s)) \in \Omega(t_0^*, T^*, \beta)$, and $V(t,\psi(t)) > V(t+s, \psi(t+s))$ for $s \in [-r, 0)$. Then using the Lipshitz conditions for functions $V_1(t,x)$ and $V_2^{(\alpha)}(t,x)$, and condition (iv) of Theorem 8, we obtain

$$\begin{aligned}&(\varphi_0 \cdot D_{(2)}V(t,\psi(t)))\\ &\quad= \left(\varphi_0 \cdot \left(D_{(2)}V_1(t,\psi(t)) + D_{(2)}V_2^{(\alpha)}(t,\psi(t))\right)\right)\\ &\quad\le \left(\varphi_0 \cdot D_{(1)}V_1(t,\psi(t)) + D_{(1)}V_2^{(\alpha)}(t,\psi(t))\right)\\ &\qquad + (M_1 + M_2)\left\| G\left(t, \psi(t), \sup_{s\in[-r,0]} \psi(t+s)\right)\right\|\\ &\quad\le g_2(t, (\varphi_0 \cdot V(t,\psi(t)))) + (M_1 + M_2)\\ &\qquad\times \sup_{(x,y)\in\Omega(t_0^*,T^*,\beta)} \|G(t,x,y^*)\|,\end{aligned} \tag{41}$$

where $T^* = t^* - t_0^*$.

Let $\tau_k \in (t_0^*, t^*)$, $x \in R^n$ be such that $(\tau_k, x) \in S(h,\beta,\varphi_0) \cap S^c(h_0,\alpha,\varphi_0)$. According to condition (v) of Theorem 8, we have

$$\begin{aligned}&(\varphi_0 \cdot V(t_k + 0, I_k(x) + J_k(x)))\\ &\quad= (\varphi_0 \cdot V(t_k + 0, I_k(x)))\\ &\qquad + (\varphi_0 \cdot (V(t_k + 0, I_k(x) + J_k(x)) - V(t_k + 0, I_k(x))))\\ &\quad\le \eta_k(\varphi_0 \cdot V(t_k, x)) + (M_1 + M_2)\|J_k(x)\|\\ &\quad\le \eta_k(\varphi_0 \cdot V(t_k, x)) + (M_1 + M_2)\\ &\qquad\times \sup_{x:h(\tau_k,x)<\beta} \|J_k(x)\|.\end{aligned} \tag{42}$$

According to inequalities (41), (42) and Lemma 5, the inequality

$$(\varphi_0 \cdot V(t, y^*(t))) \le r^*(t;t_0^*,w_0^*), \quad t \in [t_0^*, t^*] \tag{43}$$

holds.

Consider the scalar impulsive differential equation (9), where

$$q(t) = (M_1 + M_2) \sup_{(x,y)\in\Omega(t_0^*, T^*, \beta)} \|G(t, x, y^*)\|, \qquad \gamma_k = (M_1 + M_2) \sup_{x: h(\tau_k, x) < \beta} \|J_k(x)\|. \tag{44}$$

According to above notations and inequality (31) for $T^* = t^* - t_0^*$, we obtain

$$\int_{t_0^*}^{t^*} q(s)\, ds + \sum_{t_0^* \le \tau_k \le t^*} \gamma_k \le (M_1 + M_2)\alpha = \alpha_1. \tag{45}$$

Let $r^*(t; t_0^*, w_0^*)$ be the maximal solution of (9) through the point (t_0^*, w_0^*), where $w_0^* = V(t_0^* + s, y^*(t_0^* + s))$, and perturbations $q(t)$ and γ_k are defined above and satisfy inequality (45).

Choose a point $T^* > t^*$ such that

$$\int_{t_0^*}^{t^*} q(s)\, ds + \frac{1}{2}(T^* - t^*) q(t^*) < \alpha_1. \tag{46}$$

Now define the continuous function $q^*(t) : [t_0^*, \infty) \to R$:

$$q^*(t) = \begin{cases} q(t) & \text{for } t \in [t_0^*, t^*] \\ \dfrac{q(t^*)}{t^* - T^*}(t - T^*) & \text{for } t \in [t^*, T^*] \\ 0 & \text{for } t \ge T^*, \end{cases} \tag{47}$$

and the sequence of numbers $\{\gamma_k^*\}_1^\infty$:

$$\gamma_k^* = \begin{cases} \gamma_k & \text{for } k : \tau_k \in (t_0^*, t^*] \\ 0 & \text{for } k : \tau_k > t^*. \end{cases} \tag{48}$$

From (45), it follows that for every $T > 0$

$$\int_{t_0^*}^{t_0^*+T} q^*(s)\, ds + \sum_{t_0^* \le \tau_k \le t_0^*+T} \gamma_k^* \le \int_{t_0^*}^{t_0^*+T} q(s)\, ds + \sum_{t_0^* \le \tau_k \le t_0^*+T} \gamma_k \le \alpha_1. \tag{49}$$

Let $R(t; t_0^*, w_0^*)$ be the maximal solution of the scalar impulsive differential equation (9) through the point (t_0^*, w_0^*), where perturbations of the right parts are defined above function $q^*(t)$ and numbers γ_k^*. We note that

$$R(t; t_0^*, w_0^*) = r^*(t; t_0^*, w_0^*), \quad t \in [t_0^*, t^*). \tag{50}$$

From inequalities (38) and (39), the definition of point w_0^*, and inequality (49) follows the validity of (24) for the solution $R(t; t_0^*, w_0^*)$; that is,

$$R(t; t_0^*, w_0^*) < \beta_1, \quad t \ge t_0^*. \tag{51}$$

From inequalities (43) and (51), equality (50), the choice of point t^*, and condition (iii) of Theorem 8, we obtain

$$\begin{aligned} b(\beta) &\ge \beta_1 > R(t^*; t_0^*, w_0^*) \\ &\ge (\varphi_0 \cdot V(t^*, y^*(t^*; t_0, \phi))) \\ &\ge (\varphi_0 \cdot V_2^{(\alpha)}(t^*, y^*(t^*; t_0, \phi))) \\ &\ge b((\varphi_0 \cdot h(t^*, y^*(t^*; t_0, \phi)))) \\ &= b(\beta). \end{aligned} \tag{52}$$

The obtained contradiction proves the validity of the inequality (32) for $t \ge t_0$.

Case 1.2. Let there exist a point $\tau_k \in (t_0, t^*)$ such that $\delta_3 < (\varphi_0 \cdot h_0(\tau_k + 0, y^*(\tau_k + 0; t_0, x_0)))$, $\delta_3 > (\varphi_0 \cdot h_0(\tau_k, y^*(\tau_k; t_0, x_0)))$, and (35) is true.

We choose a number $\widetilde{\delta_3} : \delta_3 < \widetilde{\delta_3} < \beta$ such that $\widetilde{\delta_3} = (\varphi_0 \cdot h_0(t_0^*, y^*(t_0^*; t_0, x_0)))$ and $t_0^* \in (t_0, t^*)$, $t_0^* \ne \tau_k$, $k = 1, 2, \ldots$. We repeat the proof of Case 1.1, where instead of δ_3 we use $\widetilde{\delta_3}$ and obtain a contradiction.

Case 2. Let there exist a natural number k such that $(\varphi_0 \cdot h(t, y^*(t))) < \beta$ for $t \in [t_0, \tau_k]$ and $(\varphi_0 \cdot h(\tau_k, y^*(\tau_k + 0))) = (\varphi_0 \cdot h(\tau_k, I_k(y^*(\tau_k)) + J_k(y^*(\tau_k)))) > \beta$.

We repeat the proof of Case 1 as in this case we choose the constant $\beta = \beta(\beta_1) > 0$, such that $b(\beta) \ge \sup_k\{\eta_k(\beta_1)\}$.

As in the proof of Case 1.1, we obtain the validity of inequalities (51) and (43). We apply conditions (iii) and (v) of Theorem 8 and obtain

$$\begin{aligned} b(\beta) &\ge \eta_k(r^*(\tau_k; t_0^*, w_0^*)) \\ &\ge \eta_k(\varphi_0 \cdot V(\tau_k, y^*(\tau_k))) \\ &= \eta_k((\varphi_0 \cdot (V_1(\tau_k, y^*(\tau_k)) + V_2^{(\alpha)}(\tau_k, y^*(\tau_k))))) \\ &\ge (\varphi_0 \cdot V_1(\tau_k, I_k(y^*(\tau_k)) + J_k(y^*(\tau_k))) \\ &\qquad + V_2^{(\alpha)}(\tau_k, I_k(y^*(\tau_k)) + J_k(y^*(\tau_k)))) \\ &\ge (\varphi_0 \cdot V_2^{(\alpha)}(\tau_k, I_k(y^*(\tau_k)) + J_k(y^*(\tau_k)))) \\ &\ge b(\varphi_0 \cdot h((\tau_k, I_k(y^*(\tau_k)) + J_k(y^*(\tau_k))))) \\ &> b(\beta), \end{aligned} \tag{53}$$

and the obtained contradiction proves the validity of inequality (32) in this case. Inequality (32) proves (H_0, h)-uniform-integral φ_0-stabilities of the considered system of the impulsive differential equations with "supremum." □

Next, we will provide an example which satisfies all the hypotheses of Theorem 8.

Example 9. Consider the system of impulsive differential equations with "supremum"

$$\begin{aligned}
x' &= -e^{-t}x(t) + 2y(t) + e^{-t}\max_{s\in[t-r,t]} x(s), \quad t \neq k,\\
y' &= -x(t) - e^{-t}y(t) + \frac{1}{2}e^{-t}\max_{s\in[t-r,t]} y(s), \quad t \neq k,\\
x(k+0) &= \frac{1}{2^{k/2}}x(k),\\
y(k+0) &= \frac{1}{2^{k/2}}y(k),\\
k &= 1,2,\ldots,\\
x(t) &= \phi_1(t-t_0),\\
y(t) &= \phi_2(t-t_0)\\
t &\in [t_0-r, t_0],
\end{aligned} \tag{54}$$

and its perturbed impulsive differential equations with "supremum"

$$\begin{aligned}
x' &= -e^{-t}x(t) + 2y(t) + e^{-t}\max_{s\in[t-r,t]} x(s) + e^{-t}\max_{s\in[t-r,t]} x^2(s),\\
&\qquad t \neq k,\\
y' &= -x(t) - e^{-t}y(t) + \frac{1}{2}e^{-t}\max_{s\in[t-r,t]} y(s) + e^{-t}\max_{s\in[t-r,t]} y^2(s),\\
&\qquad t \neq k,\\
x(k+0) &= \frac{1}{2^{k/2}}x(k),\\
y(k+0) &= \frac{1}{2^{k/2}}y(k),\\
k &= 1,2,\ldots,\\
x(t) &= \phi_1(t-t_0), \quad y(t) = \phi_2(t-t_0) \quad t \in [t_0-r, t_0],
\end{aligned} \tag{55}$$

where $x, y \in R$, $r > 0$ is enough small constant, $t \geq t_0 \geq 0$. Without loss of generality we will assume further that $1 \geq t_0 \geq 0$.

Let $h_0(t,x,y) = (\|x\|, \|y\|)$, $h(t,x,y) = (x^2, y^2)$.

Consider function $V : R^2 \to \mathscr{K}, V = (V_1, V_2)$, $V_1(x,y) = (1/2)x^2$, $V_2(x,y) = (1/2)y^2$, where $\mathscr{K} = \{(x,y) : x \geq 0,\ y \geq 0\} \subset R^2$ is a cone.

Now, let us consider the vector $\varphi_0 = (1,2)$. It is easy to check that the function $V_1(t,x,y) = V(x,y)$ is φ_0-strongly h_0-decrescent with a function $\psi_2 = x \in K$ and the condition (iii) is satisfied for the function $V_2^{(\mu)} = V(x,y)$, where $b(u) = (1/2)u$ and $a(u) = u^2$.

Let $t \geq 0$, $t \neq k$, $k = 1,2\ldots$ $\psi \in PC([t-r,t], R^2)$, $\psi = (\psi_1, \psi_2)$ be such that the inequality

$$\begin{aligned}
&(\varphi_0 \cdot V(\psi_1(t), \psi_2(t)))\\
&\quad \geq (\varphi_0 \cdot V(\psi_1(t+s), \psi_2(t+s))), \quad s \in [-r, 0]
\end{aligned} \tag{56}$$

or

$$\frac{1}{2}\psi_1^2(t) + \psi_2^2(t) \geq \frac{1}{2}\psi_1^2(t+s) + \psi_2^2(t+s), \quad s \in [-r,0]; \tag{57}$$

then

$$\begin{aligned}
\psi_1(t)\max_{s\in[t-r,t]}\psi_1(s) &\leq 2(\varphi_0 \cdot V(\psi_1(t), \psi_2(t))),\\
\psi_2(t)\max_{s\in[t-r,t]}\psi_2(s) &\leq (\varphi_0 \cdot V(\psi_1(t), \psi_2(t))).
\end{aligned} \tag{58}$$

Therefore if inequality (57) is satisfied then

$$\begin{aligned}
&(\varphi_0 \cdot D_{(54)}V(\psi_1(t), \psi_2(t)))\\
&\quad = e^{-t}\Big(-(\psi_1(t))^2 - 2(\psi_2(t))^2\\
&\qquad + \psi_1(t)\max_{s\in[t-r,t]}\psi_1(s) + \psi_2(t)\max_{s\in[t-r,t]}\psi_2(s)\Big)\\
&\quad \leq e^{-t}\big(-(\psi_1(t))^2 - 2(\psi_2(t))^2 + 2(\varphi_0 \cdot V(\psi_1(t), \psi_2(t)))\\
&\qquad + (\varphi_0 \cdot V(\psi_1(t), \psi_2(t)))\big)\\
&\quad = e^{-t}(\varphi_0 \cdot V(\psi_1(t), \psi_2(t)))
\end{aligned} \tag{59}$$

or

$$\begin{aligned}
&(\varphi_0 \cdot D_{(54)}V(\psi_1(t), \psi_2(t)))\\
&\quad \leq e^{-t}(\varphi_0 \cdot V(\psi_1(t), \psi_2(t))).
\end{aligned} \tag{60}$$

Inequality (60) proves the validity of condition (i) of Theorem 8 for the function $V_1(t,x,y) = V(x,y)$, where $g_1(t,u) = ue^{-t}$. Meanwhile, inequality (60) proves the validity of condition (iv) of Theorem 8 for the function $V_2^{\mu}(t,x,y) = V(x,y)$, where $g_2(t,u) = 2ue^{-t}$.

From jump conditions (54) and the choice of vector φ_0 and function V we obtain the validity of conditions (ii) and (v) of Theorem 8 for the functions $V_1(t,x,y) = V(x,y)$ and $V_2^{\mu}(t,x,y) = V(x,y)$, where $\xi_k(u) = (1/2^k)u$ and $\eta_k(u) = (1/2^k)u$.

Consider following comparison scalar impulsive differential equation:

$$u' = ue^{-t}, \quad t \neq k, \quad u(k+0) = \frac{1}{2^k}u(k), \tag{61}$$

$$w' = 2we^{-t}, \quad t \neq k, \quad w(k+0) = \frac{1}{2^k}w(k). \tag{62}$$

The solutions of the impulsive differential equation (61) and (62), correspondingly, are equi-stable and uniform-integrally stable. Thus, according to Theorem 8 the system of impulsive differential equations with "supremum" (54) is (H_0, h)-uniform-integrally φ_0-stable.

4. Conclusion

This paper extends the notions of φ_0-stability in terms of two measures to integral φ_0-stability in terms of two measures for impulsive differential equations with "supremum" and establishes a criterion on integral φ_0-stability in terms of two measures for such system by using the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method. Finally, an example is given to illustrate our result.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

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