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Research Article

Conditional Lie-Bäcklund Symmetries and Reductions of the Nonlinear Diffusion Equations with Source

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Conditional Lie-Bäcklund symmetry approach is used to study the invariant subspace of the nonlinear diffusion equations with source $u_t = e^{-qx}(e^{px}P(u)u_x^m)_x + Q(x,u)$, $m \neq 1$. We obtain a complete list of canonical forms for such equations admit multidimensional invariant subspaces determined by higher order conditional Lie-Bäcklund symmetries. The resulting equations are either solved exactly or reduced to some finite-dimensional dynamic systems.

1. Introduction

The classical symmetry theory for studying differential equations is presented firstly by Lie, which has been universally used and proved to be very effective in similarity reductions and group classifications [1–7]. However, there exist some important equations with very small Lie point symmetry groups. For example, the Fisher equation and Fitzhugh-Nagumo equation, which are widely used in mathematical biology, are invariant only under the time and space translations. This means that the classical symmetry reduction method is not a proper tool for dealing with these equations. To overcome this difficulty, several generalized methods have been developed and established, including the nonclassical symmetry method (or referred to as the conditional symmetry method) [8], the weak symmetry method [9, 10], iteration of the nonclassical method [11], the Clarkson-Kruskal direct method [12, 13], and the conditional Lie-Bäklund symmetry (CLBS) method (or referred to as the generalized conditional symmetry method) [14–16].

CLBS can be regarded as a natural generalization of the nonclassical symmetry. Therefore, the procedure for computing the CLBSs is about the same as for the nonclassical method. Furthermore, Galaktionov and Svirshchevski have shown that the CLBS method is closely related to the invariant subspace (IS) method; namely, exact solutions defined on ISs for differential equations or their variant forms can be obtained by using the CLBS method [17–19]. For nonlinear diffusion equations (NLDEs), symmetry-related methods, especially the CLBS method, have been proved to be very powerful to classify and reduce the considered equations [20–34]. For example, NLDEs

$$f(x)\,u_t = \left(g(x)w(u)u_x\right)_x \tag{1}$$

can be used to describe not only the process by which matter is transported from one part of a system to another, as a result of random molecular motion, but they can also represent many other apparently unrelated phenomena such as heat conduction in solids or even the stationary notion of a boundary layer of fluid over a flat plate [35]. In [36], the Lie point symmetry method has been used to obtain the similarity solutions of the inhomogeneous NLDEs

$$u_t = x^{-q}\left(x^p u^n u_x\right)_x \tag{2}$$

belonging to the above equations, where nonzero constants p and q have several applications such as propagation of a thermal wave in an exponential atmosphere. A complete classification of the symmetry reductions of these equations using the nonclassical method is given by Saied in [37]. The second-order CLBSs of these equations have also been studied in [33]. Furthermore, the generalized porous medium equations

$$u_t = \left(P(u)u_x^m\right)_x + Q(u), \quad m \neq 1, \tag{3}$$

are considered by using the CLBS method in [34]. Some exact solutions, defined on the polynomial, trigonometric, and exponential ISs determined by the CLBSs, are constructed.

In this paper, we mainly discuss the following NLDEs:

$$u_t = e^{-qx}\left(e^{px}P(u)u_x^m\right)_x + Q(x,u), \quad m \neq 1, \tag{4}$$

by means of the CLBS method. Here, $P(u)$ and $Q(x,u)$ are, respectively, referred to as the diffusion and source terms. Equation (4) has a wide range of applications in physics, diffusion process, and engineering sciences and has been applied to describe several situations such as heat conduction by electrons in a plasma, heat conduction by radiation in a fully ionized gas, axisymmetric flow of a very viscous fluid, and turbulent diffusion [38, 39].

The remainder of this paper is organized as follows. In the following section, we review some necessary notations, definitions, and fundamental theorems on the CLBS method. Equations of the form (4) admitting CLBSs and the corresponding ISs are classified in Section 3. Exact solutions and reductions of some examples in the resulting equations are obtained in Section 4. The last section is devoted to conclusions and discussions.

2. Preliminaries

Let us give a brief discussion on the CLBS method. For the mth-order equation

$$u_t = E(t,x,u,u_1,\ldots,u_m), \tag{5}$$

we set

$$V = \sum_{k=0}^{\infty} D_x^k \eta \frac{\partial}{\partial u_k} \tag{6}$$

as an evolutionary vector field with characteristic η. Here, we use the following notations:

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1}\frac{\partial}{\partial u_k}, \qquad D_x^{j+1} = D_x\left(D_x^j\right),$$
$$D_x^0 = 1, \qquad u_k = \frac{\partial^k u}{\partial x^k}. \tag{7}$$

Definition 1. The evolutionary vector field (6) is said to be a Lie-Bäcklund symmetry of (5) if and only if

$$V\left(u_t - E\right)\big|_L = 0, \tag{8}$$

where L is the set of all differential consequences of the equation; that is,

$$u_t - E = 0, \qquad D_x^j D_t^k\left(u_t - E\right) = 0, \quad j,k = 0,1,2,\ldots. \tag{9}$$

Definition 2. The evolutionary vector field (6) is said to be a CLBS of (5) if and only if

$$V\left(u_t - E\right)\big|_{L\cap M} = 0, \tag{10}$$

where M denotes the set of all differential consequences of equation $\eta = 0$ with respect to x; that is, $D_x^j\eta = 0$, $j = 0,1,2,\ldots$.

Proposition 3 (Zhdanov [14] and Fokas and Liu [15, 16]). *Equation* (5) *admits the CLBS* (6) *if there exists a function* $W(t,x,u,\eta)$ *such that*

$$\frac{\partial\eta}{\partial t} = [E,\eta] + W(t,x,u,\eta), \quad W(t,x,u,0) = 0, \tag{11}$$

where $[E,\eta] = E'\eta - \eta' E$, *the prime denotes the Fréchet derivative, and* W *is an analytic function of* t, x, u, u_1, *and* η, $D_x\eta$, $D_x^2\eta,\ldots$.

An obvious conclusion of this proposition is that (5) admits the CLBS with the characteristic η if

$$D_t\eta\big|_{L\cap M} = 0. \tag{12}$$

Here, L and M are given as in Definitions 1 and 2.

For (4), we set the characteristic

$$\sigma = [g(u)]_{lx} + a_1(x)[g(u)]_{(l-1)x} + \cdots + a_l(x)g(u), \tag{13}$$

where $[g(u)]_{ix} = \partial^i[g(u)]/\partial x^i$, $i = 1,2,\ldots,l$, and $l \in N$. It is important to note that if (4) admits CLBS (13), then equation

$$v_t = e^{(p-q)x}\left[A(v)v_x^{m-1}\left(v_{xx} + \frac{p}{m}v_x\right) + B(v)v_x^{m+1}\right] + C(x,v) \tag{14}$$

admits CLBS

$$\eta = v_{lx} + a_1(x)v_{(l-1)x} + \cdots + a_l(x)v. \tag{15}$$

In fact, (4) and (14) are related as

$$A(v) = mP\left[f(v), f'(v)\right]^{m-1},$$
$$B(v) = \frac{1}{m}\left[\frac{f''(v)}{f'(v)}A(v) + A'(v)\right], \tag{16}$$
$$C(x,v) = \frac{Q(x,f(v))}{f'(v)},$$

where $u = f(v)$ denotes the inverse function of $v = g(u)$.

From (12), we can see that (14) admitting of the CLBS with the characteristic (15) is equivalent to $\eta' E[v]|_M = 0$; namely,

$$(E[v])_{lx} + a_1(x)(E[v])_{(l-1)x} + \cdots + a_l(x)(E[v])|_M = 0, \tag{17}$$

where

$$E[v] \equiv e^{(p-q)x}\left[A(v)v_x^{m-1}\left(v_{xx} + \frac{p}{m}v_x\right) + B(v)v_x^{m+1}\right] + C(x,v) \tag{18}$$

and M is given as in Definition 2. Thus, the linear solution space

$$W_l = W\{f_1(x), f_2(x), \ldots, f_l(x)\} \tag{19}$$

of linear ordinary differential equation (ODE) $\eta = 0$ is invariant with respect to the above operator $E[v]$; that is,

$$E\left[\sum_{i=1}^{l} C_i f_i(x)\right] = \sum_{i=1}^{l} \Psi_i\left(C_1, C_2, \ldots, C_l\right) f_i(x). \tag{20}$$

It follows that if (14) admits CLBS (15), then (14) has an exact solution of the generalized separation of variables form

$$v(x,t) = \sum_{i=1}^{l} C_i(t) f_i(x), \tag{21}$$

where the coefficients $C_i(t)$ satisfy the l-dimensional dynamic system

$$C_i'(t) = \Psi_i\left(C_1(t), C_2(t), \ldots, C_l(t)\right), \quad i = 1, 2, \ldots, l. \tag{22}$$

Thus, we will determine the forms of (14) so that (14) admits the CLBS (15), which is equivalent to classify (14) in terms of the IS (19), or it is equivalent to study the CLBS (13) of (4).

The following theorem provides us with the estimate to the maximal dimension of the IS admitted by an mth-order nonlinear differential operator.

Theorem 4 (Galaktionov and Svirshchevski [19]). *If a linear space W_l is invariant with respect to the nonlinear differential operator E of order m, then there exists an inequality*

$$l \leq 2m + 1. \tag{23}$$

It follows that the order of linear ODE $\eta = 0$ is not greater than five if (14) admits the CLBS (15). This allows us to classify (14) based on the existence of the generalized variable separation solutions (GVSSs) (21), which are generated by the solution space (19) determined by the linear ODE $\eta = 0$, or it is equivalent to study the GVSSs $u = f(v)$ of (4) generated by $\sigma = 0$.

3. CLBSs and Corresponding ISs of (14)

In view of Theorem 4, it suffices to consider CLBSs (15) of (14) with $2 \leq l \leq 5$. We first consider the case $l = 2$. It implies from (17) that (14) admits the CLBS (15) if there holds

$$\begin{aligned}
&\eta' E[v]\big|_M \\
&= e^{(p-q)x} B'' v_x^{m+3} + e^{(p-q)x} \\
&\quad \times \left\{\left[\frac{p}{m} - a_1\right] A'' + 2\left[p - q - (m+1) a_1\right] B'\right\} v_x^{m+2} \\
&\quad + e^{(p-q)x} \Big\{ m\left(2A' + (m+1) B\right) a_1^2 - \left[(m+1) B + 2A'\right] a_1' \\
&\qquad - \left[A'' v + 2A' + (2m+3) B' v + mB\right] a_2' \\
&\qquad + p\left[\frac{2}{m}(p-q) - 2a_1\right] A' + (p-q) \\
&\qquad \times \left[B\left(p - (2m+1) a_1\right) - 2a_1 A'\right] + Bq^2 \Big\} v_x^{m+1} \\
&\quad + e^{(p-q)x} \Big\{ -\left(m^2 - m\right) A a_1^3 + \left[(3m-2) p - (2m-1) q\right] \\
&\qquad \times A a_1^2 + \Big[(4m-1) A' v + 2m(m+1) Bv \\
&\qquad + 3(m-1) A\Big] a_1 a_2 \\
&\qquad + \left[-(p-q)\left(\frac{3m-1}{m} p - q\right) + (3m-1) a_1'\right] A a_1 \\
&\qquad + \Big[-\frac{(2m+1) A' v + (m-1) A}{m} p \\
&\qquad - 2(p-q)\left(A + A' v + (2m+1) Bv\right)\Big] a_2 \\
&\qquad - \left(2A' v + 2A + (m+1) Bv\right) a_2' \\
&\qquad + \left[\frac{p(p-q)^2}{m} - a_1'' + (2q - 3p) a_1'\right] A \Big\} v_x^m \\
&\quad + e^{(p-q)x} \Big\{ -(3m-2)(m-1) A a_1^2 a_2 v \\
&\qquad + \left[(6m-5) p - (4m-3) q\right] A v a_1 a_2 \\
&\qquad + \left[\left(m^2 + m\right) Bv + (3m-3) A + (2m-1) A' v\right] \\
&\qquad \times a_2^2 v + \left[(3m-1) a_1' - (3p-q)(p-q)\right] a_2 A v \\
&\qquad + \left[2q - 3p + (3m-3) a_1\right] A v a_2' \Big\} v_x^{m-1} \\
&\quad + (m-1) e^{(p-q)x} \left[-(3m-4) a_1 a_2 + 3a_2' + a_2(3p - 2q)\right] \\
&\quad \times a_2 v^2 A v_x^{m-2} - (m-1)(m-2) A a_2^3 v^3 v_x^{m-3} + v_x^2 C_{vv} \\
&\quad + 2C_{xv} v_x + C_{xx} + a_2\left(C - vC_v\right) + a_1 C_x = 0,
\end{aligned} \tag{24}$$

where the primes and subscripts denote the derivatives and the partial derivatives with respect to the indicated variables, respectively. To vanish all the coefficients of (24), we will have the following overdetermined system:

$$\begin{aligned}
&B'' = 0, \\
&\left[\frac{p}{m} - a_1\right] A'' + 2\left[p - q - (m+1) a_1\right] B' = 0, \\
&m\left(2A' + (m+1) B\right) a_1^2 - \left[(m+1) B + 2A'\right] a_1' \\
&\quad - \left[A'' v + 2A' + (2m+3) B' v + mB\right] a_2' \\
&\quad + p\left[\frac{2}{m}(p-q) - 2a_1\right] A' \\
&\quad + (p-q)\left[B\left(p - (2m+1) a_1\right) - 2a_1 A'\right] + Bq^2 = 0,
\end{aligned}$$

$$\begin{gathered}
[(3m-2)p-(2m-1)q]Aa_1^2-\left(m^2-m\right)Aa_1^3 \\
+\left[(4m-1)A'v+2m(m+1)Bv+3(m-1)A\right]a_1a_2 \\
+\left[-(p-q)\left(\frac{3m-1}{m}p-q\right)+(3m-1)a_1'\right]Aa_1 \\
+\left[-\frac{(2m+1)A'v+(m-1)A}{m}p\right. \\
\left.-2(p-q)\left(A+A'v+(2m+1)Bv\right)\right]a_2 \\
-\left(2A'v+2A+(m+1)Bv\right)a_2' \\
+\left[\frac{p(p-q)^2}{m}-a_1''+(2q-3p)a_1'\right]A=0, \\
[(6m-5)p-(4m-3)q]Ava_1a_2-(3m-2)(m-1) \\
\times Aa_1^2a_2v+\left[\left(m^2+m\right)Bv+(3m-3)A+(2m-1)A'v\right] \\
\times a_2^2v+\left[(3m-1)a_1'-(3p-q)(p-q)\right]a_2Av \\
+[2q-3p+(3m-3)a_1]Ava_2'=0, \\
\left[-(3m-4)a_1a_2+3a_2'+a_2(3p-2q)\right]a_2v^2A=0, \\
(m-1)(m-2)Aa_2^3v^3=0, \\
C_{vv}=0, \\
C_{xv}=0, \\
C_{xx}+a_2(C-vC_v)+a_1C_x=0.
\end{gathered} \tag{25}$$

For general m, it is apparent that $B(u)=au+b$ and $a_2=0$. Substituting $B(u)$ into the second of the above system, we arrive at

$$\frac{p}{m}A''+2(p-q)a-a_1\left[A''+2(m+1)a\right]=0. \tag{26}$$

To solve (26), we can derive three possibilities: $a_1(x)=s\neq p/m$, $a_1(x)=p/m$, and $a_1(x)\neq$ constant.

Assume that the diffusion coefficient $P(u)$ takes power or exponential forms. From (16), without loss of generality, it is reasonable to consider the following four cases: (i) $A(v)=k_1v^2$, $B(v)=k_2v$; (ii) $A(v)=k_1v$, $B(v)=k_2$; (iii) $A(v)=k_1$, $B(v)=k_2$; and (iv) $A(v)=v^k$ $(k\neq 0,1)$, $B(v)=0$, where k_1 and k_2 are arbitrary constants.

Case 1 ($a_1(x)=s\neq p/m$). In this case, we can derive $A''=2ma[p-q-s(m+1)]/(ms-p)$. Correspondingly, there exist the following cases: (a) $A(v)=(ma[p-q-s(m+1)]/(ms-p))v^2$, $B(v)=av$; (b) $A(v)=k_1v$, $B(v)=k_2$; and (c) $A(v)=k_1$, $B(v)=k_2$. If $A(v)=(ma[p-q-s(m+1)]/(ms-p))v^2$, $B(v)=av$, the third and fourth equations are simplified as

$$\begin{gathered}
\left(m^2+m\right)s^2-(p-q)(2m+1)s+(p-q)^2=0, \\
\left[\left(m^2-m\right)s^2-(p-q)(2m-1)s+(p-q)^2\right] \\
\times(p-q-s(m+1))=0.
\end{gathered} \tag{27}$$

Then, we derive $s=(p-q)/m$ or $s=(p-q)/(m+1)$. However, if $s=(p-q)/(m+1)$, function $A(v)$ turns to zero, which should be omitted. By the similar calculation, we obtain $a_1(x)=(p-q)/m$ or $a_1(x)=(p-q)/(m-1)$ with case (b), while we derive $a_1(x)=(p-q)/m$ with case (c). Therefore, we have results listed as the 1–6th entries in Table 1 with Case 1.

Case 2 ($a_1(x)=p/m$). In this case, (26) becomes $2a(p-q-((m+1)p/m))=0$, which implies $a=0$ or $q=-p/m$. When $q\neq -p/m$, we have $a=0$; the corresponding solution is listed as the seventh entry in Table 1 with arbitrary $A(v)$. And results in the case of $q=-p/m$ are presented as the 8–11th entries in Table 1.

Case 3 ($a_1(x)\neq$ constant). In this case, we can derive

$$\begin{gathered}
\frac{p}{m}A''+2(p-q)a=0, \\
A''+2(m+1)a=0,
\end{gathered} \tag{28}$$

which arrives at $a=0$ or $a\neq 0$, $q=-p/m$. When $a=0$, it is easy to solve out $A(v)=cv+e$. Without loss of generality, we consider two cases (a) $A(v)=k_1v$, $B(v)=k_2$, and (b) $A(v)=k_1$, $B(v)=k_2$. If $A(v)=k_1v$, $B(v)=k_2$, we can formulate $a_1(x)$ according to the third equation of (25). Then, substituting $a_1(x)$ into the fourth equation, we have the following condition:

$$\begin{gathered}
(2m+1)p^2+\left(2m^2-m-1\right)pq-(2m+1)mq^2=0, \\
(m+1)p+(m+1)mq=0.
\end{gathered} \tag{29}$$

Therefore, for general m, we have $q=-p/m$. So, we derive the corresponding result as the 12-13th entries in the Table 1. By similar calculation, we obtain solution listed as 14-15th entries in Table 1 with the case of $A(v)=k_1$, $B(v)=k_2$. When $a\neq 0$, $q=-p/m$, it is apparent that $A(v)=-(m+1)av^2+cv+e$; we assume $A(v)=-(m+1)av^2$, $B(v)=av$ $(m\neq -1)$ without loss of generality. Then, we substitute $A(v)$ and $B(v)$ into the third equation of (25), acquiring $a_1(x)=p[c(m+1)e^{-px/m}-1]/m[cme^{-px/m}-1]$. After translation transformation of x, we have $a_1(x)=p[(m+1)e^{-px/m}-1]/m[me^{-px/m}-1]$, which can be seen as the same case with the former one. Substituting $a_1(x)$ into the fourth equation, we find it identical spontaneously. The corresponding result is listed as the 16th entry in Table 1.

Finally, we have all the solutions listed in the Table 1 with case of $l=2$ for general m including the special relation between p and q, where the unknown functions are given as

Table 1: CLBS (15) and IS (19) with $l = 2$ of (14) for general m.

Number	Equation (14)	CLBS (15)	IS (19)
1	$v_t = e^{(p-q)x}\left[mv\left(\frac{m}{m+k-1}v_x\right)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\frac{m+k-1}{m}\left(\frac{m}{m+k-1}v_x\right)^{m+1}\right] + \frac{m+k-1}{m}\left(\alpha v+\beta+\gamma f_1^{(1)}(x)\right)$	$\eta = v_{xx}+\frac{p-q}{m}v_x$	$W\left\{f_1^{(1)}(x),1\right\}$
2	$v_t = e^{(p-q)x}mvv_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\alpha v+\beta+\gamma f_1^{(1)}(x)$	$\eta = v_{xx}+\frac{p-q}{m}v_x$	$W\left\{f_1^{(1)}(x),1\right\}$
3	$v_t = e^{(p-q)x}mv_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\alpha v+\beta+\gamma f_1^{(1)}(x)$	$\eta = v_{xx}+\frac{p-q}{m}v_x$	$W\left\{f_1^{(1)}(x),1\right\}$
4	$v_t = e^{(p-q)x}mv^2\left(-\frac{p-(m+1)q}{p-q}v_x\right)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\frac{p-q}{mq(p-(m+1)q)^2}e^{(p-q)x} \times v\left(-\frac{p-(m+1)q}{p-q}v_x\right)^{m+1} - \frac{p-q}{p-(m+1)q}\left(\alpha v+\beta+\gamma e^{-((p-q)/m)x}\right),\quad q\neq p,\ \frac{p}{m+1}$	$\eta = v_{xx}+\frac{p-q}{m}v_x$	$W\left\{e^{-((p-q)/m)x},1\right\}$
5	$v_t = e^{(p-q)x}mv_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\alpha v+\beta+\gamma f_1^{(2)}(x)$	$\eta = v_{xx}+\frac{p-q}{m-1}v_x$	$W\left\{f_1^{(2)}(x),1\right\}$
6	$v_t = e^{(p-q)x}\left[m\left(-\frac{p-mq}{p-q}v_x\right)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\frac{p-q}{p-mq}\left(-\frac{p-mq}{p-q}v_x\right)^{m+1}\right] - \frac{p-q}{p-mq}\left(\alpha v+\beta+\gamma e^{-((p-q)/(m-1))x}\right),\quad q\neq p,\ \frac{p}{m}$	$\eta = v_{xx}+\frac{p-q}{m-1}v_x$	$W\left\{e^{-((p-q)/(m-1))x},1\right\}$
7	$v_t = e^{(p-q)x}A(v)v_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\alpha v+\beta+\gamma e^{-(p/m)x}$	$\eta = v_{xx}+\frac{p}{m}v_x$	$W\left\{e^{-(p/m)x},1\right\}$
8	$v_t = e^{((m+1)/m)px}\left[mv_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+v_x^{m+1}\right]+\alpha v+\beta+\gamma e^{-(p/m)x}$	$\eta = v_{xx}+\frac{p}{m}v_x$	$W\left\{e^{-(p/m)x},1\right\}$
9	$v_t = e^{((m+1)/m)px}\left[mv\left(\frac{m}{m+k-1}v_x\right)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\frac{m+k-1}{m}\left(\frac{m}{m+k-1}v_x\right)^{m+1}\right] + \frac{m+k-1}{m}\left(\alpha v+\beta+\gamma e^{-(p/m)x}\right)$	$\eta = v_{xx}+\frac{p}{m}v_x$	$W\left\{e^{-(p/m)x},1\right\}$
10	$v_t = e^{((m+1)/m)px}\left[mv^2(-v_x)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)-(-v_x)^{m+1}\right]+\alpha v+\beta+\gamma e^{-(p/m)x}$	$\eta = v_{xx}+\frac{p}{m}v_x$	$W\left\{e^{-(p/m)x},1\right\}$

Table 1: Continued.

Number	Equation (14)	CLBS (15)	IS (19)
11	$v_t = e^{((m+1)/m)px}\left[mv^2\left(\frac{m+1}{m+k-1}v_x\right)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\frac{(2m+k)(m+k-1)}{(m+1)^2}v\left(\frac{m+1}{m+k-1}v_x\right)^{m+1}\right] + \frac{m+k-1}{m+1}\left(\alpha v+\beta+\gamma e^{-(p/m)x}\right)$	$\eta = v_{xx}+\frac{p}{m}v_x$	$W\left\{e^{-(p/m)x}, 1\right\}$
12	$v_t = e^{((m+1)/m)px}\left[mv\left(\frac{m}{m+k-1}v_x\right)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\frac{m+k-1}{m}\left(\frac{m}{m+k-1}v_x\right)^{m+1}\right] + \frac{m+k-1}{m}(\alpha v+\beta+\gamma f_1^{(3)}(x))$	$\eta = v_{xx}+\frac{p\left[(m+1)\,e^{-(px/m)}-1\right]}{m\left[me^{-(px/m)}-1\right]}v_x$	$W\left\{f_1^{(3)}(x), 1\right\}$
13	$v_t = e^{((m+1)/m)px}mvv_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\alpha v+\beta+\gamma f_1^{(3)}(x)$	$\eta = v_{xx}+\frac{p\left[(m+1)\,e^{-(px/m)}-1\right]}{m\left[me^{-(px/m)}-1\right]}v_x$	$W\left\{f_1^{(3)}(x), 1\right\}$
14	$v_t = e^{((m+1)/m)px}\left[mv_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+v_x^{m+1}\right]+\alpha v+\beta+\gamma f_1^{(3)}(x)$	$\eta = v_{xx}+\frac{p\left[(m+1)\,e^{-(px/m)}-1\right]}{m\left[me^{-(px/m)}-1\right]}v_x$	$W\left\{f_1^{(3)}(x), 1\right\}$
15	$v_t = e^{((m+1)/m)px}mv_x^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)+\alpha v+\beta+\gamma f_1^{(3)}(x)$	$\eta = v_{xx}+\frac{p\left[(m+1)\,e^{-(px/m)}-1\right]}{m\left[me^{-(px/m)}-1\right]}v_x$	$W\left\{f_1^{(3)}(x), 1\right\}$
16	$v_t = e^{((m+1)/m)px}mv^2\left(\frac{3m+2}{m+1}v_x\right)^{m-1}\left(v_{xx}+\frac{p}{m}v_x\right)-\frac{m(m+1)}{(3m+2)^2}e^{((m+1)/m)px} \times v\left(-\frac{p-(m+1)q}{p-q}v_x\right)^{m+1}-\frac{p-q}{p-(m+1)q}\left(\alpha v+\beta+\gamma\left[me^{-(px/m)}-1\right]^{(m+1)/m}\right), \quad m\neq -1$	$\eta = v_{xx}+\frac{p\left[(m+1)\,e^{-(px/m)}-1\right]}{m\left[me^{-(px/m)}-1\right]}v_x$	$W\left\{\left[me^{-(px/m)}-1\right]^{(m+1)/m}, 1\right\}$

$f_1^{(1)}(x) = e^{-((p-q)/m)x}$, for $q \neq p$, and $f_1^{(1)}(x) = x$, for $q = p$; $f_1^{(2)}(x) = e^{-((p-q)/(m-1))x}$, for $q \neq p$, and $f_1^{(2)}(x) = x$, for $q = p$; $f_1^{(3)}(x) = [me^{-px/m} - 1]^{(m+1)/m}$, for $m \neq -1$ and $f_1^{(3)}(x) = \ln(1 + e^{px})$, for $m = -1$.

For special m, it is noted that we can combine some terms in (24). This leads to new overdetermined system which is different from (25). Therefore, there exist new forms of (14) admitting CLBS (15) and IS (19) with $l = 2$, which are not included in Table 1.

For $m = 3$, it follows from (24) that the coefficients in (14) and (15) satisfy

$$\begin{gathered}
B'' = 0, \\
\left(\frac{p}{3} - a_1\right) A'' + 2\left(p - q - 4a_1\right) B' = 0, \\
3\left(2A' + 4B\right) a_1^2 - \left(4B + 2A'\right) a_1' \\
- \left(A''v + 2A' + 9B'v + 3B\right) a_2' + p\left[\frac{2}{3}(p - q) - 2a_1\right] A' \\
+ (p - q)\left[B\left(p - 7a_1\right) - 2a_1 A'\right] + Bq^2 = 0, \\
(7p - 5q) Aa_1^2 - 6Aa_1^3 + \left(11A'v + 24Bv + 6A\right) a_1 a_2 \\
+ \left[-(p - q)\left(\frac{8}{3}p - q\right) + 8a_1'\right] Aa_1 \\
+ \left[-\frac{7A'v + 2A}{3} p - 2(p - q)\left(A + A'v + 7Bv\right)\right] a_2 \\
- \left(2A'v + 2A + 4Bv\right) a_2' \\
+ \left[\frac{p(p - q)^2}{3} - a_1'' + (2q - 3p) a_1'\right] A = 0, \\
(13p - 9q) Ava_1 a_2 - 14Aa_1^2 a_2 v + \left(12Bv + 6A + 5A'v\right) \\
\times a_2^2 v + \left[8a_1' - (3p - q)(p - q)\right] a_2 Av \\
+ \left(2q - 3p + 6a_1\right) Ava_2' + C_{vv} e^{-(p-q)x} = 0, \\
2e^{(p-q)x}\left[-5a_1 a_2 + 3a_2' + a_2(3p - 2q)\right] a_2 v^2 A \\
+ 2C_{xv} = 0, \\
C_{xx} - 2Aa_2^3 v^3 + a_2\left(C - vC_v\right) + a_1 C_x = 0.
\end{gathered} \tag{30}$$

Solving the above system by the same approach, we can obtain new results which are presented in Table 2. Similarly, we also list new results in Table 2 for other special m.

The unknown functions in Table 2 are given in the following:

$$f_1^{(4)}(x) = e^{-(p/2)x}, \quad f_2^{(4)}(x) = xe^{-(p/2)x}, \quad \text{for } q = -\frac{1}{2}p,$$
$$f_1^{(4)}(x) = e^{-((4p-q)/9)x}, \quad f_2^{(4)}(x) = e^{-((p-q)/3)x}, \quad \text{for } q \neq -\frac{1}{2}p;$$

$$f_1^{(5)}(x) = e^{-(p/4)x}, \quad f_2^{(5)}(x) = xe^{-(p/4)x}, \quad \text{for } q = \frac{1}{2}p,$$
$$f_1^{(5)}(x) = e^{(1/2)(-1+\sqrt{-3(p-2q)/(5p-6q)})(p-q)x},$$
$$f_2^{(5)}(x) = e^{-(1/2)(1+\sqrt{-3(p-2q)/(5p-6q)})(p-q)x},$$
$$\text{for } (p - 2q)(5p - 6q) < 0,$$
$$f_1^{(5)}(x) = e^{-(1/2)(p-q)x} \sin\left(\sqrt{\frac{3(p - 2q)}{5p - 6q}}\, \frac{p - q}{2} x\right),$$
$$f_2^{(5)}(x) = e^{-(1/2)(p-q)x} \cos\left(\sqrt{\frac{3(p - 2q)}{5p - 6q}}\, \frac{p - q}{2} x\right),$$
$$\text{for } (p - 2q)(5p - 6q) > 0;$$

$$f_1^{(6)}(x) = e^{-(p/4)x}, \quad f_2^{(6)}(x) = xe^{-(p/4)x}, \quad \text{for } q = \frac{1}{2}p,$$
$$f_1^{(6)}(x) = e^{-((p-q)/2)x}, \quad f_2^{(6)}(x) = e^{-((p+q)/6)x}, \quad \text{for } q \neq \frac{1}{2}p;$$

$$f_1^{(7)}(x) = e^{-(p/6)x}, \quad f_2^{(7)}(x) = xe^{-(p/6)x}, \quad \text{for } q = \frac{5}{6}p,$$
$$f_1^{(7)}(x) = e^{-(p-q)x}, \quad f_2^{(7)}(x) = e^{((2p-3q)/3)x},$$
$$\text{for } q \neq \frac{5}{6}p;$$

$$f_1^{(8)}(x) = e^{-(p/2)x}, \quad f_2^{(8)}(x) = xe^{-(p/2)x}, \quad \text{for } q = -\frac{1}{2}p,$$
$$f_1^{(8)}(x) = e^{-(p/2)x}, \quad f_2^{(8)}(x) = e^{-((p-q)/3)x}, \quad \text{for } q \neq -\frac{1}{2}p;$$

$$f_1^{(9)}(x) = e^{-((p-q)/2)x}, \quad f_2^{(9)}(x) = xe^{-((p-q)/2)x},$$
$$\text{for } pk - 3qk + 4p - 6q = 0,$$
$$f_1^{(9)}(x) = e^{(1/2)(-p+q+\sqrt{(-(p-q)(pk-3qk+4p-6q))/(k+2)})x},$$
$$f_2^{(9)}(x) = e^{-(1/2)(p-q+\sqrt{(-(p-q)(pk-3qk+4p-6q))/(k+2)})x},$$
$$\text{for } (pk - 3qk + 4p - 6q)(p - q)(k + 2) < 0,$$
$$f_1^{(9)}(x) = e^{-(1/2)(p-q)x} \sin\left(\frac{1}{2}\sqrt{\frac{(p - q)(pk - 3qk + 4p - 6q)}{k + 2}}\, x\right),$$
$$f_2^{(9)}(x) = e^{-(1/2)(p-q)x} \cos\left(\frac{1}{2}\sqrt{\frac{(p - q)(pk - 3qk + 4p - 6q)}{k + 2}}\, x\right),$$
$$\text{for } (pk - 3qk + 4p - 6q)(p - q)(k + 2) > 0;$$

$$f_1^{(10)}(x) = e^{-((p-q)/2)x}, \quad f_2^{(10)}(x) = xe^{-((p-q)/2)x},$$
$$\text{for } p - q = \frac{k + 1}{2k + 5}(p + q),$$

$$f_1^{(10)}(x) = e^{-((p-q)/2)x}, \quad f_2^{(10)}(x) = e^{-((k+1)(p+q)/(4k+10))x}, \quad \text{for } p - q \neq \frac{k+1}{2k+5}(p+q);$$

$$f_1^{(11)}(x) = e^{-(p/2)x}, \quad f_2^{(11)}(x) = xe^{-(p/2)x}, \quad \text{for } q = \frac{1}{2}p,$$

$$f_1^{(11)}(x) = e^{-(p/2)x}, \quad f_2^{(11)}(x) = e^{-(p-q)x}, \quad \text{for } q \neq \frac{1}{2}p;$$

$$f_1^{(12)}(x) = e^{-(p/3)x}, \quad f_2^{(12)}(x) = xe^{-(p/3)x}, \quad \text{for } q = \frac{2}{3}p,$$

$$f_1^{(12)}(x) = e^{-((p+q)/5)x}, \quad f_2^{(12)}(x) = e^{-((3p-2q)/5)x}, \quad \text{for } q \neq \frac{2}{3}p;$$

$$f_1^{(13)}(x) = e^{-(p-q)x}, \quad f_2^{(13)}(x) = xe^{-(p-q)x}, \quad \text{for } s = 2p - 2q,$$

$$f_1^{(13)}(x) = e^{-(p-q)x}, \quad f_2^{(13)}(x) = e^{(p-q-s)x}, \quad \text{for } s \neq 2p - 2q;$$

$$f_1^{(14)}(x) = e^{\sqrt{(s/2p)}x}, \quad f_1^{(14)}(x) = e^{-\sqrt{(s/2p)}x}, \quad \text{for } sp < 0,$$

$$f_1^{(14)}(x) = \sin\left(\sqrt{-\frac{s}{2p}}x\right), \quad f_1^{(14)}(x) = \cos\left(\sqrt{-\frac{s}{2p}}x\right), \quad \text{for } sp > 0;$$

$$f_1^{(15)}(x) = \int \frac{e^{px}}{se^{qx} - 1} dx. \tag{31}$$

When $l = 3, 4$, by similar calculation, we can derive the classification result listed in Table 3, where $f_2^{(16)}(x) = e^{-((p-q)/2)x}$, for $q \neq p$, and $f_2^{(16)}(x) = x$, for $q = p$; $f_2^{(17)}(x) = xe^{-(p-q)x}$, $f_3^{(15)}(x) = x^2 e^{-(p-q)x}$, for $(2p - 3q)(p - q) = 0$, and $f_2^{(17)}(x) = e^{(-p+q+\sqrt{-(2p-3q)(p-q)})x}$, $f_3^{17}(x) = e^{(-p+q\sqrt{-(2p-3q)(p-q)})x}$, for $(2p - 3q)(p - q) < 0$, and $f_2^{(17)}(x) = e^{-(p-q)x}\sin(\sqrt{(2p-3q)(p-q)}x)$, $f_3^{(17)}(x) = e^{-(p-q)x}\cos(\sqrt{(2p-3q)(p-q)}x)$, for $(2p - 3q)(p - q) > 0$; $f_1^{(18)}(x) = e^{-(p/3)x}$, $f_2^{(18)}(x) = xe^{-(p/3)x}$, $f_3^{(18)}(x) = x^2 e^{-(p/3)x}$, for $q = (2/3)p$, and $f_1^{(18)}(x) = e^{-(p-q)x}$, $f_2^{(18)}(x) = e^{-((3p-2q)/5)x}$, $f_3^{(18)}(x) = e^{-((p+q)/5)x}$, for $q \neq (2/3)p$.

For $l = 5$, we find that the overdetermining system is inconsistent. So, there are no CLBSs (15) and ISs (19) for (14).

4. CLBS (13) and Reductions of (4)

For (16), we can transfer the CLBS (15) of (14) into CLBS (13) of (4); that is,

$$u = f(v) = \int \frac{1}{A(v)} \exp\left[\int m \frac{B(v)}{A(v)} dv\right] dv,$$

$$P(u) = \frac{A(g(u))}{m}\left[f'(g(u))\right]^{1-m},$$
$$Q(x,u) = f'(g(u))\, C(r, g(u)), \tag{32}$$

where $v = g(u)$ is the inverse function of $u = f(v)$ as referred above. Hence, the GVSS of (4) can be derived from the GVSS (21) with the transformation $u = f(v)$. Here, the GVSS (21) is defined on IS (19) determined by the linear ODE $\eta = 0$, where η is given by CLBS (15). The coefficient functions $C_i(t)$ $(i = 1, 2, \ldots, l)$ are determined by a finite-dimensional dynamic system. For simplification, we just give some examples to illustrate our approach. Here, we pointed out that the selection of examples is random.

Example 5. Equation

$$u_t = e^{-px}\left(e^{px}u^2 u_x^{-1}\right)_x + \alpha \ln(u)\, u + \left[\beta + \gamma \ln\left(1 + e^{px}\right)\right] u \tag{33}$$

admits the CLBS

$$\sigma = [\ln(u)]_{xx} - \frac{p}{e^{px} + 1}[\ln(u)]_x. \tag{34}$$

The corresponding GVSS is given by

$$u(x,t) = e^{C_1(t) + C_2(t)\ln(1+e^{px})}, \tag{35}$$

where $C_1(t)$ and $C_2(t)$ satisfy the 2-dimensional dynamic system

$$C_1' = \frac{1}{C_2} + 1 + \alpha C_1 + \beta,$$
$$C_2' = \alpha C_2 + \gamma. \tag{36}$$

When $\alpha = 0$, we have

$$C_1 = (\beta + 1)t + \frac{\ln(\gamma t + c_2)}{\gamma} + c_1,$$
$$C_2 = \gamma t + c_2, \tag{37}$$

with arbitrary constants c_1 and c_2. If $\alpha \neq 0$, we have

$$C_1 = -\frac{\beta + 1}{\alpha} + \frac{1}{\gamma} + \frac{c_2 \alpha \ln\left(c_2 \alpha - \gamma e^{-\alpha t}\right)}{\gamma^2} + c_1,$$
$$C_2 = c_2 e^{\alpha t} - \frac{\gamma}{\alpha}, \tag{38}$$

with arbitrary constants c_1 and c_2.

Example 6. Equation

$$u_t = e^{-px}\left(e^{px}u^{-2}u_x^2\right)_x - 2s + \alpha u + \left[\beta \sin\left(\sqrt{-\frac{s}{2p}}x\right) + \gamma \cos\left(\sqrt{-\frac{s}{2p}}x\right)\right] u^{3/2} \tag{39}$$

Table 2: CLBS (15) and IS (19) with $l = 2$ of (14) for special m.

Number	Equation (14)	CLBS (15)	IS (19)
1	$v_t = e^{(p-q)x}\left[3v\left(\frac{9(p-q)}{2(4p-q)}v_x\right)^2\left(v_{xx}+\frac{p}{3}v_x\right)-\frac{2(4p-q)}{9(p-q)}\left(\frac{9(p-q)}{2(4p-q)}v_x\right)^4\right] + \frac{1}{216}(p-q)^3(4p-q)v^4e^{(p-q)x} - \frac{2(4p-q)}{9(p-q)}(\alpha v+\beta f_1^{(4)}(x)+\gamma f_2^{(4)}(x))$	$\eta = v_{xx}+\frac{7p-4q}{9}v_x+\frac{(4p-q)(p-q)}{27}v$	$W\{f_1^{(4)}(x), f_2^{(4)}(x)\}$
2	$v_t = e^{(p-q)x}\left[3v_x^2\left(v_{xx}+\frac{p}{3}v_x\right)+\frac{(p-q)^3(2p-3q)^3v^3}{(5p-6q)^2}\right]+\alpha v+\beta f_1^{(5)}(x)+\gamma f_2^{(5)}(x), \quad q\neq p$	$\eta = v_{xx}+(p-q)v_x+\frac{(2p-3q)(p-q)^2}{5p-6q}v$	$W\{f_1^{(5)}(x), f_2^{(5)}(x)\}$
3	$v_t = e^{(p-q)x}\left[3v_x^2\left(v_{xx}+\frac{p}{3}v_x\right)+\frac{(p-q)(p+q)^3v^3}{432}\right]+\alpha v+\beta f_1^{(6)}(x)+\gamma f_2^{(6)}(x)$	$\eta = v_{xx}+\frac{2p-q}{3}v_x+\frac{p^2-q^2}{12}v$	$W\{f_1^{(6)}(x), f_2^{(6)}(x)\}$
4	$v_t = e^{(p-q)x}\left[\frac{3}{v}v_x^2\left(v_{xx}+\frac{p}{3}v_x\right)+\frac{(p-q)(2p-3q)^3v^2}{9}\right]+\frac{1}{2}\left(\alpha v+\beta f_1^{(7)}(x)+\gamma f_2^{(7)}(x)\right)$	$\eta = v_{xx}+\frac{p}{3}v_x-\frac{(2p-3q)(p-q)}{3}v$	$W\{f_1^{(7)}(x), f_2^{(7)}(x)\}$
5	$v_t = \frac{3}{4}v^2v_x^2\left(v_{xx}+\frac{p}{3}v_x\right)+\frac{1}{8}vv_x^4-\frac{27}{80000}p^4v^5-2\left(\alpha v+\beta e^{(1/10)px}+\gamma e^{-(3/10)px}\right)$	$\eta = v_{xx}+\frac{p}{5}v_x-\frac{3p^2}{100}v$	$W\{e^{(1/10)px}, e^{-(3/10)px}\}$
6	$v_t = 3vv_x^2\left(v_{xx}+\frac{p}{3}v_x\right)+\frac{1}{8}vv_x^4-8\left(\frac{pv}{5}\right)^4+\alpha v+\beta e^{(1/5)px}+\gamma e^{-(2/5)px}$	$\eta = v_{xx}+\frac{p}{5}v_x-\frac{2p^2}{25}v$	$W\{e^{(1/5)px}, e^{-(2/5)px}\}$
7	$v_t = e^{-(3/13)x}\left[\frac{3}{4}vv_x^2\left(v_{xx}+\frac{p}{2}v_x\right)-\frac{8}{27}v_x^4-2\left(\frac{6v}{13}\right)^4\right]-\frac{3}{2}\left(\alpha v+\beta e^{-(9p/13)x}+\gamma e^{(6p/13)x}\right)$	$\eta = v_{xx}+\frac{3p}{13}v_x-\frac{54p^2}{169}v$	$W\{e^{-(9p/13)x}, e^{(6p/13)x}\}$
8	$v_t = e^{(p-q)x}\left[-\frac{2(13p-4q)}{9p}v^2v_x\left(v_{xx}+\frac{p}{2}v_x\right)-\frac{4(p-q)(13p-4q)}{81p^2}v_x^3\right] + \frac{(p-q)(3p-4q)p}{162}v^4e^{(p-q)x}-\frac{9p}{13p-4q}\left(\alpha v+\beta f_1^{(8)}(x)+\gamma f_2^{(8)}(x)\right)$	$\eta = v_{xx}+\frac{5p-2q}{6}v_x+\frac{p^2-pq}{6}v$	$W\{f_1^{(8)}(x), f_2^{(8)}(x)\}$
9	$v_t = e^{(p-q)x}\left[\frac{4}{k+1}vv_x\left(v_{xx}+\frac{p}{2}v_x\right)+\frac{4}{(k+1)^2}v_x^3\right] - \frac{(-4q-2qk+3p+pk)^2(p-q)}{(k+1)^2(k+2)}v^3e^{(p-q)x}+\frac{k+1}{2}\left(\alpha v+\beta f_1^{(9)}(x)+\gamma f_2^{(9)}(x)\right), \quad q\neq p$	$\eta = v_{xx}+\frac{p-q}{6}v_x+\frac{(-4q-2qk+3p+pk)(p-q)}{2(k+2)}v$	$W\{f_1^{(9)}(x), f_2^{(9)}(x)\}$

Table 2: Continued.

Number	Equation (14)	CLBS (15)	IS (19)
10	$v_t = e^{(p-q)x}\left[\frac{4}{k+1}vv_x\left(v_{xx}+\frac{p}{2}v_x\right)+\frac{4}{(k+1)^2}v_x^3\right]-\frac{(k+1)(3p+pk-qk-2q)}{2(2k+5)^3}\times(p+q)^2v^3e^{(p-q)x}+\frac{k+1}{2}\left(\alpha v+\beta f_1^{(10)}(x)+\gamma f_2^{(10)}(x)\right)$	$\eta = v_{xx}+\frac{-4q-qk+6p+3pk}{4k+10}v_x+\frac{(k+1)(p^2-q^2)}{4(2k+5)}v$	$W\left\{f_1^{(10)}(x), f_2^{(10)}(x)\right\}$
11	$v_t = e^{(p-q)x}\left[-\frac{4(3p-2q)}{9p}vv_x\left(v_{xx}+\frac{p}{2}v_x\right)+\frac{4(3p-q)^2}{81p^2}v_x^3\right]+4p\left(\frac{(3p-2q)v}{9}\right)^3e^{(p-q)x}-\frac{9p}{2(3p-2q)}\left(\alpha v+\beta e^{-(p/2)x}+\gamma e^{-((3p-2q)/6)x}\right)$	$\eta = v_{xx}+(p-\frac{q}{3})v_x+\frac{p(3p-2q)}{12}v$	$W\left\{e^{-(p/2)x}, e^{-((3p-2q)/6)x}\right\}$
12	$v_t = 2e^{(p-q)x}v_x\left(v_{xx}+\frac{p}{2}v_x\right)+\alpha v+\beta f_1^{(11)}(x)+\gamma f_2^{(11)}(x)$	$\eta = v_{xx}+\left(\frac{3p}{2}-q\right)v_x+\frac{p^2-pq}{2}v$	$W\left\{f_1^{(11)}(x), f_2^{(11)}(x)\right\}$
13	$v_t = e^{(p-q)x}\left[2v_x\left(v_{xx}+\frac{p}{2}v_x\right)-\frac{1}{125}(3p-2q)(p+q)^2v^2\right]+\alpha v+\beta f_1^{(12)}+\gamma f_2^{(12)}$	$\eta = v_{xx}+\frac{4p-q}{5}v_x+\frac{1}{25}(p+q)(3p-2q)v$	$W\left\{f_1^{(12)}(x), f_2^{(12)}(x)\right\}$
14	$v_t = e^{(p-q)x}\left[2v_x\left(v_{xx}+\frac{p}{2}v_x\right)-(3p-2q-2s)\times(p-q-s)^2v^2\right]+\alpha v+\beta f_1^{(13)}(x)+\gamma f_2^{(13)}(x)$	$\eta = v_{xx}+sv_x-(p-q)(p-q-s)v$	$W\left\{f_1^{(13)}(x), f_2^{(13)}(x)\right\}$
15	$v_t = e^{(p-q)x}\left[-\frac{4(p-q)}{2p-q}vv_x\left(v_{xx}+\frac{p}{2}v_x\right)+\frac{4(p-q)^2}{(2p-q)^2}v_x^3\right]-\frac{2p-q}{2(p-q)}\left(\alpha v+\beta e^{-((p-q)/2)x}+\gamma e^{-((2p-q)/2)x}\right)$	$\eta = v_{xx}+\frac{3p-2q}{2}v_x+\frac{1}{25}(p+q)(3p-2q)v$	$W\left\{e^{-((p-q)/2)x}, e^{-((2p-q)/2)x}\right\}$
16	$v_t = -4vv_x\left(v_{xx}+\frac{p}{2}v_x\right)+4v_x^3+sv^3-\frac{1}{2}\left(\alpha v+\beta f_1^{(14)}(x)+\gamma f_2^{(14)}(x)\right), \quad s\neq 0$	$\eta = v_{xx}-\frac{s}{2p}v$	$W\left\{f_1^{(14)}(x), f_2^{(14)}(x)\right\}$
17	$v_t = -e^{(p-q)x}v^2v_x^{-2}(v_{xx}-pv_x)-sqf_1^{(15)}(x)v+\alpha v+\beta+\gamma f_1^{(15)}(x)$	$\eta = v_{xx}+\frac{p-s(p-q)e^{qx}}{se^{qx}-1}v_x$	$W\left\{f_1^{(15)}(x), 1\right\}$

Table 2: Continued.

Number	Equation (14)	CLBS (15)	IS (19)
18	$v_t = -e^{(p-q)x} v^2 v_x^{-2} \left(v_{xx} - p v_x\right) - \dfrac{q e^{(p-q)x}}{p-q} v + \alpha v + \beta + \gamma e^{(p-q)x}$	$\eta = v_{xx} + (-p+q) v_x$	$W\left\{e^{(p-q)x}, 1\right\}$
19	$v_t = -v^2 v_x^{-2} \left(v_{xx} - p v_x\right) - p x v + \alpha v + \beta + \gamma x$	$\eta = v_{xx}$	$W\{x, 1\}$
20	$v_t = -e^{(p-q)x} v v_x^{-2} \left(v_{xx} - p v_x\right) + \alpha v + \beta + \gamma f_1^{(15)}(x)$	$\eta = v_{xx} + \dfrac{p - s(p-q) e^{qx}}{s e^{qx} - 1} v_x$	$W\left\{f_1^{(15)}(x), 1\right\}$
21	$v_t = -e^{(p-q)x} v_x^{-2} \left(v_{xx} - p v_x\right) + \alpha v + \beta + \gamma f_1^{(15)}(x)$	$\eta = v_{xx} + \dfrac{p - s(p-q) e^{qx}}{s e^{qx} - 1} v_x$	$W\left\{f_1^{(15)}(x), 1\right\}$
22	$v_t = e^{(p-q)x} \left[-v v_x^{-2} \left(v_{xx} - p v_x\right) - \dfrac{p+q}{p-q}\right] + \alpha v + \beta + \gamma e^{-((p-q)/2)x}, \quad q \neq p$	$\eta = v_{xx} - \dfrac{p-q}{2} v_x$	$W\left\{e^{-((p-q)/2)x}, 1\right\}$
23	$v_t = e^{(p-q)x} \left[-2 v v_x^{-3} \left(v_{xx} - \dfrac{p}{2} v_x\right) + (v+1) v_x^{-1}\right] - \dfrac{2}{p-q} e^{(p-q)x} + \alpha v + \beta + \gamma e^{((p-q)/2)x}, \quad q \neq p$	$\eta = v_{xx} - \dfrac{p-q}{2} v_x$	$W\left\{e^{((p-q)/2)x}, 1\right\}$
24	$v_t = e^{(p-q)x} \left[-2(k-3)^3 v^2 v_x^{-3} \left(v_{xx} - \dfrac{p}{2} v_x\right) - 2(k-3)^2 v v_x^{-1}\right] + \left(\dfrac{4(k-2)^2}{p-q} - \dfrac{4q(k-3)^3}{(p-q)^2}\right) e^{(p-q)x} + (k-3)\left(\alpha v + \beta + \gamma e^{((p-q)/2)x}\right), \quad q \neq p$	$\eta = v_{xx} - \dfrac{p-q}{2} v_x$	$W\left\{e^{((p-q)/2)x}, 1\right\}$
25	$v_t = e^{(p-q)x} \left[2\left(\dfrac{p-q}{3(p+q)}\right)^3 v^2 v_x^{-3} \left(v_{xx} - \dfrac{p}{2} v_x\right) + \dfrac{2(p-q)^2 (p+2q)}{27(p+q)^3} v_x^{-1}\right] - \dfrac{(p-q)(p+2q)}{9(p+q)^3} + \dfrac{p-q}{p+2q}\left(\alpha v + \beta + \gamma e^{((p-q)/3)x}\right), \quad q \neq p$	$\eta = v_{xx} - \dfrac{p-q}{3} v_x$	$W\left\{e^{((p-q)/3)x}, 1\right\}$
26	$v_t = e^{(p-q)x} \left[2\left(\dfrac{p-q}{2(p+q)}\right)^3 v^3 v_x^{-3} \left(v_{xx} - \dfrac{p}{2} v_x\right) + \dfrac{q(p-q)^2}{2(p+q)^3} v^2 v_x^{-1}\right] - \dfrac{(p-q) q}{2(p+q)^3} v e^{(p-q)x} - \dfrac{p-q}{2p+2q}\left(\alpha v + \beta + \gamma e^{((p-q)/2)x}\right), \quad q \neq p$	$\eta = v_{xx} - \dfrac{p-q}{2} v_x$	$W\left\{e^{((p-q)/2)x}, 1\right\}$

Table 2: Continued.

Number	Equation (14)	CLBS (15)	IS (19)
27	$v_t = e^{(p/2)x}\left[2v^3 v_x^{-3}\left(v_{xx} - \frac{p}{2}v_x\right) - v^2 v_x^{-1} + \frac{2}{p}v\right] - \left(\alpha v + \beta + \gamma e^{(p/2)x}\right)$	$\eta = v_{xx} - \frac{p}{2}v_x$	$W\left\{e^{(p/2)x}, 1\right\}$
28	$v_t = e^{(p-q)x}\left[-3v^3\left(\frac{2(p-q)}{(4p+5q)v_x}\right)^4\left(v_{xx} - \frac{p}{3}v_x\right) - \frac{72q(p-q)^3}{(4p+5q)^4}v^2 v_x^{-2}\right] + \frac{216q(p-q)}{(4p+5q)^4}e^{(p-q)x} - \frac{2(p-q)}{4p+5q}\left(\alpha v + \beta + \gamma e^{((p-q)/3)x}\right), \quad q \neq p$	$\eta = v_{xx} - \frac{p-q}{3}v_x$	$W\left\{e^{((p-q)/3)x}, 1\right\}$
29	$v_t = e^{(2p/3)x}\left[-3(k-4)^4 v^3 v_x^{-4}\left(v_{xx} - \frac{p}{3}v_x\right) - (2k-9)(k-4)^3 v^2 v_x^{-2}\right] - \frac{9(2k-9)(k-4)^3}{p^2}e^{(2p/3)x} - (k-4)\left(\alpha v + \beta + \gamma e^{(p/3)x}\right)$	$\eta = v_{xx} - \frac{p}{3}v_x$	$W\left\{e^{(p/3)x}, 1\right\}$

TABLE 3: CLBS (15) and IS (19) with $l = 3, 4$ of (14).

Number	Equation (14)	CLBS (15)	IS (19)
1	$v_t = 2e^{(p-q)x} v_x \left(v_{xx} + \frac{p}{2} v_x\right) + \alpha v + \beta + \gamma e^{-(p/2)x} + \lambda \left(f_2^{(16)}(x)\right)^2$	$\eta = v_{xxx} + \frac{3p-2q}{2} v_{xx} + \frac{p^2 - pq}{2} v_x$	$W\left\{e^{-(p/2)x}, (f_2^{(16)}(x))^2, 1\right\}$
2	$v_t = 2e^{(p-q)x} v_x \left(v_{xx} + \frac{p}{2} v_x\right) + \alpha v + \beta + \gamma \left(f_2^{(16)}(x)\right)^2 + \lambda f_2^{(16)}(x)$	$\eta = v_{xxx} + \frac{3p-3q}{2} v_{xx} + \frac{(p-q)^2}{2} v_x$	$W\left\{(f_2^{(14)}(x))^2, f_2^{(16)}(x), 1\right\}$
3	$v_t = 2e^{(p-q)x} v_x \left(v_{xx} + \frac{p}{2} v_x\right) - (p-q)(3p-4q)^2 v^2 e^{(p-q)x}$ $+ \alpha v + \beta e^{-(p-q)x} + \gamma f_2^{(17)}(x) + \lambda f_3^{(17)}(x)$	$\eta = v_{xxx} + 3(p-q) v_{xx} + (5p-6q)(p-q) v_x$ $+ (3p-4q)(p-q)^2 v$	$W\left\{e^{-(p-q)x}, f_2^{(17)}(x), f_3^{(17)}(x)\right\}$
4	$v_t = 2e^{(p-q)x} v_x \left(v_{xx} + \frac{p}{2} v_x\right) - \frac{1}{125}(3p-2q)(p+q)^2 v^2 e^{(p-q)x}$ $+ \alpha v + \beta f_1^{(18)}(x) + \gamma f_2^{(18)}(x) + \lambda f_3^{(18)}(x)$	$\eta = v_{xxx} + \frac{9p-6q}{5} v_{xx} + \frac{23p^2 - 24pq + 3q^2}{25} v_x$ $+ \frac{1}{25}(p-q)(3p-2q)(p+q) v$	$W\left\{f_1^{(18)}(x), f_2^{(18)}(x), f_3^{(18)}(x)\right\}$
5	$v_t = 2e^{(4p/5)x} v_x \left(v_{xx} + \frac{p}{2} v_x\right) + \alpha v + \beta + \gamma e^{-(2p/5)x} + \lambda e^{-(3p/5)x}$	$\eta = v_{xxx} + p v_{xx} + \frac{6p^2}{25} v_x$	$W\left\{e^{-(2p/5)x}, e^{-(3p/5)x}, 1\right\}$
6	$v_t = e^{(3p/2)x} \left[-2v v_x \left(v_{xx} + \frac{p}{2} v_x\right) + v_x^3\right] - \left(\alpha v + \beta + \gamma e^{-(p/2)x} + \lambda e^{-px}\right)$	$\eta = v_{xxx} + \frac{3p}{2} v_{xx} + \frac{p^2}{2} v_x$	$W\left\{e^{-(p/2)x}, e^{-px}, 1\right\}$
7	$v_t = 2e^{(3p/2)x} v_x \left(v_{xx} + \frac{p}{2} v_x\right) + \alpha v + \beta + \gamma e^{-(p/2)x} + \lambda e^{-px}$	$\eta = v_{xxx} + \frac{3p}{2} v_{xx} + \frac{p^2}{2} v_x$	$W\left\{e^{-(p/2)x}, e^{-px}, 1\right\}$
8	$v_t = 3e^{(p/2)x} v_x^2 \left(v_{xx} + \frac{p}{3} v_x\right) + \alpha v + \beta + \gamma e^{-(p/3)x} + \lambda e^{-(p/6)x}$	$\eta = v_{xxx} + \frac{p}{2} v_{xx} + \frac{p^2}{18} v_x$	$W\left\{e^{-(p/3)x}, e^{-(p/6)x}, 1\right\}$

TABLE 3: Continued.

Number	Equation (14)	CLBS (15)	IS (19)
9	$v_t = 3e^{(4p/3)x}v_x^2\left(v_{xx} + \frac{p}{3}v_x\right) + \alpha v + \beta + \gamma e^{-(p/3)x} + \lambda e^{-(2p/3)x}$	$\eta = v_{xxx} + pv_{xx} + \frac{2p^2}{9}v_x$	$W\left\{e^{-(p/3)x}, e^{-(2p/3)x}, 1\right\}$
10	$v_t = e^{(4p/3)x}\left[\frac{27}{4}vv_x^2\left(v_{xx} + \frac{p}{3}v_x\right) - \frac{27}{8}v_x^4\right] - \frac{2}{3}\left(\alpha v + \beta + \gamma e^{-(p/3)x}\lambda e^{-(2p/3)x}\right)$	$\eta = v_{xxx} + pv_{xx} + \frac{2p^2}{9}v_x$	$W\left\{e^{-(p/3)x}, e^{-(2p/3)x}, 1\right\}$
11	$v_t = 3e^{(6p/7)x}v_x^2\left(v_{xx} + \frac{p}{3}v_x\right) + \alpha v + \beta + \gamma e^{-(2p/7)x} + \lambda e^{-(5p/7)x}$	$\eta = v_{xxx} + \frac{5p}{7}v_{xx} + \frac{6p^2}{49}v_x$	$W\left\{e^{-(2p/7)x}, e^{-(5p/7)x}, 1\right\}$
12	$v_t = 2e^{(3p/2)x}v_x\left(v_{xx} + \frac{p}{2}v_x\right) + \alpha v + \beta + \gamma e^{-px} + \lambda e^{-(p/2)x} + \mu e^{-(p/3)x}$	$\eta = v_{xxxx} + 3pv_{xxx} + \frac{11p^2}{4}v_{xx} + \frac{3p^3}{4}v_x$	$W\left\{e^{-px}, e^{-(p/2)x}, e^{-(p/3)x}, 1\right\}$
13	$v_t = 2e^{(p/3)x}v_x\left(v_{xx} + \frac{p}{2}v_x\right) + \alpha v + \beta + \gamma e^{-(p/2)x} + \lambda e^{-(p/3)x} + \mu e^{-(p/6)x}$	$\eta = v_{xxxx} + pv_{xxx} + \frac{11p^2}{36}v_{xx} + \frac{p^3}{36}v_x$	$W\left\{e^{-(p/2)x}, e^{-(p/3)x}, e^{-(p/6)x}, +1\right\}$
14	$v_t = 2e^{(4p/5)x}v_x\left(v_{xx} + \frac{p}{2}v_x\right) + \alpha v + \beta + \gamma e^{-(2p/5)x} + \lambda e^{-(3p/5)x} + \mu e^{-(4p/5)x}$	$\eta = v_{xxxx} + \frac{9p}{5}v_{xxx} + \frac{26p^2}{25}v_{xx} + \frac{24p^3}{125}v_x$	$W\left\{e^{-(2p/5)x}, e^{-(3p/5)x}, e^{-(4p/5)x}, 1\right\}$

with $s \neq 0$ admits the CLBS

$$\sigma = \left(u^{-1/2}\right)_{xx} - \frac{s}{2p} u^{-1/2}. \tag{40}$$

The corresponding GVSS is given by

$$u(x,t) = \left[C_1(t)\sin\left(\sqrt{-\frac{s}{2p}}x\right) + C_2(t)\cos\left(\sqrt{-\frac{s}{2p}}x\right)\right]^{-2}, \tag{41}$$

where $C_1(t)$ and $C_2(t)$ satisfy the 2-dimensional dynamic system

$$\begin{aligned} C_1' &= \left(C_1^2 + C_2^2\right)\left[-4C_2\left(-\frac{s}{2p}\right)^{3/2} + sC_1\right] - \frac{1}{2}\left(\alpha C_1 + \beta\right), \\ C_2' &= \left(C_1^2 + C_2^2\right)\left[4C_1\left(-\frac{s}{2p}\right)^{3/2} + sC_2\right] - \frac{1}{2}\left(\alpha C_2 + \gamma\right). \end{aligned} \tag{42}$$

Example 7. Equation

$$\begin{aligned} u_t &= e^{-qx}\left(e^{px}u^3u_x^{-2}\right)_x + \frac{q}{(p+q)^2}ue^{(p-q)x} + \alpha u \\ &\quad + \left(\beta + \gamma e^{((p-q)/2)x}\right)u^{(3p+q)/2(p+q)} \end{aligned} \tag{43}$$

with $p \neq q, -q$ admits the CLBS

$$\sigma = \left(u^{-((p-q)/2(p+q))}\right)_{xx} - \frac{p-q}{2}\left(u^{-((p-q)/2(p+q))}\right)_x. \tag{44}$$

The corresponding GVSS is given by

$$u(x,t) = \left[C_1(t) + C_2(t)e^{((p-q)/2)x}\right]^{-(2(p+q)/(p-q))}, \tag{45}$$

where $C_1(t)$ and $C_2(t)$ satisfy the 2-dimensional dynamic system

$$\begin{aligned} C_1' &= -\frac{q(p-q)C_1^3}{2(p+q)^3C_2^2} - \frac{p-q}{2(p+q)}\left(\alpha C_1 + \beta\right), \\ C_2' &= -\frac{q(p-q)C_1^2}{2(p+q)^3C_2} - \frac{p-q}{2(p+q)}\left(\alpha C_2 + \gamma\right). \end{aligned} \tag{46}$$

Example 8. Equation

$$u_t = e^{-(2p/3)x}\left(e^{px}u_x^2\right)_x - \frac{p^3}{27}e^{(p/3)x}u^2 \tag{47}$$

admits the CLBS

$$\sigma = u_{xxx} + pu_{xx} + \frac{p^2}{3}u_x + \frac{p^3}{27}u. \tag{48}$$

The corresponding GVSS is given by

$$u(x,t) = \left[C_1(t) + C_2(t)x + C_3(t)x^2\right]e^{-(p/3)x}, \tag{49}$$

where $C_1(t)$, $C_2(t)$, and $C_3(t)$ satisfy the 3-dimensional dynamic system

$$\begin{aligned} C_1' &= -\frac{1}{3}\left(4pC_1C_3 + pC_2^2 - 12C_2C_3\right), \\ C_2' &= -\frac{1}{3}\left(8pC_2C_3 - 24C_3^2\right), \\ C_3' &= -\frac{8pC_3^2}{3}. \end{aligned} \tag{50}$$

This system has the following exact solution:

$$\begin{aligned} C_1 &= \frac{1}{12}\frac{c_2^2p^2 + 12c_1p^2\sqrt{8pt+c_3} + (6561 + 18c_2p)\ln\left[8pt+c_3\right]}{p^2(8pt+c_3)}, \\ C_2 &= \frac{pc_2 + 9\ln\left[8pt + c_3\right]}{p(8pt + c_3)}, \\ C_3 &= \frac{3}{8pt + c_3}. \end{aligned} \tag{51}$$

Example 9. Equation

$$u_t = e^{(p/2)x}\left(e^{px}u^{-3}u_x^2\right)_x + \alpha u + \left(\beta + \gamma e^{-(p/2)x} + \lambda e^{-px}\right)u^2 \tag{52}$$

admits the CLBS

$$\sigma = \left(\frac{1}{u}\right)_{xxx} + \frac{3p}{2}\left(\frac{1}{u}\right)_{xx} + \frac{p^2}{2}\left(\frac{1}{u}\right)_x. \tag{53}$$

The corresponding GVSS is given by

$$u(x,t) = \frac{1}{C_1(t) + C_2(t)e^{-(p/2)x} + C_3(t)e^{-px}}, \tag{54}$$

where $C_1(t)$, $C_2(t)$, and $C_3(t)$ satisfy the 3-dimensional dynamic system

$$\begin{aligned} C_1' &= -\frac{p^3}{8}C_2^3 + \frac{p^3}{2}C_1C_2C_3 - \left(\alpha C_1 + \beta\right), \\ C_2' &= -\frac{p^3}{4}C_2^2C_3 + p^3C_1C_3^2 - \left(\alpha C_2 + \gamma\right), \\ C_3' &= -\left(\alpha C_3 + \lambda\right). \end{aligned} \tag{55}$$

Example 10. Equation

$$\begin{aligned} u_t &= e^{(p/3)x}\left(e^{px}u^{-4}u_x^3\right)_x + \alpha u \\ &\quad + \left(\beta + \gamma e^{-(p/3)x} + \lambda e^{-(2p/3)x}\right)u^{5/3} \end{aligned} \tag{56}$$

admits the CLBS

$$\sigma = \left(u^{-2/3}\right)_{xxx} + p\left(u^{-2/3}\right)_{xx} + \frac{2p^2}{9}\left(u^{-2/3}\right)_x. \tag{57}$$

The corresponding GVSS is given by

$$u(x,t) = \left[C_1(t) + C_2(t)e^{-(p/3)x} + C_3(t)e^{-(2p/3)x}\right]^{-3/2}, \tag{58}$$

where $C_1(t)$, $C_2(t)$, and $C_3(t)$ satisfy the 3-dimensional dynamic system

$$\begin{aligned} C_1' &= \frac{1}{24}p^4C_2^2\left(4C_1C_3 - C_2^2\right) - \frac{2}{3}\left(\alpha C_1 + \beta\right), \\ C_2' &= \frac{1}{6}p^4C_2C_3\left(4C_1C_3 - C_2^2\right) - \frac{2}{3}\left(\alpha C_2 + \gamma\right), \\ C_3' &= \frac{1}{6}p^4C_3^2\left(4C_1C_3 - C_2^2\right) - \frac{2}{3}\left(\alpha C_3 + \lambda\right). \end{aligned} \tag{59}$$

Example 11. Equation

$$u_t = e^{(p/2)x}\left(e^{px}u_x^2\right)_x + \alpha u + \beta + \gamma e^{-px} + \lambda e^{-(p/2)x} + \mu e^{-(3p/2)x} \tag{60}$$

admits the CLBS

$$\sigma = u_{xxxx} + 3pu_{xxx} + \frac{11p^2}{4}u_{xx} + \frac{3p^3}{4}u_x. \tag{61}$$

The corresponding GVSS is given by

$$u(x,t) = C_1(t) + C_2(t)e^{-px} + C_3(t)e^{-(p/2)x} + C_4(t)e^{-(3p/2)x}, \tag{62}$$

where $C_1(t)$, $C_2(t)$, $C_3(t)$, and $C_4(t)$ satisfy the following 4-dimensional dynamic system:

$$\begin{aligned} C_1' &= -\frac{p^3}{2}C_2C_3 + \alpha C_1 + \beta, \\ C_2' &= -\frac{9p^3}{2}C_2C_4 + \alpha C_2 + \gamma, \\ C_3' &= -p^3C_2^2 - \frac{3p^3}{2}C_3C_4 + \alpha C_3 + \lambda, \\ C_4' &= -\frac{9p^3}{2}C_4^2 + \alpha C_4 + \mu. \end{aligned} \tag{63}$$

5. Conclusions and Discussions

In this paper, we have applied the CLBS method with ISs to study (4). The transformed equations (14) admitting CLBSs (15) are listed in Tables 1, 2, and 3. The corresponding reduced equations of the resulting equations are finite-dimensional dynamic systems defined on W_l. Some concrete examples are illustrated in Section 4. Generally speaking, these reductions cannot be obtained in the frameworks within the Lie point symmetry method and the nonclassical symmetry method. Of course, we mention that the asymptotical behavior, blow-up, extinction, and geometric properties for these finite-dimensional dynamic systems are worthy of further study.

For NLDEs, we find that the CLBS method plays a key role in the study of their asymptotical behavior, blow-up, extinction, and geometric properties because of the diversity of solutions obtained by this method. Although this is an effective and complete method, there are still some important problems to be studied. How to study higher-dimensional NLDEs and systems via the CLBS method? How to deal with the initial value problems by means of the CLBS method? Is it possible to apply the CLBS method to other types of evolution equations?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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