Research Article

Generalized Composition Operators from $B_\mu$ Spaces to $Q_{K,\omega}(p, q)$ Spaces

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Let $0 < p < \infty$, let $−2 < q < \infty$, and let $\varphi$ be an analytic self-map of $D$ and $g \in H(D)$. The boundedness and compactness of generalized composition operators $(C_\varphi f) (z) = \int_0^z f'(\varphi(\xi)) g(\xi) \, d\xi$, $z \in D$, $f \in H(D)$, from $B_\mu$ spaces to $Q_{K,\omega}(p, q)$ spaces are investigated.

1. Introduction and Preliminaries

Let $\varphi$ be an analytic self-map of the open unit disc $D$ of the complex plane $\mathbb{C}$. Let $H(D)$ be the space of all analytic functions in $D$ and $g \in H(D)$. If $X$ is a Banach space, then we denote the unit ball in $X$ by $B_X$. For $0 < r < 1$, $\Omega_r = \{z \in D : |\varphi(z)| > r\}$.

A positive continuous function $\mu$ on the interval $[0, 1)$ is called normal if there exist three constants $0 \leq \delta < 1$ and $0 < a < b$ such that

\[
\frac{\mu(r)}{1-r^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{r \to 1} \frac{\mu(r)}{1-r^a} = 0;
\]

\[
\frac{\mu(r)}{1-r^b} \text{ is increasing on } [\delta, 1), \quad \lim_{r \to 1} \frac{\mu(r)}{1-r^b} = \infty.
\]

(1)

A function $f \in H(D)$ belongs to the Bloch type space $\mathcal{B}_\mu$ if

\[
\|f\|_{\mathcal{B}_\mu} = \left| f(0) \right| + \sup_{z \in D} \mu(|z|) \left| f'(z) \right| < \infty,
\]

(2)

where $\mu$ is normal and radial and $\mu(|z|) = \mu(z)$. The space $\mathcal{B}_\mu$ is a Banach space with the norm $\|f\|_{\mathcal{B}_\mu}$.

The little Bloch type space $\mathcal{B}_{\mu,0}$ consists of all $f \in \mathcal{B}_\mu$ such that

\[
\lim_{|z| \to 1^{-}} \mu(|z|) \left| f'(z) \right| = 0.
\]

(3)

For $\alpha > 0$, $\mu(\alpha) = (1 - |z|^2)\alpha$, $\mathcal{B}_\mu$ is the $\alpha$-Bloch space $\mathcal{B}_\alpha$; for $\alpha = 1$, $\mathcal{B}_1$ is the classical Bloch space; for example, see [1].

For $0 < p < \infty$, $−2 < q < \infty$, $a \in D$, $K : [0, \infty) \to [0, \infty)$ is a nondecreasing function, and $\omega : (0, 1) \to (0, \infty)$ is a given reasonable function. An analytic function $f$ on $D$ is said to belong to $Q_{K,\omega}(p, q)$ in [2] if

\[
\left\| f \right\|_{Q_{K,\omega}(p, q)} = \left\{ \sup_{a \in D} \int_D \left| f'(z) \right|^p \left( 1 - |z|^2 \right)^q K(g(z,a)) \omega^p \left( 1 - |z| \right) dA(z) \right\}^{1/p} < \infty
\]

(4)

and an analytic function $f \in Q_{K,\omega,0}(p, q)$ if

\[
\lim_{|z| \to 1^{-}} \int_D \left| f'(z) \right|^p \left( 1 - |z|^2 \right)^q K(g(z,a)) \omega^p \left( 1 - |z| \right) dA(z) = 0,
\]

(5)

where $dA$ denotes the normalized Lebesgue area measure on $D$, $g(z,a) = \log(1/|\phi_a(z)|)$ is a green function, and $\phi_a(z) = (a - z)/(1-\overline{a}z)$.

$Q_{K,\omega}(p, q)$ classes are more general than many classes of analytic functions and have attracted a lot of attention in recent years. When $\omega \equiv 1$, $Q_{K,\omega}(p, q) = Q_K(p, q)$. When $p = q = 2$, $\omega(t) = t$, $K(t) = t^p$, and $Q_{K,\omega}(p, q) = Q_p$. When
\(\omega \equiv 1, K(t) = t^r\) and \(Q_{K,\omega}(p, q) = F(p, q, s)\). Moreover, the following results hold:

1. \(Q_{K,\omega}(p, q) \in B^p_{\omega}^{(p+2)/p}\);
2. \(Q_{K,\omega}(p, q) = B^p_{\omega}^{(p+2)/p}\) if and only if

\[
\int_0^1 K \left( \log \frac{1}{r} \right) \frac{r}{(1-r^2)} dr < \infty, \quad (6)
\]

where

\[
B^p_{\omega} = \left\{ f \in H(D) : \|f\|_{B^p_{\omega}} = \sup_{z \in D} \omega(1-|z|)^{\alpha} |f'(z)| < \infty, 0 < \alpha < \infty \right\}. \quad (7)
\]

The composition operator is defined by \(C_{\varphi} f(z) = f(\varphi(z)), f \in H(D)\). This operator has been studied for many years. The first setting was in the Hardy space \(H^2\), the space of functions analytic on \(D\) (see [3]). Madigan and Matheson (see [1]) gave a characterization of the compact composition operators on the Bloch space \(B\). For more details, see [4–12]. In [13], Li and Stević defined the generalized composition operator as follows:

\[
\left( C_{\varphi} f \right)(z) = \int_0^z f'(\xi) g(\xi) \ d\xi, \quad z \in D, \ f \in H(D). \quad (8)
\]

The operator \(C_{\varphi} g\) induces many known operators. When \(g = \varphi',\) the operator \(C_{\varphi} g\) is essentially (up to a constant) the composition operator \(C_{\varphi}\). When \(\varphi(z) = z,\) the operator \(C_{\varphi} g\) coincides with the operator \(I_g\) defined by

\[
\left( I_g f \right)(z) = \int_0^z f'(\xi) g(\xi) \ d\xi, \quad z \in D, \ f \in H(D). \quad (9)
\]

So the generalized composition operator \(C_{\varphi} g\) can be considered as a generalization of the composition operator \(C_{\varphi}\) and the operator \(I_g\).

A fundamental problem in the study of generalized composition operators \(C_{\varphi} g\) is to investigate the relations between function theoretic properties of \(\varphi\) and \(g\) and operator theoretic properties of the restriction of \(C_{\varphi} g\) to various Banach spaces of analytic functions. A lot of attention has been attracted to study the problem on many Banach spaces of analytic functions in recent years. In [9], the authors studied composition operators from Bloch type spaces into \(T_K(p, q)\) spaces. In [14], the authors characterized the boundedness and compactness of generalized composition operators on \(Q_{K,\omega}(p, q)\) spaces. In [15], Rezaei and Mahyar studied generalized composition operators from logarithmic Bloch type spaces to \(Q_K\) type spaces. In [16], essential norms of generalized composition operators from Bloch type spaces to \(Q_K\) type spaces were given. In [17], generalized composition operators from \(F(p, q, s)\) spaces to Bloch-type spaces were characterized. In [18], Stević investigated generalized composition operators between mixed-norm space and some weighted spaces and from logarithmic Bloch spaces to mixed-norm spaces. In [3], Zhang and Liu studied generalized composition operators from Bloch type spaces to \(Q_K\) type spaces. In [19], generalized composition operator acting from Bloch-type spaces to mixed-norm space was studied. In [12], generalized composition operators from generalized weighted Bergman spaces to Bloch type spaces were investigated. In [20], generalized composition operators and Volterra composition operators on Bloch spaces on the unit ball were studied. This paper is devoted to investigating the boundedness and compactness of generalized composition operators \(C_{\varphi} g\) from \(\mathcal{B}_\mu(\mathcal{B}_{\mu,0})\) spaces to \(Q_{K,\omega}(p, q)\) spaces. Throughout this paper, constants are denoted by \(C;\) they are positive and may differ from one occurrence to the other.

## 2. Main Results and Their Proofs

To derive our results, we need the following lemmas.

**Lemma 1.** Assume that \(0 < p < \infty, -2 < q < \infty, K\) is a nonnegative nondecreasing function on \([0, \infty),\) and \(\omega : (0,1] \to (0,\infty)\) is a given reasonable function. Assume that \(\mu\) is a normal function, \(\varphi\) is an analytic self-map of \(D,\) and \(g \in H(D)\). Then \(C_{\varphi} g : \mathcal{B}_\mu(\mathcal{B}_{\mu,0}) \to Q_{K,\omega}(p, q)\) is compact if and only if, for every bounded sequence \(\{f_k\}\) in \(\mathcal{B}_\mu(\mathcal{B}_{\mu,0})\) which converges to 0 uniformly on compact subsets of \(D,\) \(\lim_{k \to \infty} \|C_{\varphi} g f_k\|_{Q_{K,\omega}(p, q)} = 0.\)

**Lemma 2.** Let \(\mu : [0,1) \to [0,\infty)\) be a nonincreasing radial weight function and normal on the interval \([0,1).\) Then there exist two functions \(f_1, f_2 \in \mathcal{B}_\mu\) such that, for each \(z \in D,\)

\[
|f_1'(z)| + |f_2'(z)| \geq \frac{c}{\mu(|z|)}. \quad (10)
\]

**Theorem 3.** Assume that \(0 < p < \infty, -2 < q < \infty, \varphi\) is an analytic self-map of \(D,\) \(\mu\) is a normal function, \(K\) is nonnegative and nondecreasing in \([0,\infty),\) and \(\omega : (0,1] \to (0,\infty)\) is a given reasonable function. Then the following statements are equivalent:

(a) \(C_{\varphi} g : \mathcal{B}_\mu \to Q_{K,\omega}(p, q)\) is bounded;
(b) \(C_{\varphi} g : \mathcal{B}_{\mu,0} \to Q_{K,\omega}(p, q)\) is bounded;
(c) \(\sup_{a \in D} \int_0^1 \left| g(z) \right|^p (1-|z|^2)^q K(g(z, a)) \frac{dA(z)}{\mu^p(\varphi(z)) \omega^q(1-|z|)} dA(z) < \infty. \quad (11)\)
Theorem 4. Assume that \( 0 < p < \infty, \quad 0 < q < \infty, \quad \phi \) is an analytic self-map of \( \mathbb{D}, \mu \) is a normal function, \( K \) is nonnegative and nondecreasing in \([0, \infty)\), and \( \omega : (0, 1) \to (0, \infty) \) is a given reasonable function. Then the following statements are equivalent:

(a) \( C^g_\phi : \mathcal{B}_\mu \to \mathcal{Q}_{K,\omega}(p, q) \) is compact;
(b) \( C^g_\phi : \mathcal{B}_{\mu,0} \to \mathcal{Q}_{K,\omega}(p, q) \) is compact;
(c) \( M = \sup_{a \in \mathbb{D}} \left| \frac{g(z)}{\omega (1 - |z|)} \right| K (g(z,a)) dA(z) < \infty \),

\[ M = \sup_{a \in \mathbb{D}} \left| \frac{g(z)}{\omega (1 - |z|)} \right| K (g(z,a)) dA(z) < \infty, \quad \ldots \]

\[ \lim_{r \to 1} \sup_{a \in \mathbb{D}} \left| \frac{g(z)}{\omega (1 - |z|)} \right| K (g(z,a)) dA(z) = 0. \]

Proof. (a) \( \Rightarrow \) (b) Since \( \mathcal{B}_{\mu,0} \subset \mathcal{B}_\mu \), then (a) implies (b).
(b) \( \Rightarrow \) (c) Assume that (b) holds; then we have (14). Let

\[ f_n(z) = \frac{z^n}{n\mu(1-(1/n))}, \quad z \in \mathbb{D}. \]

Then \( f_n(z) \) is bounded in \( \mathcal{B}_{\mu,0} \) and \( f_n \to 0 \) uniformly on the compact subsets of \( \mathbb{D} \) as \( n \to \infty \). Since \( C^g_\phi : \mathcal{B}_{\mu,0} \to \mathcal{Q}_{K,\omega}(p, q) \) is compact, then by Lemma 1

\[ \lim_{n \to \infty} \| C^g_\phi \ f_n \|_{Q_{K,\omega}(p, q)} = 0. \]

This means, for any given \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies

\[ \sup_{a \in \mathbb{D}} \int_{\Omega_n} \left| \frac{\varphi^{N-1} (z)}{\omega (1 - |z|)} \right| \left| \frac{g(z)}{\omega (1 - |z|)} \right| K (g(z,a)) dA(z) < \varepsilon. \]

Hence, for \( 0 < r < 1 \),

\[ \sup_{a \in \mathbb{D}} \mu^p (1 - (1/r)) \int_{\mathbb{D}} \left| \frac{\varphi^{N-1} (z)}{\omega (1 - |z|)} \right| \left| \frac{g(z)}{\omega (1 - |z|)} \right| K (g(z,a)) dA(z) \]

\[ \geq \sup_{a \in \mathbb{D}} \mu^p (1 - (1/r)) \int_{\Omega_n} \left| \frac{\varphi^{N-1} (z)}{\omega (1 - |z|)} \right| \left| \frac{g(z)}{\omega (1 - |z|)} \right| K (g(z,a)) dA(z). \]

Choosing \( r \) such that \( r^{(N-1)/p}/\mu^p (1 - (1/r)) > 1 \), then

\[ \sup_{a \in \mathbb{D}} \int_{\Omega_n} \left| \frac{g(z)}{\omega (1 - |z|)} \right| K (g(z,a)) dA(z) < \varepsilon. \]
By the triangle inequality, then
\[
\sup_{a \in \overline{D}} \int_{\Omega} \left[ \left| f'(\varphi(z)) \right|^p |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} \\
\leq 2^p \sup_{a \in \overline{D}} \int_{\Omega} \left( \left[ \left| f'_n'(\varphi(z)) \right|^p - f'(\varphi(z)) \right]^p |g(z)|^p \\
\times (1 - |z|^2)^q K(g(z,a)) \right) \\
\times (\omega^p (1 - |z|)^{-1}) dA(z) \\
+ 2^p \sup_{a \in \overline{D}} \int_{\Omega} \left( \left[ \left| f'_n(\varphi(z)) \right|^p |g(z)|^p \\
\times (1 - |z|^2)^q K(g(z,a)) \right) \\
\times (\omega^p (1 - |z|)^{-1}) dA(z) \\
< 2^p \epsilon + 2^p \left\| f'_n \right\|_{L^p}^p \\
\times \int_{\Omega} \left[ |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} \\
< 2^p \left( 1 + \left\| f'_n \right\|_{L^p}^p \right) \epsilon, 
\]
which means, for any $\epsilon > 0$ and $f \in B_{\beta_{\mu}}$, there exists $\delta = \delta(\epsilon, f) > 0$ such that for $r \in [\delta, 1)$
\[
\sup_{a \in \overline{D}} \int_{\Omega} \left[ \left[ \left| f'(\varphi(z)) - f'_n(\varphi(z)) \right| |g(z)|^p \\
\times (1 - |z|^2)^q K(g(z,a)) \right) \\
\times (\omega^p (1 - |z|)^{-1}) dA(z) < \epsilon. 
\]
(22)

Since $C_\varphi^g$ is compact, $C_\varphi^g(B_{\beta_{\mu}})$ is relatively compact in $Q_{K,\omega}(p,q)$; then there are finite functions $f_1, f_2, \ldots, f_m \in B_{\beta_{\mu}}$ such that, for any $\epsilon > 0$ and $f \in B_{\beta_{\mu}}$, we can find $f_k (1 \leq k \leq m)$ satisfying
\[
\sup_{a \in \overline{D}} \int_{\Omega} \left( \left[ \left| f'(\varphi(z)) - f'_k(\varphi(z)) \right|^p |g(z)|^p \\
\times (1 - |z|^2)^q K(g(z,a)) \right) \\
\times (\omega^p (1 - |z|)^{-1}) dA(z) < \epsilon. 
\]
(24)

Take $\delta = \max_{1 \leq j \leq m} \delta(\epsilon, f_j)$. Then for $r \in [\delta, 1)$
\[
\sup_{a \in \overline{D}} \int_{\Omega} \left[ |f'(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} \\
< \epsilon. 
\]
(25)

Then
\[
\sup_{a \in \overline{D}} \int_{\Omega} \left[ \left| f'(\varphi(z)) \right|^p |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} < 2\epsilon. 
\]
(26)

Hence, we have shown that for any $\epsilon > 0$ there exists $\delta \in [0, 1)$ such that for all $f \in B_{\beta_{\mu}}$
\[
\sup_{a \in \overline{D}} \int_{\Omega} \left[ \left| f'(\varphi(z)) \right|^p |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} < 2\epsilon. 
\]
(27)

Let $f_j, j = 1, 2$, be the functions in Lemma 2; then, for $0 < t < 1$, the functions $f_j(tz) = f_j(tz)$ are included in $\mathcal{B}_{\mu}$.

Thus by Lemma 2 and Fatou’s Lemma, we get (15).

(c) $\Rightarrow$ (a) Assume that (14) and (15) hold. Assume that $\{f_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $\mathcal{B}_{\mu}$ such that $f_n \rightarrow 0$ uniformly on compact subsets of $\overline{D}$. Assume $\|f_n\|_{\beta_{\mu}} \leq 1$; by (15), for any given $\epsilon > 0$, there exists $r \in [0, 1)$ such that
\[
\sup_{a \in \overline{D}} \int_{\Omega} \left[ |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} < \epsilon. 
\]
(28)

Since $f_n \rightarrow 0$ uniformly on compact subsets of $\overline{D}$, then $f'_n \rightarrow 0$ uniformly on compact subsets of $\overline{D}$. Then for above $\epsilon$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|f'_n| < \epsilon$ for $|z| \leq r$. Thus,
\[
\int_{\mathbb{D}} \left[ |f'_n(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} \\
\leq \left\{ \int_{\Omega} + \int_{\mathbb{D} \setminus \Omega} \right\} \left( \left[ |f'_n(\varphi(z))|^p |g(z)|^p \\
\times (1 - |z|^2)^q K(g(z,a)) \right) \\
\times (\omega^p (1 - |z|)^{-1}) dA(z) \\
\leq \|f_n\|_{\beta_{\mu}} \int_{\Omega} \left[ |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} \\
+ \epsilon^p \int_{\mathbb{D}} \left[ |g(z)|^p (1 - |z|^2)^q K(g(z,a)) \right] \frac{dA(z)}{\omega^p (1 - |z|)} \\
\leq \epsilon + \epsilon^p M. 
\]

Hence, $\|C_\varphi^g f_n\|_{Q_{K,\omega}(p,q)} \rightarrow 0$ as $n \rightarrow \infty$. Thus $C_\varphi^g : \mathcal{B}_{\mu} \rightarrow Q_{K,\omega}(p,q)$ is compact.

Remark 5. For $\alpha > 0$, $\mu(|z|) = (1 - |z|^2)^\alpha$, $\mathcal{B}_\mu$ is the $\alpha$-Bloch space $\mathcal{B}_\alpha$. Let $\mu(|z|) = (1 - |z|^2)^\alpha$ and $\omega \equiv 1$ in Theorems 3 and 4; we easily obtain the following results in [3].

Corollary 6. Assume that $0 < p < \infty$, $-2 < q < \infty$, $\alpha > 0$, $\varphi$ is an analytic self-map of $\overline{D}$, and $K$ is a nonnegative nondecreasing function on $[0, \infty)$. Then the following statements are equivalent:

(a) $C_\varphi^g : \mathcal{B}_\mu^\alpha \rightarrow Q_\mu(p,q)$ is bounded;

(b) $C_\varphi^g : \mathcal{B}_0^{\alpha} \rightarrow Q_\mu(p,q)$ is bounded;
\[ \sup_{a \in D} \int_{D} \left| g(z) \right|^p \left(1 - |z|^2 \right)^q K(g(z, a)) \frac{1}{(1 - |\varphi(z)|^2)^\alpha} dA(z) < \infty. \] (30)

**Corollary 7.** Assume that 0 < \( p < \infty \), \(-2 < q < \infty\), \( \alpha > 0 \), \( \varphi \) is an analytic self-map of \( D \), and \( K \) is a nonnegative and nondecreasing function on \([0, \infty)\). Then the following statements are equivalent:

(a) \( \mathcal{C}_p^\varphi : \mathcal{B}_\mu \to Q_K(p, q) \) is compact;
(b) \( \mathcal{C}_p^\varphi : \mathcal{B}_0 \to Q_K(p, q) \) is compact;
(c) \[ \sup_{a \in D} \int_{D} \left| g(z) \right|^p \left(1 - |z|^2 \right)^q K(g(z, a)) \frac{1}{(1 - |\varphi(z)|^2)^\alpha} dA(z) < \infty, \] \[ \lim_{r \to 1^-} \sup_{a \in D} \int_{D} \left| g(z) \right|^p \left(1 - |z|^2 \right)^q K(g(z, a)) \frac{1}{(1 - |\varphi(z)|^2)^\alpha} dA(z) = 0. \] (31)

**Remark 8.** As \( g = \varphi' \), the operator \( \mathcal{C}_p^\varphi \) is essentially the composition operator \( \mathcal{C}_\varphi \), since the difference \( \mathcal{C}_p^\varphi - \mathcal{C}_\varphi \) is constant. Moreover, \( \omega \equiv 1 \); \( Q_{K,\omega}(p, q) = Q_K(p, q) \). Let \( g = \varphi' \) and \( \omega \equiv 1 \) in Theorems 3 and 4; we easily obtain the following results in [9].

**Corollary 9.** Assume that 0 < \( p < \infty \), \(-2 < q < \infty\), \( \varphi \) is an analytic self-map of \( D \), \( \mu \) is a normal function, and \( K \) is nonnegative and nondecreasing in \([0, \infty)\). Then the following statements are equivalent:

(a) \( \mathcal{C}_\varphi : \mathcal{B}_\mu \to Q_K(p, q) \) is bounded;
(b) \( \mathcal{C}_\varphi : \mathcal{B}_{\mu,0} \to Q_K(p, q) \) is bounded;
(c) \[ \sup_{a \in D} \int_{D} \left| \varphi'(z) \right|^p \left(1 - |z|^2 \right)^q K(g(z, a)) \frac{1}{\mu^p \left(|\varphi(z)| \right)} dA(z) < \infty. \] (32)

**Corollary 10.** Assume that 0 < \( p < \infty \), \(-2 < q < \infty\), \( \varphi \) is an analytic self-map of \( D \), \( \mu \) is a normal function, and \( K \) is nonnegative and nondecreasing in \([0, \infty)\). Then the following statements are equivalent:

(a) \( \mathcal{C}_\varphi : \mathcal{B}_\mu \to Q_K(p, q) \) is compact;
(b) \( \mathcal{C}_\varphi : \mathcal{B}_{\mu,0} \to Q_K(p, q) \) is compact;
(c) \( \varphi \in Q_K(p, q) \) and
\[ \lim_{r \to 1^-} \sup_{a \in D} \int_{D} \left| \varphi'(z) \right|^p \left(1 - |z|^2 \right)^q K(g(z, a)) \frac{1}{\mu^p \left(|\varphi(z)| \right)} dA(z) = 0. \] (33)

**Problem II.** Can the boundedness and compactness of the generalized composition operator \( \mathcal{C}_p^\varphi : Q_{K,\omega}(p, q) \to \mathcal{B}_\mu \) be characterized by use of function theoretic properties of \( \varphi \) and \( g \)?

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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