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Research Article

Generalized Composition Operators from $\mathscr{B}_\mu$ Spaces to $Q_{K,\omega}(p,q)$ Spaces

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Let $0 < p < \infty$, let $-2 < q < \infty$, and let φ be an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. The boundedness and compactness of generalized composition operators $(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi$, $z \in \mathbb{D}$, $f \in H(\mathbb{D})$, from $\mathscr{B}_\mu$ $(\mathscr{B}_{\mu,0})$ spaces to $Q_{K,\omega}(p,q)$ spaces are investigated.

1. Introduction and Preliminaries

Let φ be an analytic self-map of the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ and $g \in H(\mathbb{D})$. If X is a Banach space, then we denote the unit ball in X by B_X. For $0 < r < 1$, $\Omega_r = \{z \in \mathbb{D} : |\varphi(z)| > r\}$.

A positive continuous function μ on the interval $[0, 1)$ is called normal if there exist three constants $0 \le \delta < 1$ and $0 < a < b$ such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1), \qquad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0;$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1), \qquad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = \infty. \tag{1}$$

A function $f \in H(\mathbb{D})$ belongs to the Bloch type space $\mathscr{B}_\mu$ if

$$\|f\|_{\mathscr{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}}\mu(z)\left|f'(z)\right| < \infty, \tag{2}$$

where μ is normal and radial and $\mu(|z|) = \mu(z)$. The space $\mathscr{B}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathscr{B}_\mu}$.

The little Bloch type space $\mathscr{B}_{\mu,0}$ consists of all $f \in \mathscr{B}_\mu$ such that

$$\lim_{|z|\to 1^-}\mu(|z|)\left|f'(z)\right| = 0. \tag{3}$$

For $\alpha > 0$, $\mu(|z|) = (1-|z|^2)^\alpha$, $\mathscr{B}_\mu$ is the α-Bloch space $\mathscr{B}^\alpha$; for $\alpha = 1$, $\mathscr{B}^\alpha$ is the classical Bloch space; for example, see [1].

For $0 < p < \infty$, $-2 < q < \infty$, $a \in D$, $K : [0,\infty) \to [0,\infty)$ is a nondecreasing function, and $\omega : (0, 1] \to (0,\infty)$ is a given reasonable function. An analytic function f on D is said to belong to $Q_{K,\omega}(p,q)$ in [2] if

$$\|f\|_{Q_{K,\omega}(p,q)} = \left\{\sup_{a\in D}\int_D \left|f'(z)\right|^p\left(1-|z|^2\right)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)\right\}^{1/p} < \infty \tag{4}$$

and an analytic function $f \in Q_{K,\omega,0}(p,q)$ if

$$\lim_{|a|\to 1^-}\int_D \left|f'(z)\right|^p\left(1-|z|^2\right)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z) = 0, \tag{5}$$

where dA denotes the normalized Lebesgue area measure on D, $g(z,a) = \log(1/|\phi_a(z)|)$ is a green function, and $\phi_a(z) = (a-z)/(1-\overline{a}z)$.

$Q_{K,\omega}(p,q)$ classes are more general than many classes of analytic functions and have attracted a lot of attention in recent years. When $\omega \equiv 1$, $Q_{K,\omega}(p,q) = Q_K(p,q)$. When $p = q = 2$, $\omega(t) = t$, $K(t) = t^p$, and $Q_{K,\omega}(p,q) = Q_p$. When

$\omega \equiv 1$, $K(t) = t^s$ and $Q_{K,\omega}(p,q) = F(p,q,s)$. Moreover, the following results hold:

(1) $Q_{K,\omega}(p,q) \subset B_{\omega}^{(q+2)/p}$;

(2) $Q_{K,\omega}(p,q) = B_{\omega}^{(q+2)/p}$ if and only if

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{\left(1-r^2\right)^2}dr < \infty, \tag{6}$$

where

$$B_{\omega}^{\alpha} = \left\{f \in H(D) : \|f\|_{B_{\omega}^{\alpha}} = \sup_{z\in D}\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}\left|f'(z)\right| < \infty, 0 < \alpha < \infty\right\}. \tag{7}$$

The composition operator is defined by $C_{\varphi}f(z) = f(\varphi(z))$, $f \in H(\mathbb{D})$. This operator has been studied for many years. The first setting was in the Hardy space H^2, the space of functions analytic on $\mathbb{D}$ (see [3]). Madigan and Matheson (see [1]) gave a characterization of the compact composition operators on the Bloch space $\mathcal{B}$. For more details, see [4–12]. In [13], Li and Stević defined the generalized composition operator as follows:

$$\left(C_{\varphi}^{g}f\right)(z) = \int_0^z f'(\varphi(\xi))\,g(\xi)\,d\xi, \quad z \in \mathbb{D},\ f \in H(\mathbb{D}). \tag{8}$$

The operator C_{φ}^{g} induces many known operators. When $g = \varphi'$, the operator C_{φ}^{g} is essentially (up to a constant) the composition operator C_{φ}. When $\varphi(z) = z$, the operator C_{φ}^{g} coincides with the operator I_g defined by

$$\left(I_g f\right)(z) = \int_0^z f'(\zeta)\,g(\zeta)\,d\zeta, \quad \zeta \in \mathbb{D},\ f \in H(\mathbb{D}). \tag{9}$$

So the generalized composition operator C_{φ}^{g} can be considered as a generalization of the composition operator C_{φ} and the operator I_g.

A fundamental problem in the study of generalized composition operators C_{φ}^{g} is to investigate the relations between function theoretic properties of φ and g and operator theoretic properties of the restriction of C_{φ}^{g} to various Banach spaces of analytic functions. A lot of attentions have been attracted to study the problem on many Banach spaces of analytic functions in recent years. In [9], the authors studied composition operators from Bloch type spaces into $Q_K(p,q)$ spaces. In [14], the authors characterized the boundedness and compactness of generalized composition operators on $Q_{K,\omega}(p,q)$ spaces. In [15], Rezaei and Mahyar studied generalized composition operators from logarithmic Bloch type spaces to Q_K type spaces. In [16], essential norms of generalized composition operators from Bloch type spaces to Q_K type spaces were given. In [17], generalized composition operators from $F(p,q,s)$ spaces to Bloch-type spaces were characterized. In [18], Stević investigated generalized composition operators between mixed-norm space and some weighted spaces and from logarithmic Bloch spaces to mixed-norm spaces. In [3], Zhang and Liu studied generalized composition operators from Bloch type spaces to Q_K type spaces. In [19], generalized composition operator acting from Bloch-type spaces to mixed-norm space was studied. In [12], generalized composition operators from generalized weighted Bergman spaces to Bloch type spaces were investigated. In [20], generalized composition operators and Volterra composition operators on Bloch spaces on the unit ball were studied. This paper is devoted to investigating the boundedness and compactness of generalized composition operators C_{φ}^{g} from $\mathcal{B}_{\mu}$ ($\mathcal{B}_{\mu,0}$) spaces to $Q_{K,\omega}(p,q)$ spaces. Throughout this paper, constants are denoted by C; they are positive and may differ from one occurrence to the other.

2. Main Results and Their Proofs

To derive our results, we need the following lemmas.

Lemma 1. *Assume that $0 < p < \infty$, $-2 < q < \infty$, K is a nonnegative nondecreasing function on $[0,\infty)$, and $\omega : (0,1] \to (0,\infty)$ is a given reasonable function. Assume that μ is a normal function, φ is an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$. Then $C_{\varphi}^{g} : \mathcal{B}_{\mu}(\mathcal{B}_{\mu,0}) \to Q_{K,\omega}(p,q)$ is compact if and only if, for every bounded sequence $\{f_k\}$ in $\mathcal{B}_{\mu}(\mathcal{B}_{\mu,0})$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\lim_{k\to\infty}\|C_{\varphi}^{g}f_k\|_{Q_{K,\omega}(p,q)} = 0$.*

Lemma 1 can be proved in a standard way of Theorem 3.11 in [4].

The following lemma is similar to Lemma 2.2 in [5, 7], using the results for the Hadamard gap series and following a technique used before in the Bloch space in [5, 7]. Specific details can be seen in [9].

Lemma 2. *Let $\mu : [0,1) \to [0,\infty)$ be a nonincreasing radial weight function and normal on the interval $[0,1)$. Then there exist two functions $f_1, f_2 \in \mathcal{B}_{\mu}$ such that, for each $z \in \mathbb{D}$,*

$$\left|f_1'(z)\right| + \left|f_2'(z)\right| \geq \frac{C}{\mu(|z|)}. \tag{10}$$

Theorem 3. *Assume that $0 < p < \infty$, $-2 < q < \infty$, φ is an analytic self-map of $\mathbb{D}$, μ is a normal function, K is nonnegative and nondecreasing in $[0,\infty)$, and $\omega : (0,1] \to (0,\infty)$ is a given reasonable function. Then the following statements are equivalent:*

(a) $C_{\varphi}^{g} : \mathcal{B}_{\mu} \to Q_{K,\omega}(p,q)$ *is bounded;*

(b) $C_{\varphi}^{g} : \mathcal{B}_{\mu,0} \to Q_{K,\omega}(p,q)$ *is bounded;*

(c)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\mu^p(|\varphi(z)|)\,\omega^p(1-|z|)}dA(z) < \infty. \tag{11}$$

Proof. (a) ⇒ (b) Since $\mathscr{B}_{\mu,0} \subset \mathscr{B}_\mu$, then (a) implies (b).

(b) ⇒ (c) Suppose (b) holds; then $\|C_\varphi^g f\|_{Q_{K,\omega}(p,q)} \leq \|C_\varphi^g\| \|f\|_{\mathscr{B}_\mu}$ for all $f \in \mathscr{B}_{\mu,0}$. For any given $f \in \mathscr{B}_\mu$, the function $f_t(z) = f(tz)$, $0 < t < 1$, belongs to $\mathscr{B}_{\mu,0}$ and $\|f_t\|_{\mathscr{B}_\mu} \leq \|f\|_{\mathscr{B}_\mu}$. Let f_1, f_2 be the functions from Lemma 2 and we get

$$\begin{aligned} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} & \frac{|g(z)|^p (1-|z|^2)^q K(g(z,a))}{\mu^p(t|\varphi(z)|)\,\omega^p(1-|z|)} dA(z) \\ & \leq 2^p \left\|C_\varphi^g\right\|^p \left(\|f_{1t}\|_{\mathscr{B}_\mu}^p + \|f_{2t}\|_{\mathscr{B}_\mu}^p\right) \\ & \leq 2^p \left\|C_\varphi^g\right\|^p \left(\|f_1\|_{\mathscr{B}_\mu}^p + \|f_2\|_{\mathscr{B}_\mu}^p\right). \end{aligned} \tag{12}$$

Then (11) holds with Fatou's Lemma.

(c) ⇒ (a) For $f \in \mathscr{B}_\mu$,

$$\begin{aligned} & \left\|C_\varphi^g f\right\|_{Q_{K,\omega}(p,q)}^p \\ & = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\left|f'(\varphi(z))\right|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\ & \leq \|f\|_{\mathscr{B}_\mu}^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p (1-|z|^2)^q K(g(z,a))}{\mu^p(|\varphi(z)|)\,\omega^p(1-|z|)} dA(z). \end{aligned} \tag{13}$$

□

Theorem 4. *Assume that $0 < p < \infty$, $-2 < q < \infty$, φ is an analytic self-map of $\mathbb{D}$, μ is a normal function, K is nonnegative and nondecreasing in $[0,\infty)$, and $\omega : (0,1] \to (0,\infty)$ is a given reasonable function. Then the following statements are equivalent:*

(a) $C_\varphi^g : \mathscr{B}_\mu \to Q_{K,\omega}(p,q)$ *is compact;*

(b) $C_\varphi^g : \mathscr{B}_{\mu,0} \to Q_{K,\omega}(p,q)$ *is compact;*

(c)

$$M = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p (1-|z|^2)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty, \tag{14}$$

$$\lim_{r\to 1} \sup_{a\in\mathbb{D}} \int_{\Omega_r} \frac{|g(z)|^p (1-|z|^2)^q K(g(z,a))}{\mu^p(|\varphi(z)|)\,\omega^p(1-|z|)} dA(z) = 0. \tag{15}$$

Proof. (a) ⇒ (b) Since $\mathscr{B}_{\mu,0} \subset \mathscr{B}_\mu$, then (a) implies (b).

(b) ⇒ (c) Assume that (b) holds; then we have (14), Let

$$f_n(z) = \frac{z^n}{n\mu(1-(1/n))}, \quad z \in \mathbb{D}. \tag{16}$$

Then $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{B}_{\mu,0}$ and $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$ as $n \to \infty$. Since $C_\varphi^g : \mathscr{B}_{\mu,0} \to Q_{K,\omega}(p,q)$ is compact, then by Lemma 1

$$\lim_{n\to\infty} \left\|C_\varphi^g f_n\right\|_{Q_{K,\omega}(p,q)} = 0. \tag{17}$$

This means, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} & \frac{\left|\varphi^{n-1}(z)\right|^p}{\mu^p(1-(1/n))\,\omega^p(1-|z|)} \\ & \times |g(z)|^p (1-|z|^2)^q K(g(z,a))\, dA(z) < \varepsilon. \end{aligned} \tag{18}$$

Hence, for $0 < r < 1$,

$$\begin{aligned} & \sup_{a\in\mathbb{D}} \frac{1}{\mu^p(1-(1/N))} \\ & \quad \times \int_{\mathbb{D}} \frac{\left|\varphi^{N-1}(z)\right|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\ & \geq \sup_{a\in\mathbb{D}} \frac{1}{\mu^p(1-(1/N))} \\ & \quad \times \int_{\Omega_r} \frac{\left|\varphi^{N-1}(z)\right|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\ & \geq \frac{r^{(N-1)p}}{\mu^p(1-(1/N))} \\ & \quad \times \sup_{a\in\mathbb{D}} \int_{\Omega_r} \frac{|g(z)|^p (1-|z|^2)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z). \end{aligned} \tag{19}$$

Choosing r such that $r^{(N-1)p}/\mu^p(1-(1/N)) > 1$, then

$$\sup_{a\in\mathbb{D}} \int_{\Omega_r} \frac{|g(z)|^p (1-|z|^2)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \varepsilon. \tag{20}$$

For $f \in \mathscr{B}_{\mu,0}$, let $f_t(z) = f(tz)$ for $0 < t < 1$. Then $f_t \in \mathscr{B}_{\mu,0}$ and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. Since C_φ^g is compact, then $\|C_\varphi^g f_t - C_\varphi^g f\|_{Q_{K,\omega}(p,q)} \to 0$ as $t \to 1$. Then for every $\varepsilon > 0$ there exists $t_0 \in (0,1)$ such that

$$\begin{aligned} \int_{\mathbb{D}} & \Big(\left(\left|f_{t_0}'(\varphi(z)) - f'(\varphi(z))\right|^p |g(z)|^p \right. \\ & \left. \times \left(1-|z|^2\right)^q K(g(z,a)) \right) \\ & \times \left(\omega^p(1-|z|)\right)^{-1} \Big)\, dA(z) < \varepsilon. \end{aligned} \tag{21}$$

By the triangle inequality, then

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\Omega_r}\frac{\left|f'(\varphi(z))\right|^p|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)\\
&\leq 2^p\sup_{a\in\mathbb{D}}\int_{\Omega_r}\Big(\Big(\left|f'_{t_0}(\varphi(z))-f'(\varphi(z))\right|^p|g(z)|^p\\
&\qquad\times\left(1-|z|^2\right)^q K(g(z,a))\Big)\\
&\qquad\times\left(\omega^p(1-|z|)\right)^{-1}\Big)dA(z)\\
&\quad+2^p\sup_{a\in\mathbb{D}}\int_{\Omega_r}\Big(\Big(\left|f'_{t_0}(\varphi(z))\right|^p|g(z)|^p\\
&\qquad\times\left(1-|z|^2\right)^q K(g(z,a))\Big)\\
&\qquad\times\left(\omega^p(1-|z|)\right)^{-1}\Big)dA(z)\\
&< 2^p\varepsilon+2^p\left\|f'_{t_0}\right\|_{H^\infty}^p\\
&\quad\times\int_{\Omega_r}\frac{|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)\\
&< 2^p\left(1+\left\|f'_{t_0}\right\|_{H^\infty}^p\right)\varepsilon,
\end{aligned}\tag{22}$$

which means, for any $\varepsilon > 0$ and $f \in B_{\mathscr{B}_{\mu,0}}$, there exists $\delta = \delta(\varepsilon, f) > 0$ such that for $r \in [\delta, 1)$

$$\sup_{a\in\mathbb{D}}\int_{\Omega_r}\frac{\left|f'(\varphi(z))\right|^p|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\varepsilon.\tag{23}$$

Since C_φ^g is compact, $C_\varphi^g(B_{\mathscr{B}_{\mu,0}})$ is relatively compact in $Q_{K,\omega}(p,q)$; then there are finite functions $f_1, f_2, \ldots, f_m \in B_{\mathscr{B}_{\mu,0}}$ such that, for any $\varepsilon > 0$ and $f \in B_{\mathscr{B}_{\mu,0}}$, we can find $f_k (1 \leq k \leq m)$ satisfying

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\Big(\Big(\left|f'(\varphi(z))-f'_k(\varphi(z))\right|^p|g(z)|^p\\
&\qquad\times\left(1-|z|^2\right)^q K(g(z,a))\Big)\\
&\qquad\times\left(\omega^p(1-|z|)\right)^{-1}\Big)dA(z)<\varepsilon.
\end{aligned}\tag{24}$$

Take $\delta = \max_{1\leq j\leq m}\delta(\varepsilon, f_j)$. Then for $r \in [\delta, 1)$

$$\sup_{a\in\mathbb{D}}\int_{\Omega_r}\frac{\left|f'_k(\varphi(z))\right|^p|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\varepsilon.\tag{25}$$

Then

$$\sup_{a\in\mathbb{D}}\int_{\Omega_r}\frac{\left|f'(\varphi(z))\right|^p|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)<2\varepsilon.\tag{26}$$

Hence, we have shown that for any $\varepsilon > 0$ there exists $\delta \in [0, 1)$ such that for all $f \in B_{\mathscr{B}_{\mu,0}}$

$$\sup_{a\in\mathbb{D}}\int_{\Omega_r}\frac{\left|f'(\varphi(z))\right|^p|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)<2\varepsilon.\tag{27}$$

Let f_j, $j = 1, 2$, be the functions in Lemma 2; then, for $0 < t < 1$, the functions $f_{jt}(z) = f_j(tz)$ are included in $\mathscr{B}_{\mu,0}$. Thus by Lemma 2 and Fatou's Lemma, we get (15).

(c) $\Rightarrow$ (a) Assume that (14) and (15) hold. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathscr{B}_\mu$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Assume $\|f_n\|_{\mathscr{B}_\mu} \leq 1$; by (15), for any given $\varepsilon > 0$, there exists $r \in [0, 1)$ such that

$$\sup_{a\in\mathbb{D}}\int_{\Omega_r}\frac{|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\mu^p(|\varphi(z)|)\,\omega^p(1-|z|)}dA(z)<\varepsilon.\tag{28}$$

Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $f'_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then for above ε, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|f'_n| < \varepsilon$ for $|z| \leq r$. Thus,

$$\begin{aligned}
&\int_{\mathbb{D}}\frac{\left|f'_n(\varphi(z))\right|^p|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)\\
&\leq\left\{\int_{\Omega_r}+\int_{\mathbb{D}\setminus\Omega_r}\right\}\Big(\Big(\left|f'_n(\varphi(z))\right|^p|g(z)|^p\\
&\qquad\times\left(1-|z|^2\right)^q K(g(z,a))\Big)\\
&\qquad\times\left(\omega^p(1-|z|)\right)^{-1}\Big)dA(z)
\end{aligned}\tag{29}$$

$$\begin{aligned}
&\leq\|f_n\|_{\mathscr{B}_\mu}^p\int_{\Omega_r}\frac{|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\mu^p(|\varphi(z)|)\,\omega^p(1-|z|)}dA(z)\\
&\quad+\varepsilon^p\int_{\mathbb{D}}\frac{|g(z)|^p\left(1-|z|^2\right)^q K(g(z,a))}{\omega^p(1-|z|)}dA(z)\\
&\leq\varepsilon+\varepsilon^p M.
\end{aligned}$$

Hence, $\|C_\varphi^g f_n\|_{Q_{K,\omega}(p,q)} \to 0$ as $n \to \infty$. Thus $C_\varphi^g : \mathscr{B}_\mu \to Q_{K,\omega}(p,q)$ is compact. □

Remark 5. For $\alpha > 0$, $\mu(|z|) = (1-|z|^2)^\alpha$, $\mathscr{B}_\mu$ is the α-Bloch space $\mathscr{B}^\alpha$. Let $\mu(|z|) = (1-|z|^2)^\alpha$ and $\omega \equiv 1$ in Theorems 3 and 4; we easily obtain the following results in [3].

Corollary 6. *Assume that $0 < p < \infty$, $-2 < q < \infty$, $\alpha > 0$, φ is an analytic self-map of $\mathbb{D}$, and K is a nonnegative nondecreasing function on $[0,\infty)$. Then the following statements are equivalent:*

(a) *$C_\varphi^g : \mathscr{B}^\alpha \to Q_K(p,q)$ is bounded;*

(b) *$C_\varphi^g : \mathscr{B}_0^\alpha \to Q_K(p,q)$ is bounded;*

(c)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|g(z)|^{p}\left(1-|z|^{2}\right)^{q}K\left(g(z,a)\right)}{\left(1-|\varphi(z)|^{2}\right)^{p\alpha}}dA(z)<\infty. \quad (30)$$

Corollary 7. *Assume that $0<p<\infty$, $-2<q<\infty$, $\alpha>0$, φ is an analytic self-map of $\mathbb{D}$, and K is a nonnegative nondecreasing function on $[0,\infty)$. Then the following statements are equivalent:*

(a) $C_{\varphi}^{g}:\mathscr{B}^{\alpha}\rightarrow Q_{K}(p,q)$ *is compact;*

(b) $C_{\varphi}^{g}:\mathscr{B}_{0}^{\alpha}\rightarrow Q_{K}(p,q)$ *is compact;*

(c)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g(z)|^{p}\left(1-|z|^{2}\right)^{q}K\left(g(z,a)\right)dA(z)<\infty,$$
$$\lim_{r\rightarrow 1}\sup_{a\in\mathbb{D}}\int_{\Omega_{r}}\frac{|g(z)|^{p}\left(1-|z|^{2}\right)^{q}K\left(g(z,a)\right)}{\left(1-|\varphi(z)|^{2}\right)^{p\alpha}}dA(z)=0. \quad (31)$$

Remark 8. As $g=\varphi'$, the operator C_{φ}^{g} is essentially the composition operator C_{φ}, since the difference $C_{\varphi}^{g}-C_{\varphi}$ is constant. Moreover, $\omega\equiv 1$; $Q_{K,\omega}(p,q)=Q_{K}(p,q)$. Let $g=\varphi'$ and $\omega\equiv 1$ in Theorems 3 and 4; we easily obtain the following results in [9].

Corollary 9. *Assume that $0<p<\infty$, $-2<q<\infty$, φ is an analytic self-map of $\mathbb{D}$, μ is a normal function, and K is nonnegative and nondecreasing in $[0,\infty)$. Then the following statements are equivalent:*

(a) $C_{\varphi}:\mathscr{B}_{\mu}\rightarrow Q_{K}(p,q)$ *is bounded;*

(b) $C_{\varphi}:\mathscr{B}_{\mu,0}\rightarrow Q_{K}(p,q)$ *is bounded;*

(c)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\varphi'(z)|^{p}\left(1-|z|^{2}\right)^{q}K\left(g(z,a)\right)}{\mu^{p}\left(|\varphi(z)|\right)}dA(z)<\infty. \quad (32)$$

Corollary 10. *Assume that $0<p<\infty$, $-2<q<\infty$, φ is an analytic self-map of $\mathbb{D}$, μ is a normal function, and K is nonnegative and nondecreasing in $[0,\infty)$. Then the following statements are equivalent:*

(a) $C_{\varphi}:\mathscr{B}_{\mu}\rightarrow Q_{K}(p,q)$ *is compact;*

(b) $C_{\varphi}:\mathscr{B}_{\mu,0}\rightarrow Q_{K}(p,q)$ *is compact;*

(c) $\varphi\in Q_{K}(p,q)$ *and*

$$\lim_{r\rightarrow 1}\sup_{a\in\mathbb{D}}\int_{\Omega_{r}}\frac{|\varphi'(z)|^{p}\left(1-|z|^{2}\right)^{q}K\left(g(z,a)\right)}{\mu^{p}\left(|\varphi(z)|\right)}dA(z)=0. \quad (33)$$

Problem 11. Can the boundedness and compactness of the generalized composition operator $C_{\varphi}^{g}:Q_{K,\omega}(p,q)\rightarrow\mathscr{B}_{\mu}$ be characterized by use of function theoretic properties of φ and g?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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