Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 871026, 10 pages
http://dx.doi.org/10.1155/2014/871026

Research Article

Viscosity Approximation Methods with Errors and Strong Convergence Theorems for a Common Point of Pseudocontractive and Monotone Mappings: Solutions of Variational Inequality Problems

Yan Tang

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

Correspondence should be addressed to Yan Tang; ttyy7999@163.com

Received 9 May 2014; Revised 25 June 2014; Accepted 25 June 2014; Published 15 July 2014

Academic Editor: Adrian Petrusel

We introduce two proximal iterative algorithms with errors which converge strongly to the common solution of certain variational inequality problems for a finite family of pseudocontractive mappings and a finite family of monotone mappings. The strong convergence theorems are obtained under some mild conditions. Our theorems extend and unify some of the results that have been proposed for this class of nonlinear mappings.

1. Introduction

In many problems, for example, convex optimization, linear programming, monotone inclusions, elliptic differential equations, and variational inequalities, it is quite often to seek a proximal point of a given nonlinear problem. The proximal point algorithm is recognized as a powerful and successful algorithm in finding a common point of the fixed points of pseudocontractive mappings and the solutions of monotone mappings. Let C be a closed convex subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We recall that a mapping $A : C \to H$ is called monotone if and only if

$$\langle x-y, Ax-Ay\rangle \geq 0, \quad \forall x, y \in C. \tag{1}$$

A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x, y \in C. \tag{2}$$

Obviously, the class of monotone mappings includes the class of the α-inverse strongly monotone mappings. The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. The classical variational inequality problem is formulated as finding a point $u \in C$ such that $\langle v-u, Au\rangle \geq 0$, for all $v \in C$. The set of solutions of variational inequality problems is denoted by $\mathrm{VI}(C, A)$.

A mapping $T : C \to H$ is called pseudocontractive if, for all $x, y \in C$, we have

$$\langle Tx-Ty, x-y\rangle \leq \|x-y\|^2. \tag{3}$$

A mapping $T : C \to H$ is called κ-strict pseudocontractive if there exists a constant $0 \leq \kappa \leq 1$ such that

$$\langle x-y, Tx-Ty\rangle \leq \|x-y\|^2 - \kappa\|(I-T)x-(I-T)y\|^2, \quad \forall x, y \in C. \tag{4}$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx-Ty\| \leq \|x-y\|, \quad \forall x, y \in C. \tag{5}$$

Clearly, the class of pseudocontractive mappings includes the class of strict pseudocontractive mappings and the class of nonexpansive mappings. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

A mapping $f : C \to C$ is called contractive with a contraction coefficient if there exists a constant $\rho \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \rho \|x - y\|, \quad \forall x, y \in C. \tag{6}$$

Recently viscosity approximation methods for finding fixed points of pseudocontractive mappings have received vast investigations because of their extensive applications in a variety of applied areas of partial differential equations, image recovery, and signal processing. In Hilbert spaces, many authors have studied the fixed-point problems of the nonexpansive mappings and monotone mappings by the viscosity approximation methods and obtained a series of good results; see [1–18] and the reference therein.

For finding an element of the set of fixed points of the nonexpansive mappings, Halpern [1] was the first to study the convergence of the scheme in 1967:

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) T(x_n). \tag{7}$$

In 2000, Moudafi [2] introduced the viscosity approximation methods and proved the strong convergence of the following iterative algorithm under some suitable conditions:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(x_n). \tag{8}$$

Takahashi et al. [19, 20] introduced the following scheme and studied the weak and strong convergence theorems of the elements of $F(T) \cap \mathrm{VI}(C, A)$, respectively, under different conditions:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C (x_n - \lambda_n A x_n), \tag{9}$$

where T is a nonexpansive mapping and A is an α-inverse strong monotone operator. Recently, Zegeye and Shahzad [21] introduced the mappings as follows:

$$\begin{aligned} T_r(x) &= \Big\{ z \in C : \langle y - z, Tz \rangle \\ &\qquad - \frac{1}{r} \langle y - z, (1 + r) z - x \rangle \le 0, \forall y \in C \Big\}. \\ F_r(x) &= \Big\{ z \in C : \langle y - z, Az \rangle \\ &\qquad + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \forall y \in C \Big\}. \end{aligned} \tag{10}$$

Very recently, Tang [22] introduced the following sequence and obtained the strong convergence theorems:

$$\begin{aligned} y_n &= \lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^{m} \mu_i F_{ir_n} x_n, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \sigma_i T_{ir_n} y_n. \end{aligned} \tag{11}$$

For other related results, see [11–13, 23–25]. On the other side, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. There is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job. Starting from any initial guess, $z_0 \in H$, the proximal point algorithm generates a sequence $\{z_k\}$ according to the inclusion:

$$z_k \in z_{k+1} + c_k A z_{k+1}, \tag{12}$$

where A is a maximal monotone operator and $c_k > 0$ is a parameter. For solving the original problem of finding a solution to the inclusion $0 \in Az$, Rockafellar [23] introduced the following algorithm:

$$z_k + e_k \in z_{k+1} + c_k A z_{k+1}, \tag{13}$$

where $\{e_k\}$ is a sequence of errors. Rockafellar [23] obtained the weak convergence of the algorithm. Very recently Yao and Shahzad [24] proved that sequences generated from the method of resolvent are given by

$$x_m = P_C (\alpha_m u_m + (1 - \alpha_m) T x_m), \quad m \ge 0, \tag{14}$$

where $\{\alpha_m\}$ is a sequence in $[0,1]$, the sequence $\{u_m\} \subset H$ is a small perturbation, and T is a nonexpansive mapping.

The following is our concern now: Is it possible to construct a new sequence with general errors that converges strongly to a common element of fixed points of pseudocontractive mappings and the solution set of monotone mappings and converges strongly to the unique solution of certain variational inequality?

In this paper, motivated and inspired by the above results, we introduce two iterations with perturbations which converge strongly to a common element of the set of fixed points of a finite family of pseudocontractive mappings more general than nonexpansive mappings and the solution set of a finite family of monotone mappings more general than α-inverse strongly monotone mappings or maximal monotone mappings. Our theorems presented in this paper improve and extend the corresponding results of Yao and Shahzad [24], Zegeye and Shahzad [21], and Tang [22] and some other results in this direction.

2. Preliminaries

In the sequel, we will use the following lemmas.

Lemma 1 (see [6]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \theta_n) a_n + \sigma_n, \quad n \ge 0, \tag{15}$$

where $\{\theta_n\}$ is a sequence in (0,1) and $\{\sigma_n\}$ is a real sequence such that

(i) $\sum_{n=0}^{\infty} \theta_n = \infty$;

(ii) $\limsup_{n\to\infty} (\sigma_n/\theta_n) \le 0$ *or* $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Let C be a nonempty closed and convex subset of a real Hilbert space H; a mapping $P_C : H \to C$ is called the metric *projection if, for all $x \in H$, there exists a unique point in C, denoted by $P_C x$ such that*

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{16}$$

It is well known that P_C is a nonexpansive mapping.

Lemma 2 (see [25]). *Let C be a nonempty closed and convex subset of a real smooth Hilbert space H. Let $x \in H$; then $P_C x$ have the property as follows:*

$$\begin{gathered}\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall x \in H, \ y \in C, \\ \|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, \ y \in C.\end{gathered} \tag{17}$$

Lemma 3 (see [21]). *Let C be closed convex subset of Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping, let $T : C \to C$ be a continuous pseudocontractive mapping, and define the mappings T_r and F_r as follows: for $x \in H$, $r \in (0, \infty)$,*

$$\begin{aligned} T_r(x) = \Big\{ & z \in C : \langle y - z, Tz\rangle \\ & -\frac{1}{r}\langle y - z, (1 + r) z - x\rangle \le 0, \forall y \in C \Big\}, \\ F_r(x) = \Big\{ & z \in C : \langle y - z, Az\rangle \\ & +\frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \Big\}. \end{aligned} \tag{18}$$

Then the following hold:

(i) *T_r and F_r are single valued;*

(ii) *T_r and F_r are firmly nonexpansive mappings; that is, $\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle$, $\|F_r x - F_r y\|^2 \le \langle F_r x - F_r y, x - y\rangle$;*

(iii) *$F(T_r) = F(T)$, $F(F_r) = VI(C, A)$;*

(iv) *$F(T)$ and $VI(C, A)$ are closed convex.*

Lemma 4 (see [11]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1. \tag{19}$$

Suppose

$$\begin{gathered} x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \quad n \ge 0, \\ \lim_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \end{gathered} \tag{20}$$

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 5 (see [22]). *Let C be a nonempty closed convex and bounded subset of a Hilbert space H, and let $\{\Gamma_i : C \to C, i = 1, 2, \ldots m\}$ be a finite family of nonexpansive mappings such that $\cap_i^m F(\Gamma_i) \ne \emptyset$. Suppose that $\alpha = \inf\{\alpha_i\} > 0$ and $\Sigma_{i=1}^m \alpha_i = 1$. Then there exists nonexpansive mapping $\Gamma : C \to C$ such that $F(\Gamma) = \cap_{i=1}^m F(\Gamma_i)$.*

3. Main Results

Let C be closed convex subset of Hilbert space H. Let $\{A_i : C \to H, i = 1, 2, \ldots, N\}$ be a finite family of continuous monotone mappings, and let $\{T_i : C \to C, i = 1, 2, \ldots, N\}$ be a finite family of continuous pseudocontractive mappings. For the rest of this paper, $T_{ir_n} : E \to C$ and $F_{ir_n} : E \to C$ are mappings defined as follows: for $x \in E$, $r_n \in (0, \infty)$,

$$\begin{aligned} T_{ir_n}(x) := \Big\{ & z \in C : \langle y - z, T_i z\rangle \\ & -\frac{1}{r_n}\langle y - z, (1 + r_n) z - x\rangle \le 0, \forall y \in C \Big\}, \end{aligned} \tag{21}$$

$$\begin{aligned} F_{ir_n}(x) := \Big\{ & z \in C : \langle y - z, A_i z\rangle \\ & +\frac{1}{r_n}\langle y - z, z - x\rangle \ge 0, \forall y \in C \Big\}. \end{aligned} \tag{22}$$

By using Lemmas 2.3–2.6 in Zegeye and Shahzad [21], we have that the mappings T_{ir_n} and F_{ir_n} are well defined and they are nonexpansive and $F(T_{ir_n}) = F(T_i)$, $F(F_{ir_n}) = \mathrm{VI}(C, A_i)$ are closed convex. Denote $F_1 = \cap_{i=1}^N F(T_{ir_n})$, $F_2 = \cap_{i=1}^N F(F_{ir_n})$.

Theorem 6. *Let C be a nonempty closed convex subset of uniformly smooth strictly convex real Hilbert space H. Let $\{T_i : C \to C, i = 1, 2, \ldots, N\}$ be a finite family of continuous pseudocontractive mappings, let $\{A_i : C \to H, i = 1, 2, \ldots, N\}$ be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \ne \emptyset$, and let $f : C \to C$ be contraction with a contraction coefficient $\rho \in (0, 1)$. T_{ir_n} and F_{ir_n} are defined as (21) and (22), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$:*

$$\begin{aligned} y_n &= P_C\left(\varepsilon_n u_n + (1 - \varepsilon_n)\sum_{i=1}^N \mu_i F_{ir_n} x_n\right), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^N \sigma_i T_{ir_n} y_n, \end{aligned} \tag{23}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\varepsilon_n\}$ are sequences of nonnegative real numbers in $[0, 1]$ and $\mu_i \ge 0$, $\sigma_i \ge 0$, $i = 1, 2, \ldots, N$, and the sequence $\{u_n\} \subset H$ is a small perturbation such that

(i) *$\alpha_n + \beta_n + \gamma_n = 1$, $n \ge 0$, $\sum_{i=1}^N \mu_i = 1$, and $\sum_{i=1}^N \sigma_i = 1$;*

(ii) *$\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\lim_{n\to\infty}\varepsilon_n = 0$;*

(iii) *$0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$;*

(iv) *$\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$, $\lim_{n\to\infty}\|u_n\| = 0$, and $\sum_{n=0}^\infty \varepsilon_n \|u_n\| < \infty$.*

Then the sequence $\{x_n\}$ converges strongly to an element $w = \Pi_F f(w)$ and also w is the unique solution of the variational inequality

$$\langle (f - I) w, y - w\rangle \le 0, \quad \forall y \in F. \tag{24}$$

Proof. By using Lemmas 3 and 5, the mappings $\sum_{i=1}^{N}\mu_i F_{ir_n}$ and $\sum_{i=1}^{N}\sigma_i T_{ir_n}$ are well defined. First we prove that $\{x_n\}$ is bounded. Take $p \in F$, because F_{ir_n}, P_C are nonexpansive; then we have that

$$\begin{aligned}\|y_n - p\| &= \left\|P_C\left(\varepsilon_n u_n + (1-\varepsilon_n)\sum_{i=1}^{N}\mu_i F_{ir_n}x_n - p\right)\right\| \\ &\le \varepsilon_n\left[\|u_n\| + \|p\|\right] + (1-\varepsilon_n)\sum_{i=1}^{N}\mu_i\left\|F_{ir_n}x_n - F_{ir_n}p\right\| \\ &\le \varepsilon_n\left[\|u_n\| + \|p\|\right] + (1-\varepsilon_n)\|x_n - p\|.\end{aligned} \tag{25}$$

For $n \ge 0$, because T_{ir_n} and F_{ir_n} are nonexpansive and f is contractive, we have from (25) that

$$\begin{aligned}\|x_{n+1} - p\| &= \left\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n\sum_{i=1}^{N}\sigma_i T_{ir_n}y_n - p\right\| \\ &\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| \\ &\quad + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\le (\rho\alpha_n + \beta_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\quad + \gamma_n\left[\varepsilon_n(\|u_n\| + \|p\|) + (1-\varepsilon_n)\|x_n - p\|\right] \\ &\le (1 - \alpha_n + \rho\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\quad + \varepsilon_n(\|u_n\| + \gamma_n\varepsilon_n\|p\|) \\ &\le (1 - (1-\rho)\alpha_n - \gamma_n\varepsilon_n)\|x_n - p\| + \gamma_n\varepsilon_n\|p\| \\ &\quad + (1-\rho)\alpha_n\left(\frac{1}{1-\rho}\|f(p) - p\|\right) + \varepsilon_n\|u_n\| \\ &\le \max\left\{\|x_n - p\|, \|p\|, \frac{1}{1-\rho}\|f(p) - p\|\right\} \\ &\quad + \varepsilon_n\|u_n\|.\end{aligned} \tag{26}$$

This implies that

$$\begin{aligned}\|x_n - p\| \le \max&\left\{\|x_0 - p\|, \|p\|, \frac{1}{1-\rho}\|f(p) - p\|\right\} \\ &+ \Sigma_{i=0}^{n-1}\varepsilon_n\|u_n\|.\end{aligned} \tag{27}$$

Notice condition (iv); therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{F_{ir_n}x_n\}$, $\{T_{ir_n}y_n\}$ and $\{y_n\}$, $\{f(x_n)\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$. We have from (23) that

$$\begin{aligned}\|y_{n+1} - y_n\| \le\ &\varepsilon_{n+1}\|u_{n+1} - u_n\| \\ &+ (1-\varepsilon_{n+1})\Sigma_{i=1}^{N}\mu_i\left\|F_{ir_{n+1}}x_{n+1} - F_{ir_n}x_n\right\| \\ &+ \left|\varepsilon_{n+1} - \varepsilon_n\right|\left\|u_n - \Sigma_{i=1}^{N}\mu_i F_{ir_n}x_n\right\|.\end{aligned} \tag{28}$$

Let $v_{i,n} = F_{ir_n}x_n$, $v_{i,n+1} = F_{ir_{n+1}}x_{n+1}$; by the definition of mapping F_{ir_n}, we have that

$$\left\langle y - v_{i,n}, A_i v_{i,n}\right\rangle + \frac{1}{r_n}\left\langle y - v_{i,n}, v_{i,n} - x_n\right\rangle \ge 0, \quad \forall y \in C. \tag{29}$$

$$\left\langle y - v_{i,n+1}, A_i v_{i,n+1}\right\rangle + \frac{1}{r_{n+1}}\left\langle y - v_{i,n+1}, v_{i,n+1} - x_{n+1}\right\rangle \ge 0, \quad \forall y \in C. \tag{30}$$

Putting $y := v_{i,n+1}$ in (29) and letting $y := v_{i,n}$ in (30), we have that

$$\left\langle v_{i,n+1} - v_{i,n}, A_i v_{i,n}\right\rangle + \frac{1}{r_n}\left\langle v_{i,n+1} - v_{i,n}, v_{i,n} - x_n\right\rangle \ge 0, \tag{31}$$

$$\left\langle v_{i,n} - v_{i,n+1}, A_i v_{i,n+1}\right\rangle + \frac{1}{r_{n+1}}\left\langle v_{i,n} - v_{i,n+1}, v_{i,n+1} - x_{n+1}\right\rangle \ge 0. \tag{32}$$

Adding (31) and (32), we have that

$$\begin{aligned}&\left\langle v_{i,n+1} - v_{i,n}, A_i v_{i,n} - A_i v_{i,n+1}\right\rangle \\ &\quad + \left\langle v_{i,n+1} - v_{i,n}, \frac{v_{i,n} - x_n}{r_n} - \frac{v_{i,n+1} - x_{n+1}}{r_{n+1}}\right\rangle \ge 0.\end{aligned} \tag{33}$$

Since $\{A_i, i = 1, 2, \ldots, N\}$ are monotone mappings, which implies that

$$\left\langle v_{i,n+1} - v_{i,n}, \frac{v_{i,n} - x_n}{r_n} - \frac{v_{i,n+1} - x_{n+1}}{r_{n+1}}\right\rangle \ge 0, \tag{34}$$

we have that

$$\begin{aligned}&\left\langle v_{i,n+1} - v_{i,n}, v_{i,n} - x_n - \frac{r_n(v_{i,n+1} - x_{n+1})}{r_{n+1}} + v_{i,n+1} - v_{i,n+1}\right\rangle \\ &\quad \ge 0;\end{aligned} \tag{35}$$

that is,

$$\begin{aligned}&\left\|v_{i,n+1} - v_{i,n}\right\|^2 \\ &\le \left\langle v_{i,n+1} - v_{i,n}, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(v_{i,n+1} - x_{n+1})\right\rangle \\ &\le \left\|v_{i,n+1} - v_{i,n}\right\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\left\|v_{i,n+1} - x_{n+1}\right\|\right\}.\end{aligned} \tag{36}$$

Without loss of generality, let b be a real number such that $r_n > b > 0$, for all $n \in N$; then we have that

$$\begin{aligned}\left\|v_{i,n+1} - v_{i,n}\right\| &\le \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\left\|v_{i,n+1} - x_{n+1}\right\| \\ &\le \|x_{n+1} - x_n\| + \frac{1}{b}\left|r_{n+1} - r_n\right|K,\end{aligned} \tag{37}$$

where $K = \sup\{\|v_{i,n+1} - x_{n+1}\|, i = 1, 2, \ldots, N\}$. Then we have from (37) and (28) that

$$\begin{aligned}\|y_{n+1} - y_n\| \leq{} & (1 - \varepsilon_{n+1}) \|x_{n+1} - x_n\| + \varepsilon_{n+1} \|u_{n+1} - u_n\| \\ & + \frac{(1 - \varepsilon_{n+1}) |r_{n+1} - r_n|}{b} K \\ & + |\varepsilon_{n+1} - \varepsilon_n| \left\|u_n - \Sigma_{i=1}^N \mu_i F_{ir_n} x_n\right\|.\end{aligned} \tag{38}$$

On the other hand, let $w_{i,n} = T_{ir_n} y_n$, $w_{i,n+1} = T_{ir_{n+1}} y_{n+1}$; we have that

$$\langle y - w_{i,n}, T_i w_{i,n}\rangle - \frac{1}{r_n} \langle y - w_{i,n}, (1 + r_n) w_{i,n} - y_n\rangle \leq 0, \quad \forall y \in C, \tag{39}$$

$$\begin{aligned}&\langle y - w_{i,n+1}, T_i w_{i,n+1}\rangle \\ &\quad - \frac{1}{r_{n+1}} \langle y - w_{i,n+1}, (1 + r_{n+1}) w_{i,n+1} - y_{n+1}\rangle \leq 0, \quad \forall y \in C.\end{aligned} \tag{40}$$

Let $y := w_{i,n+1}$ in (39) and let $y := w_{i,n}$ in (40); we have that

$$\begin{aligned}&\langle w_{n+1} - w_{i,n}, T_i w_{i,n}\rangle \\ &\quad - \frac{1}{r_n} \langle w_{n+1} - w_{i,n}, (1 + r_n) w_{i,n} - y_n\rangle \leq 0,\end{aligned} \tag{41}$$

$$\begin{aligned}&\langle w_{i,n} - w_{i,n+1}, T_i w_{i,n+1}\rangle \\ &\quad - \frac{1}{r_{n+1}} \langle w_{i,n} - w_{i,n+1}, (1 + r_{n+1}) w_{i,n+1} - y_{n+1}\rangle \leq 0.\end{aligned} \tag{42}$$

Adding (41) and (42) and because $\{T_i, i = 1, 2, \ldots, N\}$ are pseudocontractive mappings, we have that

$$\left\langle w_{i,n+1} - w_{i,n}, \frac{w_{i,n} - y_n}{r_n} - \frac{w_{i,n+1} - y_{n+1}}{r_{n+1}}\right\rangle \geq 0. \tag{43}$$

Therefore we have

$$\begin{aligned}&\Bigg\langle w_{i,n+1} - w_{i,n}, w_{i,n} - y_n - \frac{r_n (w_{i,n+1} - y_{n+1})}{r_{n+1}} \\ &\qquad + w_{i,n+1} - w_{i,n+1}\Bigg\rangle \geq 0.\end{aligned} \tag{44}$$

Hence we have that

$$\|w_{i,n+1} - w_{i,n}\| \leq \|y_{n+1} - y_n\| + \frac{1}{b} |r_{n+1} - r_n| M, \tag{45}$$

where $M = \sup\{\|w_{i,n} - y_n\|, i = 1, 2, \ldots, N\}$.

Let $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$. Hence we have that

$$\begin{aligned}z_{n+1} - z_n ={} & \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (f(x_{n+1}) - f(x_n)) \\ & + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right) f(x_n) \\ & + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \Sigma_{i=1}^N \sigma_i (w_{i,n+1} - w_{i,n}) \\ & + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right) \Sigma_{i=1}^N \sigma_i w_{i,n}.\end{aligned} \tag{46}$$

Then we have from (46), (45), and (38) that

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\quad \leq \frac{(\rho - 1) \alpha_{n+1}}{1 - \beta_{n+1}} \|x_{n+1} - x_n\| \\ &\qquad + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right| \left\{\|f(x_n)\| + \left\|\Sigma_{i=1}^N \sigma_i w_{i,n}\right\|\right\} \\ &\qquad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \frac{|r_{n+1} - r_n|}{b} ((1 - \varepsilon_{n+1}) K + M) \\ &\qquad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} |\varepsilon_{n+1} - \varepsilon_n| \left\|u_n - \Sigma_{i=1}^N \mu_i F_{ir_n} x_n\right\| \\ &\qquad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|u_{n+1} - u_n\|.\end{aligned} \tag{47}$$

Notice conditions (ii), (iii), and (iv); we have that

$$\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) = 0. \tag{48}$$

Hence we have from Lemma 4 that

$$\limsup_{n \to \infty} \|z_n - x_n\| = 0. \tag{49}$$

Therefore we have that

$$\|x_{n+1} - x_n\| = |1 - \beta_n| \|z_n - x_n\| \longrightarrow 0. \tag{50}$$

Hence we have from (37), (38), and (45) that

$$\begin{gathered}\|y_{n+1} - y_n\| \longrightarrow 0, \qquad \|w_{i,n+1} - w_{i,n}\| \longrightarrow 0, \\ \|v_{i,n+1} - v_{i,n}\| \longrightarrow 0.\end{gathered} \tag{51}$$

In addition, since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \Sigma_{i=1}^N \sigma_i w_{i,n}$, $y_n = P_C(\varepsilon_n u_n + (1 - \varepsilon_n) \Sigma_{i=1}^N \mu_i v_{i,n})$, for all $p \in F$, we have from the monotonicity of A_i, the nonexpansivity of T_{ir_n}, and the convexity of $\|\cdot\|^2$ that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad = \left\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n \Sigma_{i=1}^N \sigma_i w_{i,n}-p\right\|^2 \\
&\quad \le \left\|\alpha_n (f(x_n)-p)+\beta_n (x_n-p)\right\|^2+\gamma_n\left\|\Sigma_{i=1}^N \sigma_i w_{i,n}-p\right\|^2 \\
&\quad \le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2 \\
&\quad \le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\varepsilon_n\|u_n-p\|^2 \\
&\qquad + (1-\varepsilon_n)\gamma_n\Sigma_{i=1}^N \mu_i\|v_{i,n}-p\|^2 \\
&\quad \le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\varepsilon_n\|u_n-p\|^2 \\
&\qquad + (1-\varepsilon_n)\gamma_n\Sigma_{i=1}^N \mu_i\left(\|x_n-p\|^2-\|x_n-v_{i,n}\|^2\right) \\
&\quad \le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2 \\
&\qquad - (1-\varepsilon_n)\gamma_n\Sigma_{i=1}^N \mu_i\|x_n-v_{i,n}\|^2+\gamma_n\varepsilon_n\|u_n-p\|^2.
\end{aligned} \tag{52}$$

So we have that

$$\begin{aligned}
(1-\varepsilon_n)\gamma_n\Sigma_{i=1}^N \mu_i\|x_n-v_{i,n}\|^2 &\le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2 \\
&\quad -\|x_{n+1}-p\|^2+\gamma_n\varepsilon_n\|u_n-p\|^2 \\
&\le \alpha_n\|f(x_n)-p\|^2+\|x_n-x_{n+1}\| \\
&\quad \times(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\quad +\gamma_n\varepsilon_n\|u_n-p\|^2.
\end{aligned} \tag{53}$$

Since $\alpha_n \to 0$, $\varepsilon_n \to 0$, we have from (50) that

$$\|x_n-v_{i,n}\| \longrightarrow 0. \tag{54}$$

In a similar way, we have that

$$\|x_n-w_{i,n}\| \longrightarrow 0. \tag{55}$$

Consequently, we have that

$$\begin{aligned}
&\|y_n-x_n\| \le |(1-\varepsilon_n)|\,\Sigma_{i=1}^N \mu_i\,\|x_n-v_{i,n}\| \longrightarrow 0, \\
&\|y_n-w_{i,n}\| \le \|y_n-x_n\|+\|x_n-w_{i,n}\| \longrightarrow 0.
\end{aligned} \tag{56}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ and $w \in C$ such that $x_{nk} \to w$ weakly. And because $x_n \to v_{i,n}$, $v_{i,nk} \to w$ weakly. Next we show that $w \in F$.

Because $v_{i,n} = F_{ir_n}x_n$, by the definition of mapping F_{ir_n}, we have that

$$\begin{aligned}
&\langle y-v_{i,n}, A_i v_{i,n}\rangle+\frac{1}{r_n}\langle y-v_{i,n}, v_{i,n}-x_n\rangle \ge 0, \quad \forall y \in C, \\
&\langle y-v_{i,nk}, A_i v_{i,nk}\rangle+\left\langle y-v_{i,nk}, \frac{v_{i,nk}-x_{nk}}{r_n}\right\rangle \ge 0, \quad \forall y \in C.
\end{aligned} \tag{57}$$

Let $v_t = tv+(1-t)w$, $t \in [0,1]$, for all $v \in C$; we have that

$$\begin{aligned}
\langle v_t-v_{i,nk}, A_i v_t\rangle &\ge \langle v_t-v_{i,nk}, A_i v_t\rangle-\langle v_t-v_{i,nk}, A_i v_{i,nk}\rangle \\
&\quad -\left\langle v_t-v_{i,nk}, \frac{v_{i,nk}-x_{nk}}{r_n}\right\rangle \\
&= \langle v_t-v_{i,nk}, A_i v_t-A_i v_{i,nk}\rangle \\
&\quad -\left\langle v_t-v_{i,nk}, \frac{v_{i,nk}-x_{nk}}{r_n}\right\rangle.
\end{aligned} \tag{58}$$

Because $\{A_i, i=1,2,\ldots,N\}$ are monotone and because $x_{nk}-v_{i,nk} \to 0$, we have that

$$0 \le \lim_{k\to\infty}\langle v_t-v_{i,nk}, A_i v_t\rangle = \langle v_t-w, A_i v_t\rangle. \tag{59}$$

Consequently we have that

$$\langle v-w, A_i v_t\rangle \ge 0. \tag{60}$$

If $t \to 0$, by the continuity of A_i, we have that $\langle v-w, A_i w\rangle \ge 0$; that is, $w \in \mathrm{VI}(C, A_i)$ and then $w \in F_2$.

Similarly, because $w_{i,n} = T_{ir_n}y_n$, by the definition of mapping T_{ir_n}, we have that

$$\begin{aligned}
&\langle y-w_{i,n}, T_i w_{i,n}\rangle-\frac{1}{r_n}\langle y-w_{i,n}, (1+r_n)w_{i,n}-y_n\rangle \le 0, \\
&\qquad\qquad \forall y \in C. \\
&\langle y-w_{i,nk}, T_i w_{i,nk}\rangle-\frac{1}{r_n}\langle y-w_{i,nk}, (1+r_n)w_{i,nk}-y_{nk}\rangle \le 0, \\
&\qquad\qquad \forall y \in C.
\end{aligned} \tag{61}$$

Let $v_t = tv+(1-t)w$, $t \in [0,1]$, for all $v \in C$. Because $\{T_i, i=1,2,\ldots,N\}$ are pseudocontractive mappings, we have that

$$\begin{aligned}
&\langle w_{i,nk}-v_t, T_i v_t\rangle \\
&\quad \ge \langle w_{i,nk}-v_t, T_i v_t\rangle+\langle v_t-w_{i,nk}, T_i w_{i,nk}\rangle \\
&\qquad -\frac{1}{r_n}\langle v_t-w_{i,nk}, (1+r_n)w_{i,nk}-y_{nk}\rangle \\
&\quad = \langle v_t-w_{i,nk}, T_i w_{i,nk}-T_i v_t\rangle \\
&\qquad -\left\langle v_t-w_{i,nk}, \frac{1+r_n}{r_n}w_{i,nk}-\frac{1}{r_n}y_{nk}\right\rangle \\
&\quad \ge -\|v_t-w_{i,nk}\|^2-\frac{1}{r_n}\langle v_t-w_{i,nk}, w_{i,nk}-y_{nk}\rangle \\
&\qquad -\langle v_t-w_{i,nk}, w_{i,nk}\rangle \\
&\quad = -\langle v_t-w_{i,nk}, v_t\rangle-\frac{1}{r_n}\langle v_t-w_{i,nk}, w_{i,nk}-y_{nk}\rangle.
\end{aligned} \tag{62}$$

Because $y_{nk} - w_{i,nk} \to 0$, so we have that

$$\lim_{k\to\infty} \langle w_{i,nk} - v_t, T_i v_t \rangle \geq \lim_{k\to\infty} \langle w_{i,nk} - v_t, v_t \rangle . \quad (63)$$

Consequently we have that

$$\langle w - v_t, T_i v_t \rangle \geq \langle w - v_t, v_t \rangle ; \quad (64)$$

that is,

$$\langle v - w, T_i v_t \rangle \leq \langle v - w, v_t \rangle . \quad (65)$$

If $t \to 0$, by the continuity of T_i, we have that $\langle v - w, T_i w - w \rangle \leq 0$, for all $v \in C$; we conclude that $w = T_i w$; that is, $w \in F(T_i)$ and then $w \in F_1$. Consequently $w \in F = F_1 \cap F_2$.

Denote $x^* = \Pi_F f(w)$; then $x^* \in F$ is the unique element that satisfies $\inf_{x\in F}\|x - f(w)\| = \|x^* - f(w)\|$. From Lemma 1, we have that $\langle f(w) - x^*, y - x^* \rangle \leq 0$, for all $y \in C$. If we take $y = f(w)$, then $\langle f(w) - x^*, f(w) - x^* \rangle \leq 0$; consequently we have that $f(w) = x^*$.

By using the weakly lower semicontinuity of the norm on H, we get that

$$\begin{aligned}\left\|x^* - f(w)\right\| &\leq \left\|w - f(w)\right\| \leq \liminf_{k\to\infty} \left\|x_{n_k} - f(w)\right\| \\ &\leq \limsup_{k\to\infty} \left\|x_{n_k} - f(w)\right\| \\ &\leq \inf_{x\in F} \left\|x - f(w)\right\| = \left\|x^* - f(w)\right\| ,\end{aligned} \quad (66)$$

which implies that

$$\begin{aligned}\lim_{n\to\infty} \left\|x_n - f(w)\right\| &= \left\|x^* - f(w)\right\| = \left\|w - f(w)\right\| \\ &= \inf_{x\in F} \left\|x - f(w)\right\| .\end{aligned} \quad (67)$$

Thus, from Lemma 1, we have that

$$\langle z - x^*, x^* - f(w) \rangle \geq 0, \quad \forall z \in C, \quad (68)$$

$$\langle z - w, w - f(w) \rangle \geq 0, \quad \forall z \in C. \quad (69)$$

Putting $z := w$ in (68) and $z := x^*$ in (69), we get that

$$\langle w - x^*, x^* - f(w) \rangle \geq 0, \quad \forall z \in C, \quad (70)$$

$$\langle x^* - w, w - f(w) \rangle \geq 0, \quad \forall z \in C. \quad (71)$$

Adding (70) and (71) we get that $\langle x^* - w, x^* - w \rangle \leq 0$; that is, $\|w - x^*\|^2 \leq 0$; thus $w = x^*$. Furthermore, from (67), we get that the sequence $x_n \to w = P_F f(w)$ strongly and w is the solution of the following variational inequality:

$$\langle z - w, (f - I) w \rangle \leq 0, \quad \forall z \in C. \quad (72)$$

Now we show that w is the unique solution of the variational inequality $\langle z - w, (f - I)w \rangle \leq 0$, for all $z \in C$. Suppose that $\overline{w} \in F$ is another solution of the variational inequality; that is,

$$\langle z - \overline{w}, (f - I) \overline{w} \rangle \leq 0, \quad \forall y \in C. \quad (73)$$

Let $z := \overline{w}$ in (72) and let $z := w$ in (73); we have that

$$\langle \overline{w} - w, f(w) - w \rangle \leq 0, \quad (74)$$

$$\langle w - \overline{w}, f(\overline{w}) - \overline{w} \rangle \leq 0. \quad (75)$$

Adding (74) and (75), we have that

$$\langle \overline{w} - w - (f(\overline{w}) - f(w)), \overline{w} - w \rangle \leq 0. \quad (76)$$

Hence

$$(1 - \rho) \|\overline{w} - w\|^2 \leq 0. \quad (77)$$

Because $\rho \in (0, 1)$, we conclude that $\overline{w} = w$; the uniqueness of the solution is obtained. The proof is complete. □

Theorem 7. *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Hilbert space H. Let $\{T_i : C \to C, i = 1, 2, \ldots, N\}$ be a finite family of continuous pseudocontractive mappings, let $\{A_i : C \to H, i = 1, 2, \ldots, N\}$ be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$, and let $f : C \to C$ be a contraction with a contraction coefficient $\rho \in (0, 1)$. T_{ir_n} and F_{ir_n} are defined as (21) and (22), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$,*

$$\begin{aligned} y_n &= (1 - \lambda_n) x_n + \lambda_n P_C \left(\varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{ir_n} x_n \right), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i T_{ir_n} y_n, \end{aligned} \quad (78)$$

where $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\varepsilon_n\}$ are sequences of nonnegative real numbers in $[0, 1]$ and $\mu_i \geq 0$, $\sigma_i \geq 0$, $i = 1, 2, \ldots, N$, and the sequence $\{u_n\} \subset H$ is a small perturbation such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$, $\sum_{i=1}^{N} \mu_i = 1$, *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\lim_{n\to\infty}\varepsilon_n = 0$;
(iii) $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$;
(iv) $\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\lim_{n\to\infty}\|u_n\| = 0$, *and* $\sum_{n=0}^{\infty} \varepsilon_n \|u_n\| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element $w = \Pi_F f(w)$ and also $\overline{x}$ is the unique solution of the variational inequality

$$\langle (f - I)(w), y - w \rangle \leq 0, \quad \forall y \in F. \quad (79)$$

Proof. Take $p \in F$; because F_{ir_n}, P_C are nonexpansive, then we have that

$$\begin{aligned} &\|y_n - p\| \\ &\quad = \left\| (1 - \lambda_n) x_n + \lambda_n P_C \left(\varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{ir_n} x_n - p \right) \right\| \\ &\quad \leq \varepsilon_n \left[\|u_n\| + \lambda_n \|p\| \right] + (1 - \lambda_n \varepsilon_n) \|x_n - p\| . \end{aligned} \quad (80)$$

For $n \geq 0$, because T_{ir_n} and F_{ir_n} are nonexpansive and f is contractive, we have from (28) that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \left\| \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i T_{ir_n} y_n - p \right\| \\
&\quad \leq \alpha_n \| f(x_n) - f(p) \| + \alpha_n \| f(p) - p \| \\
&\qquad + \beta_n \| x_n - p \| + \gamma_n \| y_n - p \| \\
&\quad \leq (\rho \alpha_n + \beta_n) \| x_n - p \| + \alpha_n \| f(p) - p \| \\
&\qquad + \gamma_n \left[\varepsilon_n \left(\|u_n\| + \lambda_n \|p\| \right) + (1 - \lambda_n \varepsilon_n) \| x_n - p \| \right] \\
&\quad \leq (1 - \alpha_n + \rho \alpha_n) \| x_n - p \| + \alpha_n \| f(p) - p \| \\
&\qquad + \varepsilon_n \left(\|u_n\| + \gamma_n \varepsilon_n \|p\| \right) \\
&\quad \leq (1 - (1 - \rho) \alpha_n - \gamma_n \lambda_n \varepsilon_n) \| x_n - p \| + \gamma_n \varepsilon_n \lambda_n \|p\| \\
&\qquad + (1 - \rho) \alpha_n \left(\frac{1}{1 - \rho} \| f(p) - p \| \right) + \varepsilon_n \|u_n\| \\
&\quad \leq \max \left\{ \| x_n - p \|, \|p\|, \frac{1}{1 - \rho} \| f(p) - p \| \right\} + \varepsilon_n \|u_n\|.
\end{aligned} \tag{81}$$

This implies that

$$\begin{aligned}
\| x_n - p \| \leq{}& \max \left\{ \| x_0 - p \|, \|p\|, \frac{1}{1 - \rho} \| f(p) - p \| \right\} \\
&+ \textstyle\sum_{i=0}^{n-1} \varepsilon_n \|u_n\|.
\end{aligned} \tag{82}$$

Notice condition (iv); therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{F_{ir_n} x_n\}$, $\{T_{ir_n} y_n\}$ and $\{y_n\}$, $\{f(x_n)\}$, and $P_C(\varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{ir_n} x_n)$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$. Denote $\tau_n = \varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{ir_n} x_n$; then we get that

$$\begin{aligned}
\| y_{n+1} - y_n \| \leq{}& (1 - \lambda_{n+1}) \| x_{n+1} - x_n \| + \lambda_{n+1} \| \tau_{n+1} - \tau_n \| \\
&+ |\lambda_{n+1} - \lambda_n| \, \| x_n - P_C \tau_n \|, \\
\| \tau_{n+1} - \tau_n \| \leq{}& \varepsilon_{n+1} \left[\|u_{n+1}\| + \|u_n\| \right] \\
&+ (1 - \varepsilon_{n+1}) \sum_{i=1}^{N} \mu_i \left\| F_{ir_{n+1}} x_{n+1} - F_{ir_n} x_n \right\| \\
&+ |\varepsilon_{n+1} - \varepsilon_n| \left\| u_n - \sum_{i=1}^{N} \mu_i F_{ir_n} x_n \right\|.
\end{aligned} \tag{83}$$

Repeating equations from (29) to (38), we have that

$$\begin{aligned}
\| \tau_{n+1} - \tau_n \| \leq{}& \varepsilon_{n+1} \left[\|u_{n+1}\| + \|u_n\| \right] \\
&+ (1 - \varepsilon_{n+1}) \| x_{n+1} - x_n \| \\
&+ (1 - \varepsilon_{n+1}) \frac{|r_{n+1} - r_n| K}{b} \\
&+ |\varepsilon_{n+1} - \varepsilon_n| \left\| u_n - \sum_{i=1}^{N} \mu_i F_{ir_n} x_n \right\|.
\end{aligned} \tag{84}$$

Therefore,

$$\begin{aligned}
\| y_{n+1} - y_n \| \leq{}& (1 - \lambda_{n+1}) \| x_{n+1} - x_n \| \\
&+ \lambda_{n+1} \Bigg(\varepsilon_{n+1} \left[\|u_{n+1}\| + \|u_n\| \right] \\
&\qquad + (1 - \varepsilon_{n+1}) \| x_{n+1} - x_n \| \\
&\qquad + (1 - \varepsilon_{n+1}) \frac{|r_{n+1} - r_n| K}{b} \\
&\qquad + |\varepsilon_{n+1} - \varepsilon_n| \left\| u_n - \sum_{i=1}^{N} \mu_i F_{ir_n} x_n \right\| \Bigg) \\
&+ |\lambda_{n+1} - \lambda_n| \, \| x_n - P_C \tau_n \| \\
\leq{}& (1 - \lambda_{n+1} \varepsilon_{n+1}) \| x_{n+1} - x_n \| \\
&+ \lambda_{n+1} \varepsilon_{n+1} \left[\|u_{n+1}\| + \|u_n\| \right] \\
&+ \lambda_{n+1} (1 - \varepsilon_{n+1}) \frac{|r_{n+1} - r_n| K}{b} \\
&+ \lambda_{n+1} |\varepsilon_{n+1} - \varepsilon_n| \left\| u_n - \sum_{i=1}^{N} \mu_i F_{ir_n} x_n \right\| \\
&+ |\lambda_{n+1} - \lambda_n| \, \| x_n - P_C \tau_n \|.
\end{aligned} \tag{85}$$

Similar to the rest of the proof of Theorem 6, we obtain the result. □

If, in Theorems 6 and 7, we let $f :\equiv u \in C$ be a constant mapping, we have the following corollaries.

Corollary 8. *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Hilbert space H. Let $\{T_i : C \to C, i = 1, 2, \ldots, N\}$ be a finite family of continuous pseudocontractive mappings, let $\{A_i : C \to H, i = 1, 2, \ldots, N\}$ be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$, and let $u \in C$ be a constant.*

T_{ir_n} and F_{ir_n} are defined as (21) and (22), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$,

$$y_n = P_C\left(\varepsilon_n u_n + (1-\varepsilon_n)\sum_{i=1}^{N}\mu_i F_{ir_n} x_n\right),$$
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{N}\sigma_i T_{ir_n} y_n, \quad (86)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\varepsilon_n\}$ are sequences of nonnegative real numbers in $[0,1]$ and $\mu_i \geq 0$, $\sigma_i \geq 0$, $i = 1,2,\ldots,N$, and the sequence $\{u_n\} \subset H$ is a small perturbation such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$, $\sum_{i=1}^{N}\mu_i = 1$, *and* $\sum_{i=1}^{N}\sigma_i = 1$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, *and* $\lim_{n\to\infty}\varepsilon_n = 0$;

(iii) $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$, $\lim_{n\to\infty}\|u_n\| = 0$, *and* $\sum_{n=0}^{\infty}\varepsilon_n\|u_n\| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element $w = \Pi_F u$ and also $\overline{x}$ is the unique solution of the variational inequality

$$\langle u - w, y - w\rangle \leq 0, \quad \forall y \in F. \quad (87)$$

Corollary 9. *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Hilbert space H. Let $\{T_i : C \to C, i = 1,2,\ldots,N\}$ be a finite family of continuous pseudocontractive mappings, let $\{A_i : C \to H, i = 1,2,\ldots,N\}$ be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$, and let $u \in C$ be a constant. T_{ir_n} and F_{ir_n} are defined as (21) and (22), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$,*

$$y_n = (1-\lambda_n)x_n + \lambda_n P_C\left(\varepsilon_n u_n + (1-\varepsilon_n)\sum_{i=1}^{N}\mu_i F_{ir_n} x_n\right),$$
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{N}\sigma_i T_{ir_n} y_n, \quad (88)$$

where $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\varepsilon_n\}$ are sequences of nonnegative real numbers in $[0,1]$ and $\mu_i \geq 0$, $\sigma_i \geq 0$, $i = 1,2,\ldots,N$, and the sequence $\{u_n\} \subset H$ is a small perturbation such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$, $\sum_{i=1}^{N}\mu_i = 1$, *and* $\sum_{i=1}^{N}\sigma_i = 1$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, *and* $\lim_{n\to\infty}\varepsilon_n = 0$;

(iii) $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\limsup_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$, $\lim_{n\to\infty}\|u_n\| = 0$, *and* $\sum_{n=0}^{\infty}\varepsilon_n\|u_n\| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element $w = \Pi_F u$ and also $\overline{x}$ is the unique solution of the variational inequality

$$\langle u - w, y - w\rangle \leq 0, \quad \forall y \in F. \quad (89)$$

Remark 10. If $\{u_n\} \subset C$, then sequence (23) reduces to

$$y_n = \varepsilon_n u_n + (1-\varepsilon_n)\sum_{i=1}^{N}\mu_i F_{ir_n} x_n,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N}\sigma_i T_{ir_n} y_n \quad (90)$$

and sequence (78) reduces to

$$y_n = (1-\lambda_n)x_n + \lambda_n\left(\varepsilon_n u_n + (1-\varepsilon_n)\sum_{i=1}^{N}\mu_i F_{ir_n} x_n\right),$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N}\sigma_i T_{ir_n} y_n. \quad (91)$$

The conclusions of Theorems 6 and 7 are true under the same conditions.

Remark 11. Our theorems extend and unify some of the results that have been proved for these important classes of nonlinear operators. In particular, Theorem 6 extends Theorem 6 of Yao and Shahzad [24] in the sense that our convergence is for the more general class of continuous pseudocontractive and continuous monotone mappings. Theorem 6 also extends Theorem 3.2 of Tang [22] in the sense that our convergence is for the more general algorithm with perturbations.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is supported by the Natural Science Foundation Project of Chongqing (CSTC, 2012jjA00039) and the Science and Technology Research Project of Chongqing Municipal Education Commission (KJ 1400614).

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