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Research Article

A Generalized System of Nonlinear Variational Inequalities in Banach Spaces

Prapairat Junlouchai, Anchalee Kaewcharoen, and Somyot Plubtieng

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Somyot Plubtieng; somyotp@nu.ac.th

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We introduce a new generalized system of nonlinear variational inequality problems (GSNVIP) by using the generalized projection method. Moreover, we introduce an iterative scheme for finding a solution to this problem. Moreover, some existence and strong convergence theorems are established in uniformly smooth and strictly convex Banach spaces under suitable conditions. The results presented in the paper improve and extend some recent results.

1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on differential equations, general equilibrium problems in economics and mechanics, control problems, and transportation. In 2005, Verma [1] introduced a general model for two-step projection methods and applied it to the approximation solvability of a system of nonlinear variational inequality problems in a Hilbert space. Based on the convergence of projection methods, Chang et al. [2] introduced and studied the approximate solvability of a generalized system for relaxed cocoercive nonlinear variational inequalities in Hilbert spaces (see, for instance, [3–5] and the references therein). Recently, Chang et al. [6] introduced a system of generalized nonlinear variational inequalities and an iterative scheme for finding a solution to a system of generalized nonlinear variational inequality problems by using the generalized projection method. Moreover, they proved some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces.

In this paper, we introduce a generalized system of nonlinear variational inequality problems (GSNVIP) by using the generalized projection approach to introduce an iterative scheme for finding a solution to this problem. Finally, we prove some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces under suitable conditions.

2. Preliminaries

Let E be a real Banach space with dual space E^*, $\langle\cdot,\cdot\rangle$ the dual pair between E and E^*, and K a nonempty closed convex subset of E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad \forall x \in E. \tag{1}$$

A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}. \tag{2}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1 (see [7]). (i) If E is a uniformly smooth Banach space, then the normalized duality mapping J is uniformly continuous on each bounded subset of E.

(ii) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is a single valued bijective mapping.

(iii) If E is a smooth, strictly convex and reflexive Banach space and $J^* : E^* \to E$ is the duality mapping in E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$, and $J^*J = I_E$.

(iv) If E is a strictly convex and reflexive Banach space, then J^{-1} is hemicontinuous; that is, J^{-1} is norm-weak-continuous.

(v) E is uniformly smooth if and only if E^* is uniformly convex.

(vi) If E is a uniformly smooth and strictly convex Banach space with the Kadec-Klee property (i.e., for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$), then both the normalized duality mappings $J : E \to E^*$ and $J^* = J^{-1} : E^* \to E$ are continuous.

(vii) Each uniformly convex Banach space E has the Kadec-Klee property.

Assume that E is a smooth, strictly convex and reflexive Banach space and K is a nonempty closed convex subset of E; $\phi : E \times E \to \mathbb{R}^+ := [0, \infty)$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{3}$$

Following Alber [8], the generalized projection $\prod_K : E \to K$ is defined by $\prod_K x = z$, where z is the unique solution to the minimization problem

$$\phi(z, x) = \min_{y \in K} \phi(y, x). \tag{4}$$

The existence and uniqueness of the mapping $\prod_K$ follow from the property of the function $\phi(x, y)$ and the strict monotonicity of the mapping J.

Lemma 2 (see [8]). *Let E be a smooth, strictly convex and reflexive Banach space and K a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) *if $x \in E$ and $z \in K$, then*

$$z = \prod_K x \Longleftrightarrow \langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in K; \tag{5}$$

(b) *$\prod_K$ is a continuous mapping from E onto K.*

Remark 3. If E is a real Hilbert space, then $J = I$ (identity mapping), $\phi(x, y) = \|x - y\|^2$, and $\prod_K$ is the metric projection P_K from E onto K.

Lemma 4 (see [9, 10]). *Let E be a uniformly convex Banach space, $r > 0$ a positive number, and $B_r(0) := \{x \in E : \|x\| \leq r\}$ a closed ball of E. Then, for any given finite subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any given positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{n=1}^{N} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$ the following holds:*

$$\left\|\sum_{n=1}^{N} \lambda_n x_n\right\|^2 \leq \sum_{n=1}^{N} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g\left(\|x_i - x_j\|\right). \tag{6}$$

Lemma 5 (see [11]). *Let E be a real reflexive, smooth, and strictly convex Banach space. Then the following inequality holds:*

$$\|f + g\|^2 \leq \|f\|^2 + 2\left\langle g, J^{-1}(f + g)\right\rangle, \quad \forall f, g \in E^*. \tag{7}$$

Lemma 6 (see [6]). *Let E be a real Banach space, K a nonempty closed convex subset of E with $0 \in K$, and $\prod_K : E \to K$ the generalized projection. Then for each $x \in E$, one has $\|\prod_K x\| \leq \|x\|$.*

3. Main Results

In this section, we assume that E is a real Banach space with dual space E^* and K is a nonempty closed convex subset of E. Let $T_1, \ldots, T_N : K^N \to E^*$ be nonlinear mappings and $f : K \to E$ a mapping. The generalized system of nonlinear variational inequality problems (GSNVIP) is to find $x_1^*, \ldots, x_N^*$ such that for all $x \in K$

$$\begin{gathered} \left\langle f(x) - f(x_1^*), T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_2^*), T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right\rangle \geq 0, \\ \vdots \\ \left\langle f(x) - f(x_N^*), T_N(x_1^*, x_2^*, \ldots, x_{N-1}^*, x_N^*)\right\rangle \geq 0. \end{gathered} \tag{8}$$

If $N = 3$, $f = I$, and $T_1, T_2, T_3 : K^3 \to E^*$ are nonlinear mappings, then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem (see [6]) to find x_1^*, x_2^*, x_3^* such that, for all $x \in K$,

$$\begin{gathered} \left\langle x - x_1^*, T_1(x_2^*, x_3^*, x_1^*)\right\rangle \geq 0, \\ \left\langle x - x_2^*, T_2(x_3^*, x_1^*, x_2^*)\right\rangle \geq 0, \\ \left\langle x - x_3^*, T_3(x_1^*, x_2^*, x_3^*)\right\rangle \geq 0. \end{gathered} \tag{9}$$

If $N = 2$ and $T_1, T_2 : K^2 \to E^*$ are nonlinear mappings and $f : K \to E$ is a mapping, then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem to find x_1^*, x_2^* such that, for all $x \in K$,

$$\begin{gathered} \left\langle f(x) - f(x_1^*), T_1(x_2^*, x_1^*)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_2^*), T_2(x_1^*, x_2^*)\right\rangle \geq 0. \end{gathered} \tag{10}$$

If $T, S : K^2 \to E^*$ are nonlinear mappings and $g, f : K \to E$ are two mappings. Define $T_1, T_2 : K^2 \to E^*$ by $T_1(x_1^*, x_2^*) = \rho_1 T(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$ and $T_2(x_1^*, x_2^*) = \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$. Then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem to find $x_1^*, x_2^* \in K$ such that, for all $x \in K$,

$$\begin{aligned}&\langle f(x) - f(x_1^*), \rho_1 T(x_2^*, x_1^*) + g(x_2^*) - g(x_1^*)\rangle \geq 0,\\ &\langle f(x) - f(x_2^*), \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)\rangle \geq 0,\end{aligned} \tag{11}$$

where ρ_1 and ρ_2 are two positive constants.

Lemma 7. *Let E be a smooth, strictly convex, and reflexive Banach space and K a nonempty closed convex subset of E. Let $T_1, \ldots, T_N : K^N \to E^*$ be mappings, $f : K \to K$ a bijective mapping, and $\rho_1, \ldots, \rho_N$ any positive real numbers. Then $(x_1^*, \ldots, x_N^*) \in K^N$ is a solution to problem (8) if and only if $(x_1^*, \ldots, x_N^*) \in K^N$ is a solution to the following system of operator equations:*

$$\begin{aligned}
x_1^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right),\\
x_2^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right),\\
&\vdots\\
x_{N-1}^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right),\\
x_N^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right).
\end{aligned} \tag{12}$$

Proof. By Lemma 2, we have that $(x_1^*, \ldots, x_N^*) \in K^N$ is a solution of problem (8),

$$\Longleftrightarrow \begin{cases}\langle f(x) - f(x_1^*), \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\rangle \geq 0,\\ \langle f(x) - f(x_2^*), \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\rangle \geq 0,\\ \quad\vdots\\ \langle f(x) - f(x_{N-1}^*), \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\rangle \geq 0,\\ \langle f(x) - f(x_N^*), \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\rangle \geq 0,\end{cases}$$

$$\Longleftrightarrow \begin{cases}\langle f(x) - f(x_1^*), Jf(x_1^*) - Jf(x_1^*) + \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\rangle \geq 0,\\ \langle f(x) - f(x_2^*), Jf(x_2^*) - Jf(x_2^*) + \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\rangle \geq 0,\\ \quad\vdots\\ \langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) - Jf(x_{N-1}^*) + \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\rangle \geq 0,\\ \langle f(x) - f(x_N^*), Jf(x_N^*) - Jf(x_N^*) + \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\rangle \geq 0,\end{cases}$$

$$\Longleftrightarrow \begin{cases}\langle f(x) - f(x_1^*), Jf(x_1^*) - J\left(J^{-1}\left(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right)\right)\rangle \geq 0,\\ \langle f(x) - f(x_2^*), Jf(x_2^*) - J\left(J^{-1}\left(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right)\right)\rangle \geq 0,\\ \quad\vdots\\ \langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) - J\left(J^{-1}\left(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1} \times (x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right)\right)\rangle \geq 0,\\ \langle f(x) - f(x_N^*), Jf(x_N^*) - J\left(J^{-1}\left(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right)\right)\rangle \geq 0,\end{cases} \tag{13}$$

for all $x \in K$,

$$\Longleftrightarrow \begin{cases}f(x_1^*) = \prod_K J^{-1}\left(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right),\\ f(x_2^*) = \prod_K J^{-1}\left(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right),\\ \quad\vdots\\ f(x_{N-1}^*) = \prod_K J^{-1}\left(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1} \times (x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right),\\ f(x_N^*) = \prod_K J^{-1}\left(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right),\end{cases} \tag{14}$$

for any $\rho_1 > 0, \ldots, \rho_N > 0$,

$$\Longleftrightarrow \begin{cases} x_1^* = f^{-1}\prod_K J^{-1}\left(Jf\left(x_1^*\right) - \rho_1 T_1\left(x_2^*, x_3^*, \ldots, x_N^*, x_1^*\right)\right), \\ x_2^* = f^{-1}\prod_K J^{-1}\left(Jf\left(x_2^*\right) - \rho_2 T_2\left(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*\right)\right), \\ \vdots \\ x_{N-1}^* = f^{-1}\prod_K J^{-1}\left(Jf\left(x_{N-1}^*\right) - \rho_{N-1} T_{N-1} \times \left(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*\right)\right), \\ x_N^* = f^{-1}\prod_K J^{-1}\left(Jf\left(x_N^*\right) - \rho_N T_N\left(x_1^*, x_2^*, \ldots, x_N^*\right)\right). \end{cases} \tag{15}$$

□

Algorithm 8. For any given initial points $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(N)} \in K$, compute the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ by the iterative processes

$$\begin{aligned} x_{n+1}^{(N)} &= f^{-1} \times \left(J^{-1}\left(\left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right) + \alpha_n^{(N)} J \times \left(\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N \times \left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right)\right)\right), \\ x_{n+1}^{(N-1)} &= f^{-1} \times \left(J^{-1}\left(\left(1-\alpha_n^{(N-1)}\right) Jf\left(x_n^{(N-1)}\right) + \alpha_n^{(N-1)} J \times \left(\prod_K J^{-1}\left(Jf\left(x_n^{(N-1)}\right) - \rho_{N-1} T_{N-1} \times \left(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}\right)\right)\right)\right)\right), \\ &\vdots \\ x_{n+1}^{(2)} &= f^{-1} \times \left(J^{-1}\left(\left(1-\alpha_n^{(2)}\right) Jf\left(x_n^{(2)}\right) + \alpha_n^{(2)} J \times \left(\prod_K J^{-1}\left(Jf\left(x_n^{(2)}\right) - \rho_2 T_2 \times \left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}\right)\right)\right)\right)\right), \\ x_{n+1}^{(1)} &= f^{-1} \times \left(J^{-1}\left(\left(1-\alpha_n^{(1)}\right) Jf\left(x_n^{(1)}\right) + \alpha_n^{(1)} J \times \left(\prod_K J^{-1}\left(Jf\left(x_n^{(1)}\right) - \rho_1 T_1 \times \left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}\right)\right)\right)\right)\right), \\ & n \geq 0, \end{aligned} \tag{16}$$

where $\prod_K$ is the generalized projection and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \ldots, \{\alpha_n^{(N)}\}$ are sequences in $[0,1]$.

Theorem 9. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and K a nonempty closed and convex subset of E with $\theta \in K$. Let $f : K \to K$ be an isometry mapping, $T_1, \ldots, T_N : K^N \to E^*$ continuous mappings, and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \ldots, \{\alpha_n^{(N)}\}$ the sequences in (a,b) with $0 < a < b < 1$ satisfying the following conditions:*

(i) *there exist a compact subset $C \subset E^*$ and constants $\rho_1 > 0, \rho_2 > 0, \ldots, \rho_N > 0$ such that*

$$\left(J(K) - \rho_N T_N\left(K^N\right)\right) \cup \left(J(K) - \rho_{N-1} T_{N-1}\left(K^N\right)\right) \cup \cdots \cup \left(J(K) - \rho_1 T_1\left(K^N\right)\right) \subset C, \tag{17}$$

where $J(x_1, x_2, \ldots, x_N) = Jx_N$*, for all* $(x_1, x_2, \ldots, x_N) \in K^N$*, and*

$$\begin{aligned}
&\left\langle T_1(x_1,x_2,\ldots,x_N), J^{-1}\left(Jx_N - \rho_1 T_1(x_1,x_2,\ldots,x_N)\right)\right\rangle \geq 0,\\
&\left\langle T_2(x_1,x_2,\ldots,x_N), J^{-1}\left(Jx_N - \rho_2 T_2(x_1,x_2,\ldots,x_N)\right)\right\rangle \geq 0,\\
&\vdots\\
&\left\langle T_N(x_1,x_2,\ldots,x_N), J^{-1}\left(Jx_N - \rho_N T_N(x_1,x_2,\ldots,x_N)\right)\right\rangle \geq 0,
\end{aligned} \tag{18}$$

for all $x_1, x_2, \ldots, x_N \in K$*;*

(ii) $\lim_{n\to\infty}\alpha_n^{(1)} = d_1 \in (a,b)$, $\lim_{n\to\infty}\alpha_n^{(2)} = d_2 \in (a,b)$, $\ldots$, $\lim_{n\to\infty}\alpha_n^{(N)} = d_N \in (a,b)$*. Let* $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ *be the sequences defined by* (16)*. Then the problem* (8) *has a solution* $(x_1^*, x_2^*, \ldots, x_N^*) \in K^N$ *and the sequences* $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ *converge strongly to* $x_1^*, x_2^*, \ldots, x_N^*$*, respectively.*

Proof.
Step 1. We first show that the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ are bounded in K. It follows from Lemma 5 where J is bijective and condition (18) that

$$\begin{aligned}
&\left\| Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right\|^2\\
&\leq \left\| Jf\left(x_n^{(N)}\right)\right\|^2 - 2\rho_N \left\langle T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right), J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right\rangle\\
&\leq \left\| Jf\left(x_n^{(N)}\right)\right\|^2 = \left\| f\left(x_n^{(N)}\right)\right\|^2.
\end{aligned} \tag{19}$$

Similarly, we note that

$$\begin{aligned}
&\left\| Jf\left(x_n^{(N-1)}\right) - \rho_{N-1} T_{N-1}\left(x_{n+1}^{(N)}, x_n^{(1)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}\right)\right\|^2 \leq \left\| f\left(x_n^{(N-1)}\right)\right\|^2,\\
&\left\| Jf\left(x_n^{(N-2)}\right) - \rho_{N-2} T_{N-2}\left(x_{n+1}^{(N-1)}, x_{n+1}^{(N)}, x_n^{(1)}, \ldots, x_n^{(N-3)}, x_n^{(N-2)}\right)\right\|^2 \leq \left\| f\left(x_n^{(N-2)}\right)\right\|^2,\\
&\vdots\\
&\left\| Jf\left(x_n^{(2)}\right) - \rho_2 T_2\left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}\right)\right\|^2 \leq \left\| f\left(x_n^{(2)}\right)\right\|^2,\\
&\left\| Jf\left(x_n^{(1)}\right) - \rho_1 T_1\left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}\right)\right\|^2 \leq \left\| f\left(x_n^{(1)}\right)\right\|^2.
\end{aligned} \tag{20}$$

By Lemma 6, we obtain that

$$\begin{aligned}
&\left\| f\left(x_{n+1}^{(N)}\right)\right\|\\
&= \left\| ff^{-1} \times \left(J^{-1}\left(\left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right) + \alpha_n^{(N)} J\left(\prod_K J^{-1} \times \left(Jf\left(x_n^{(N)}\right) - \rho_N T_N \times \left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right)\right)\right)\right\|\\
&= \left\| J^{-1}\left(\left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right) + \alpha_n^{(N)} J\left(\prod_K J^{-1} \times \left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right)\right)\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right) \\
&\quad + \alpha_n^{(N)} J\Big(\prod_K J^{-1} \\
&\qquad \times \left(Jf\left(x_n^{(N)}\right) \right. \\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big)\Big\| \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| Jf\left(x_n^{(N)}\right)\right\| \\
&\quad + \alpha_n^{(N)} \Big\| J\Big(\prod_K J^{-1} \\
&\qquad \times \left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big)\Big\| \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| Jf\left(x_n^{(N)}\right)\right\| \\
&\quad + \alpha_n^{(N)} \left\| JJ^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\right\| \\
&= \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\| \\
&\quad + \alpha_n^{(N)} \left\| Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right\| \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\| + \alpha_n^{(N)} \left\| f\left(x_n^{(N)}\right)\right\| \\
&= \left\| f\left(x_n^{(N)}\right)\right\|.
\end{aligned}
\tag{21}
$$

Since f is an isometry mapping, we have $\|x_{n+1}^{(N)}\| \le \|x_n^{(N)}\|$. By the same argument method as given above, we have $\|x_{n+1}^{(N-1)}\| \le \|x_n^{(N-1)}\|, \dots, \|x_{n+1}^{(1)}\| \le \|x_n^{(1)}\|$. Therefore, we note that $\lim_{n\to\infty}\|x_n^{(1)}\|, \dots, \lim_{n\to\infty}\|x_n^{(N)}\|$ exist and hence the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$ are bounded in K.

Step 2. By Lemmas 4 and 6, where f is an isometry mapping and (19), it follows that there exists a continuous strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that

$$
\begin{aligned}
&\left\| f(x_{n+1}^{(N)})\right\|^2 \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| Jf\left(x_n^{(N)}\right)\right\|^2 \\
&\quad + \alpha_n^{(N)} \Big\| J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) \right. \\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big\|^2 \\
&\quad - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} g \\
&\quad \times \Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\Big(Jf\left(x_n^{(N)}\right) \\
&\qquad\qquad - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\Big)\Big\|\Big) \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\|^2 \\
&\quad + \alpha_n^{(N)} \left\| Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right\|^2 \\
&\quad - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} \\
&\quad \times g\Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\Big(Jf\left(x_n^{(N)}\right) \\
&\qquad\qquad - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\Big)\Big\|\Big) \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\|^2 + \alpha_n^{(N)} \left\| f\left(x_n^{(N)}\right)\right\|^2 \\
&\quad - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} \\
&\quad \times g\Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\Big(Jf\left(x_n^{(N)}\right) \\
&\qquad\qquad - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\Big)\Big\|\Big) \\
&= \left\| f\left(x_n^{(N)}\right)\right\|^2 - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} \\
&\quad \times g\Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\Big(Jf\left(x_n^{(N)}\right) \\
&\qquad\qquad - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\Big)\Big\|\Big).
\end{aligned}
\tag{22}
$$

This implies that

$$
\begin{aligned}
&\left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} g\Big(\Big\| Jf\left(x_n^{(1)}\right) \\
&\qquad + J\prod_K J^{-1}\Big(Jf\left(x_n^{(N)}\right) \\
&\qquad\qquad - \rho_N T_N \\
&\qquad\qquad \times \left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\Big)\Big\|\Big) \\
&\le \left\| f\left(x_n^{(N)}\right)\right\|^2 - \left\| f\left(x_{n+1}^{(N)}\right)\right\|^2.
\end{aligned}
\tag{23}
$$

Since $\{\|x_n^{(k)}\|\}$ converges for all $k = 1, 2, \ldots, N$, it follows by letting $n \to \infty$ in (23), condition (ii), and the property of g that

$$\left\| Jf\left(x_n^{(N)}\right) - J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right\| \longrightarrow 0, \quad (24)$$

as $n \to \infty$. By (16) and (24), we have

$$\begin{aligned}\left\|Jf\left(x_{n+1}^{(N)}\right) - Jf\left(x_n^{(N)}\right)\right\| &= \alpha_n^{(N)} \left\|Jf\left(x_n^{(N)}\right) - J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right\| \longrightarrow 0,\end{aligned} \quad (25)$$

as $n \to \infty$. Similarly, we can prove that

$$\begin{aligned}&\left\|Jf\left(x_{n+1}^{(N-1)}\right) - Jf\left(x_n^{(N-1)}\right)\right\| \\ &= \alpha_n^{(N-1)} \left\|Jf\left(x_n^{(N-1)}\right) - J\prod_K J^{-1} \times \left(Jf\left(x_n^{(N-1)}\right) - \rho_{N-1}T_{N-1} \times \left(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}\right)\right)\right\| \longrightarrow 0,\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\left\|Jf\left(x_{n+1}^{(2)}\right) - Jf\left(x_n^{(2)}\right)\right\| \\ &= \alpha_n^{(2)} \left\|Jf\left(x_n^{(2)}\right) - J\prod_K J^{-1} \times \left(Jf\left(x_n^{(2)}\right) - \rho_2 T_2 \times \left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}\right)\right)\right\| \longrightarrow 0,\end{aligned}$$

$$\begin{aligned}&\left\|Jf\left(x_{n+1}^{(1)}\right) - Jf\left(x_n^{(1)}\right)\right\| \\ &= \alpha_n^{(1)} \left\|Jf\left(x_n^{(1)}\right) - J\prod_K J^{-1} \times \left(Jf\left(x_n^{(1)}\right) - \rho_1 T_1 \times \left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}\right)\right)\right\| \longrightarrow 0,\end{aligned} \quad (26)$$

as $n \to \infty$.

Step 3. Since $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ are bounded and there exists a compact subset $C \subset E^*$ such that $(J(K) - \rho_N T_N(K^N)) \subset C$, there exists a subsequence $\{x_{n_{i(N)}}^{(N)}\}$ of $\{x_{n_j}^{(N)}\}$ such that

$$Jf\left(x_{n_{i(N)}}^{(N)}\right) - \rho_N T_N\left(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)}\right) \longrightarrow h_1 \in E^*. \quad (27)$$

Since E is uniformly smooth and strictly convex, it follows by Lemma 2 (b) and Remark 1 that $\prod_K$ and J^{-1} are continuous. Thus

$$\prod_K J^{-1}\left(Jf\left(x_{n_{i(N)}}^{(N)}\right) - \rho_N T_N \times \left(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)}\right)\right) \longrightarrow \prod_K J^{-1}(h_1) := f\left(x_N^*\right), \quad (28)$$

$$J\prod_K J^{-1}\left(Jf\left(x_{n_{i(N)}}^{(N)}\right) - \rho_N T_N\left(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)}\right)\right) \longrightarrow Jf\left(x_N^*\right). \quad (29)$$

From (24) and (29), we get

$$Jf\left(x_{n_{i(N)}}^{(N)}\right) \longrightarrow Jf\left(x_N^*\right) \quad \left(\text{as } n_{i(N)} \longrightarrow \infty\right). \quad (30)$$

By (25) and (30), we have

$$Jf\left(x_{n_{i(N)}+1}^{(N)}\right) \longrightarrow Jf\left(x_N^*\right) \quad \left(\text{as } n_{i(N)} \longrightarrow \infty\right). \quad (31)$$

Since E is strictly convex and reflexive, it follows by Remark 1 (iv) that J^{-1} is norm-weak-continuous. Therefore, from (30) and (31), we note that

$$f\left(x_{n_{i(N)}}^{(N)}\right) \rightharpoonup f\left(x_N^*\right), \qquad f\left(x_{n_{i(N)}+1}^{(N)}\right) \rightharpoonup f\left(x_N^*\right) \quad (32)$$

and

$$\begin{aligned}&\left\|f\left(x_{n_{i(N)}}^{(N)}\right)\right\| \longrightarrow \left\|f\left(x_N^*\right)\right\|, \\ &\left\|f\left(x_{n_{i(N)}+1}^{(N)}\right)\right\| \longrightarrow \left\|f\left(x_N^*\right)\right\| \\ &\quad \left(\text{as } n_{i(N)} \longrightarrow \infty\right).\end{aligned} \quad (33)$$

By the Kadec-Klee property, we have

$$f\left(x_{n_{i(N)}}^{(N)}\right) \longrightarrow f\left(x_N^*\right), \qquad f\left(x_{n_{i(N)}+1}^{(N)}\right) \longrightarrow f\left(x_N^*\right) \quad \left(\text{as } n_{i(N)} \longrightarrow \infty\right). \tag{34}$$

Since f^{-1} is continuous, it implies that $\{x_{n_{i(N)}}^{(N)}\}$ is a subsequence of $\{x_{n_j}^{(N)}\}$ such that $x_{n_{i(N)}}^{(N)} \to x_N^* \in E$. Therefore $x_n^{(N)} \to x_N^*$ as $n \to \infty$. So, it follows from (16), (30), (34), and condition (ii) that

$$\begin{aligned} Jf\left(x_N^*\right) &= \lim_{n\to\infty} Jf\left(x_{n+1}^{(N)}\right) \\ &= \lim_{n\to\infty}\Big\{\left(1-\alpha_n^{(N)}\right)Jf\left(x_n^{(N)}\right)+\alpha_n^{(N)}f^{-1}J\prod_K J^{-1} \\ &\qquad \times\left(Jf\left(x_n^{(N)}\right)-\rho_N T_N\left(x_n^{(1)},x_n^{(2)},\ldots,x_n^{(N)}\right)\right)\Big\} \\ &= \left(1-d_N\right)Jf\left(x_N^*\right) \\ &\quad + d_N J\prod_K J^{-1}\left(Jf\left(x_N^*\right)-\rho_N T_N\left(x_1^*,x_2^*,\ldots,x_N^*\right)\right). \end{aligned} \tag{35}$$

Since f is a bijective mapping, we obtain that

$$x_N^* = f^{-1}\prod_K J^{-1}\left(Jf\left(x_N^*\right)-\rho_N T_N\left(x_1^*,x_2^*,\ldots,x_N^*\right)\right). \tag{36}$$

Similarly, we can prove that for every subsequence $\{x_{n_j}^{(k)}\}$ of $\{x_n^{(k)}\}$ there exist a subsequence $\{x_{n_{i(k)}}^{(k)}\}$ of $\{x_{n_j}^{(k)}\}$ and $x_k^* \in E$ such that

$$f\left(x_{n_{i(k)}}^{(k)}\right) \longrightarrow f\left(x_k^*\right) \quad \left(\text{as } n_{i(k)} \longrightarrow \infty\right), \quad \forall k = 1,2,\ldots,N-1. \tag{37}$$

Since f^{-1} is a continuous mapping, we note that

$$x_{n_{i(k)}}^{(k)} \longrightarrow x_k^* \quad \left(\text{as } n_{i(k)} \longrightarrow \infty\right). \tag{38}$$

Hence $x_n^{(k)} \to x_k^* \in E$, for all $k = 1,2,\ldots,N-1$. Therefore, we have

$$\begin{aligned} x_{N-1}^* &= f^{-1}\prod_K J^{-1}\left(Jf\left(x_{N-1}^*\right)\right. \\ &\qquad \left.-\rho_{N-1}T_{N-1}\left(x_N^*,x_1^*,\ldots,x_{N-2}^*,x_{N-1}^*\right)\right) \\ &\;\;\vdots \\ x_1^* &= f^{-1}\prod_K J^{-1}\left(Jf\left(x_1^*\right)\right. \\ &\qquad \left.-\rho_1 T_1\left(x_2^*,x_3^*,\ldots,x_N^*,x_1^*\right)\right). \end{aligned} \tag{39}$$

By Lemma 7, we can conclude that $(x_1^*,x_2^*,\ldots,x_N^*)$ is a solution of (8) and $x_n^{(1)} \to x_1^*, x_n^{(2)} \to x_2^*,\ldots,x_n^{(N)} \to x_N^*$. □

Setting $N = 3$ and $f = I$ in Theorem 9, we immediately obtain the following result.

Corollary 10 (see [6]). *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and K a nonempty closed and convex subset of E with $\theta \in K$. Let $T_1,T_2,T_3 : K^3 \to E^*$ be continuous mappings and $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, and $\{\alpha_n^{(3)}\}$ the sequences in (a,b) with $0 < a < b < 1$ satisfying the following conditions.*

(i) *There exist a compact subset $C \subset E^*$ and constants $\rho_1 > 0$, $\rho_2 > 0$, and $\rho_3 > 0$ such that*

$$\left(J(K)-\rho_3 T_3\left(K^3\right)\right)\cup\left(J(K)-\rho_2 T_2\left(K^3\right)\right) \cup\left(J(K)-\rho_1 T_1\left(K^3\right)\right)\subset C, \tag{40}$$

where $J(x_1,x_2,x_3) = Jx_3$, for all $(x_1,x_2,x_3) \in K^3$, and

$$\begin{aligned} &\left\langle T_1\left(x_1,x_2,x_3\right), J^{-1}\left(Jx_3-\rho_1 T_1\left(x_1,x_2,x_3\right)\right)\right\rangle \geq 0, \\ &\left\langle T_2\left(x_1,x_2,x_3\right), J^{-1}\left(Jx_3-\rho_2 T_2\left(x_1,x_2,x_3\right)\right)\right\rangle \geq 0, \\ &\left\langle T_3\left(x_1,x_2,x_3\right), J^{-1}\left(Jx_3-\rho_3 T_3\left(x_1,x_2,x_3\right)\right)\right\rangle \geq 0, \end{aligned} \tag{41}$$

for all $x_1,x_2,x_3 \in K$.

(ii) *$\lim_{n\to\infty}\alpha_n^{(1)} = d_1 \in (a,b)$, $\lim_{n\to\infty}\alpha_n^{(2)} = d_2 \in (a,b)$, and $\lim_{n\to\infty}\alpha_n^{(3)} = d_3 \in (a,b)$. Let $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$, and $\{x_n^{(3)}\}$ be the sequences defined by*

$$\begin{aligned} x_{n+1}^{(3)} &= J^{-1}\left(\left(1-\alpha_n^{(3)}\right)Jf\left(x_n^{(3)}\right)+\alpha_n^{(3)}J \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(3)}\right)-\rho_3 T_3\left(x_n^{(1)},x_n^{(2)},x_n^{(3)}\right)\right)\right)\right), \\ x_{n+1}^{(2)} &= J^{-1}\left(\left(1-\alpha_n^{(2)}\right)Jf\left(x_n^{(2)}\right)+\alpha_n^{(2)}J \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(2)}\right)-\rho_2 T_2\left(x_{n+1}^{(3)},x_n^{(1)},x_n^{(2)}\right)\right)\right)\right), \\ x_{n+1}^{(1)} &= J^{-1}\left(\left(1-\alpha_n^{(1)}\right)Jf\left(x_n^{(1)}\right)+\alpha_n^{(1)}J \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(1)}\right)-\rho_1 T_1\left(x_{n+1}^{(2)},x_{n+1}^{(3)},x_n^{(1)}\right)\right)\right)\right), \\ & n \geq 0. \end{aligned} \tag{42}$$

Then the problem (9) has a solution $(x_1^*,x_2^*,x_3^*) \in K^3$ and the sequences $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$ and $\{x_n^{(3)}\}$ converge strongly to x_1^*, x_2^*, and x_3^*, respectively.

Setting E as a real Hilbert space in Theorem 9, we have the following result.

Corollary 11. *Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $f : K \to K$ be an isometry mapping and $T_1, \ldots, T_N : K^N \to H$ continuous mappings and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \ldots, \{\alpha_n^{(N)}\}$ are sequences in (a, b) with $0 < a < b < 1$ satisfying the following conditions.*

(i) *There exist a compact subset $C \subset H$ and constants $\rho_1 > 0, \rho_2 > 0, \ldots, \rho_N > 0$ such that*

$$\begin{aligned}&\left(I(K) - \rho_N T_N\left(K^N\right)\right)\\&\quad\cup\left(I(K) - \rho_{N-1} T_{N-1}\left(K^N\right)\right)\\&\quad\cup\cdots\cup\left(I(K) - \rho_1 T_1\left(K^N\right)\right) \subset C,\end{aligned} \tag{43}$$

where $(x_1, x_2, \ldots, x_N) = x_N$, for all $(x_1, x_2, \ldots, x_N) \in K^N$, and

$$\begin{aligned}&\langle T_1(x_1, x_2, \ldots, x_N), x_N - \rho_1 T_1(x_1, x_2, \ldots, x_N)\rangle \geq 0,\\&\langle T_2(x_1, x_2, \ldots, x_N), x_N - \rho_2 T_2(x_1, x_2, \ldots, x_N)\rangle \geq 0,\\&\qquad\vdots\\&\langle T_N(x_1, x_2, \ldots, x_N), x_N - \rho_N T_N(x_1, x_2, \ldots, x_N)\rangle \geq 0,\end{aligned} \tag{44}$$

for all $x_1, x_2, \ldots, x_N \in K$.

(ii) $\lim_{n\to\infty}\alpha_n^{(1)} = d_1 \in (a,b)$, $\lim_{n\to\infty}\alpha_n^{(2)} = d_2 \in (a,b), \ldots, \lim_{n\to\infty}\alpha_n^{(N)} = d_N \in (a,b)$. *Let $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ be the sequences defined by*

$$\begin{aligned}x_{n+1}^{(N)} &= f^{-1}\left(\left(1-\alpha_n^{(N)}\right) f\left(x_n^{(N)}\right) + \alpha_n^{(N)} P_K\right.\\&\qquad\left.\times\left(f\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right),\\x_{n+1}^{(N-1)} &= f^{-1}\left(\left(1-\alpha_n^{(N-1)}\right) f\left(x_n^{(N-1)}\right) + \alpha_n^{(N-1)} P_K\right.\\&\qquad\times\left(f\left(x_n^{(N-1)}\right) - \rho_{N-1} T_{N-1}\right.\\&\qquad\left.\left.\times\left(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}\right)\right)\right),\\&\ \ \vdots\\x_{n+1}^{(2)} &= f^{-1}\left(\left(1-\alpha_n^{(2)}\right) f\left(x_n^{(2)}\right) + \alpha_n^{(2)} P_K\right.\\&\qquad\times\left(f\left(x_n^{(2)}\right) - \rho_2 T_2\right.\\&\qquad\left.\left.\times\left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}\right)\right)\right),\\x_{n+1}^{(1)} &= f^{-1}\left(\left(1-\alpha_n^{(1)}\right) f\left(x_n^{(1)}\right)\right.\\&\qquad + \alpha_n^{(1)} P_K\left(f\left(x_n^{(1)}\right) - \rho_1 T_1\right.\\&\qquad\left.\left.\times\left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}\right)\right)\right),\\&\qquad n \geq 0,\end{aligned} \tag{45}$$

where P_K is a metric projection on H to K. Then the problem (8) has a solution $(x_1^*, x_2^*, \ldots, x_N^*) \in K^N$ and the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ converge strongly to $x_1^*, x_2^*, \ldots, x_N^*$, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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