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Research Article

A BDDC Preconditioner for the Rotated Q_1 FEM for Elliptic Problems with Discontinuous Coefficients

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We propose a BDDC preconditioner for the rotated Q_1 finite element method for second order elliptic equations with piecewise but discontinuous coefficients. In the framework of the standard additive Schwarz methods, we describe this method by a complete variational form. We show that our method has a quasioptimal convergence behavior; that is, the condition number of the preconditioned problem is independent of the jumps of the coefficients and depends only logarithmically on the ratio between the subdomain size and the mesh size. Numerical experiments are presented to confirm our theoretical analysis.

1. Introduction

The balancing domain decomposition by constraints (BDDC) method was first introduced by Dohrmann in [1]. Then Mandel and the author Dohrmann restated the method in an abstract manner and provided its convergence theory in [2]. The BDDC method is closely related to the dual-primal FETI (FETI-DP) method [3], which is one of dual iterative substructuring methods. Each BDDC and FETI-DP method is defined in terms of a set of primal continuity; the primal continuity is enforced across the interface between the subdomains and provides a coarse space component of the preconditioner. In [4], Mandel et al. analyzed the relation between the two methods and established the corresponding theory.

In the last decades, the two methods have been widely analyzed and successfully been extended to many different types of partial differential equations. In [3], the two algorithms for elliptic problems were rederived and a brief proof of the main result was given. A BDDC algorithm for mortar finite element was developed in [5]; meanwhile, the authors also extended the FETI-DP algorithm to elasticity problems and stokes problems in [6, 7], respectively. These algorithms were based on locally conforming finite element methods, and the coarse space components of the algorithms were related to the cross-points (i.e., corners), which are often noteworthy points in domain decomposition methods (DDMs). Since the cross-points are related to more than two subregions, thus it is not convenient when designing algorithm.

The BDDC method derives from the Neumann-Neumann domain decomposition method (see [8]). The difference is that the BDDC method applies an additive rather than a multiplicative coarse grid correction, and substructure spaces have some constraints which result in nonsingular subproblems, so that we can solve each subproblem and coarse problem in parallel.

The rotated Q_1 element is an important nonconforming element. It was introduced by Rannacher and Turek in [9] for stokes equations originally, and it is the simplest example of a divergence-stable nonconforming element on quadrilaterals. Since its degree of freedom is integral average on element edge which is not related to the corners, and each degree of freedom on subdomain interfaces is only included in two neighboring subdomains, so it is easy to design algorithm.

In this paper, we consider the second order problem with discontinuous coefficients, where the discontinuities lie only along the subdomain interfaces. Such problems play an important role in scientific computing. It is well known that large jumps in the coefficients may result in bad convergence for the traditional iterative methods (such as *C-G* algorithm). To overcome this difficulty, we construct a family of weighted counting functions associated with the substructures. Our counting functions are related to only two neighboring subdomains; this brings convenience for computing. Furthermore, since the rotated Q_1 element is not related to the subdomain's vertices, we can complete our theoretical analysis conveniently. It is proved that the condition number of the preconditioned operator is independent of the jumps of the coefficients and only depends logarithmically on the ratio between the subdomain size and mesh size. Numerical experiments are presented to confirm our theoretical analysis.

The rest of this paper is organized as follows. In Section 2, we introduce the model problem and the corresponding Schur complement system. Section 3 gives the BDDC algorithm and proposes the BDDC preconditioner. Several technical tools are presented and analyzed in Section 4. In Section 5, we complete the proof of the main result. Last section provides numerical experiments. For convenience, the symbols $\preceq$, $\succeq$, and $\asymp$ are used, and $x_1 \preceq y_1$, $x_2 \succeq y_2$, and $x_3 \asymp y_3$ mean that $x_1 \le C_1 y_1$, $x_2 \ge C_2 y_2$, and $c_3 x_3 \le y_3 \le C_3 y_3$ for some constants C_1, C_2, C_3, and c_3 that are independent of discontinuous coefficients and mesh size.

2. Preliminaries

Let $\Omega \subset \mathscr{R}^2$ be a bounded, simply connect rectangular or L-shaped domain. We divide Ω into several nonoverlapping regular rectangular subdomains Ω_i $(i = 1, \ldots, N)$; that is, $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$. Consider the following model problem: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where

$$\begin{aligned} a(u, v) &= \sum_{i=1}^{N} \int_{\Omega_i} \rho_i(x) \nabla u \cdot \nabla v dx, \\ f(v) &= \sum_{i=1}^{N} \int_{\Omega_i} f v dx, \end{aligned} \tag{2}$$

where $f \in L^2(\Omega)$, and the coefficients $\rho_i(x)$ $(i = 1, \ldots, N)$ are positive constants over Ω_i $(i = 1, \ldots, N)$.

We only consider the geometrically conforming case; that is, the intersection between the closure of two different subdomains is empty, or a vertex, or an edge. The subdomains $\{\Omega_i\}_{i=1}^{N}$ together form a coarse partition $\mathscr{T}_H(\Omega)$; we denote the diameter of each Ω_i by H_i. Let $\mathscr{T}_h(\Omega_i)$ be a quasiuniform partition with mesh size $O(h_i)$, made up of rectangles in Ω_i; then $\mathscr{T}_h(\Omega) = \bigcup_{i=1}^{N} \mathscr{T}_h(\Omega_i)$ is the global quasiuniform partition on Ω. The nodes on the boundaries of neighboring subdomains match across the interface $\Gamma = (\bigcup_{i=1}^{N} \partial\Omega_i) \setminus \partial\Omega$. We define by Γ_{ij} the interface between Ω_i and Ω_j, and let $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$. We denote the sets of edges of the partition $\mathscr{T}_h(\Omega_i)$ in Ω_i, $\partial\Omega_i$, Γ, and Γ_{ij} by $\Omega_{i,h}^e$, $\partial\Omega_{i,h}^e$, Γ_h^e, and Γ_{ij}^e, respectively, and let $\Omega_{i,h}$, $\partial\Omega_{i,h}$ be the sets of vertices of the triangulation $\mathscr{T}_h(\Omega_i)$ that are in $\overline{\Omega}_i$, $\partial\overline{\Omega}_i$, respectively.

The global rotated Q_1 element space is defined as follows:

$$\begin{aligned} X_h(\Omega) &= \Big\{ v \in L^2(\Omega) \, |v|_E \\ &\qquad = a_E^1 + a_E^2 x + a_E^3 y + a_E^4 \left(x^2 - y^2\right), \\ &\qquad a_E^i \in \mathscr{R}, \\ &\qquad \int_e v ds = 0, \ \forall e \in \partial E \cap \partial\Omega, \\ &\qquad E \in \mathscr{T}_h(\Omega); \ \text{for } E_1, E_2 \in \mathscr{T}_h(\Omega), \\ &\qquad \text{if } \ \partial E_1 \cap \partial E_2 = e, \\ &\qquad \text{then } \int_e v|_{\partial E_1} ds = \int_e v|_{\partial E_2} ds \Big\}. \end{aligned} \tag{3}$$

The discrete approximation of the original problem (1) is to find $u_h \in X_h(\Omega)$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in X_h(\Omega), \tag{4}$$

where

$$\begin{aligned} a_h(u_h, v_h) &= \sum_{i=1}^{N} a_{h,i}(u_h, v_h), \\ a_{h,i}(u_h, v_h) &= \sum_{E \in \mathscr{T}_h(\Omega_i)} \int_E \rho_i \nabla u_h \nabla v_h dx, \\ (f, v_h) &= \int_\Omega f v_h \, dx. \end{aligned} \tag{5}$$

For each space $X_h(\Omega_i)$ $(X_h(\Omega_i) = X_h(\Omega)|_{\Omega_i})$, we equip the following seminorm and norm:

$$\begin{aligned} |v|_{H_h^1(\Omega_i)}^2 &= \sum_{E \in \mathscr{T}_h(\Omega_i)} |v|_{H^1(E)}^2, \qquad |v|_{H_\rho^1(\Omega_i)}^2 = a_{h,i}(v, v), \\ \|v\|_{L_\rho^2(\Omega_i)}^2 &= \int_{\Omega_i} \rho_i v^2 dx. \end{aligned} \tag{6}$$

It can be easily shown that $a_h(\cdot, \cdot)$ is positive definite on $X_h(\Omega)$, which yields the existence and uniqueness of the discrete solution.

We define a discrete harmonic operator $\mathscr{H}_i$ associated with the rotated Q_1 element: for any $v \in X_h(\Omega_i)$, let $\mathscr{H}_i v \in X_h(\Omega_i)$ such that

$$\begin{aligned} a_{h,i}(\mathscr{H}_i v, w) &= 0, \quad \forall w \in X_h^0(\Omega_i), \\ \frac{1}{|e|} \int_e \mathscr{H}_i v ds &= \frac{1}{|e|} \int_e v ds, \quad \forall e \in \partial\Omega_{i,h}^e, \end{aligned} \tag{7}$$

here $X_h^0(\Omega_i) = \{v \in X_h(\Omega_i) | \int_e v ds = 0, \ \forall e \in \partial\Omega_{i,h}^e\}$. We define a corresponding piecewise harmonic operator $\mathscr{H}$ by $\mathscr{H}|_{\Omega_i} = \mathscr{H}_i$ on the global rotated Q_1 element space $X_h(\Omega)$.

In order to introduce our domain decomposition method, we decompose the discrete space $X_h(\Omega)$ as follows:

$$X_h(\Omega) = X_h^P(\Omega) \oplus X_h(\Gamma), \qquad X_h^P(\Omega) = \bigcup_{i=1}^{N} X_h^0(\Omega_i), \tag{8}$$

where the space $X_h(\Gamma)$ is a piecewise harmonic function space defined as

$$\begin{aligned} X_h(\Gamma) &= \mathscr{H}(X_h(\Omega)) \\ &= \left\{ v \in X_h(\Omega) \mid v\big|_{\Omega_i} = \mathscr{H}_i\left(v|_{\Omega_i}\right), i = 1, 2 \ldots, N \right\}. \end{aligned} \tag{9}$$

We assume u to be the solution of (4) and $u_i \in X_h^0(\Omega_i)$ to be the solution of the local homogeneous Dirichlet problem:

$$a_{h,i}(u_i, v) = (f, v)_{\Omega_i}, \quad \forall v \in X_h^0(\Omega_i), \tag{10}$$

where $(f,v)_{\Omega_i} = \int_{\Omega_i} f v dx$. Let $u_P \in X_h^P(\Omega)$ be the function that is equal to u_i on the subdomain Ω_i; then $u_\Gamma = u - u_P$ obviously satisfies

$$a_h(u_\Gamma, v) = (f, v) - a_h(u_P, v), \quad \forall v \in X_h(\Omega), \tag{11}$$

and we get $u_\Gamma \in X_h(\Gamma)$. So we can equivalently derive the Schur complement system of (4) easily: find $u_\Gamma \in X_h(\Gamma)$ such that

$$a_h(u_\Gamma, v_\Gamma) = (f, v) - a_h(u_P, v) = (f, v_\Gamma), \quad \forall v \in X_h(\Omega), \tag{12}$$

where v_Γ is the piecewise harmonic function of v in Ω; that is, $v_\Gamma = \mathscr{H} v$.

For the sake of completeness, we define a Schur complement operator $S_h : X_h(\Gamma) \rightarrow X_h(\Gamma)$ by

$$(S_h u_\Gamma, v_\Gamma) = a_h(u_\Gamma, v_\Gamma), \quad \forall u_\Gamma, v_\Gamma \in X_h(\Gamma). \tag{13}$$

Our goal is therefore to construct a preconditioner for the operator S_h.

3. BDDC Algorithm

In this section, we introduce our BDDC preconditioner and describe the BDDC algorithm. Let $X_h(\Gamma_i) = X_h(\Gamma)|_{\Omega_i}$; we define the space $\widetilde{X}_h(\Gamma) = \{v \in \prod_{i=1}^N X_h(\Gamma_i) \mid \int_{\Gamma_{ij}} v|_{\Omega_i} ds = \int_{\Gamma_{ij}} v|_{\Omega_j} ds, \forall \Gamma_{ij} \subset \Gamma\}$. The space $\widetilde{X}_h(\Gamma)$ is intermediary between $X_h(\Gamma)$ and $\prod_{i=1}^N X_h(\Gamma_i)$; our BDDC preconditioner is constructed based on this space.

As we know, the technical aspect in DDMs is that the preconditioner includes a coarse problem which can enhance the convergence. In view of the characteristic of the space $\widetilde{X}_h(\Gamma)$, we select the standard coarse space $X_H(\Omega)$ which is the rotated Q_1 finite element space associated with the coarse partition $\mathscr{T}_H(\Omega)$, and it satisfies primal constraints on subdomain interfaces.

The substructure space $X_\Delta(\Gamma_i)$ with constraints is defined by

$$X_\Delta(\Gamma_i) = \left\{ v \in X_h(\Gamma_i) \mid \int_{\Gamma_{ij}} v ds = 0, \forall \Gamma_{ij} \subset \partial\Omega_i \right\}. \tag{14}$$

Denote $X_\Delta(\Gamma) = \prod_{i=1}^N X_\Delta(\Gamma_i)$.

The coarse space and product space $X_\Delta(\Gamma)$ play an important role in the description and analysis of our iterative method. In essence, we give a decomposition of the space $\widetilde{X}_h(\Gamma)$ as follows:

$$\widetilde{X}_h(\Gamma) = \mathscr{H}(X_H(\Omega)) + X_\Delta(\Gamma). \tag{15}$$

To present our BDDC preconditioner, we introduce several space transfer operators. Define the interpolation operator $I_H : X_h(\Gamma) \rightarrow X_H(\Omega)$ by

$$\frac{\int_{\Gamma_{ij}} I_H v ds}{|\Gamma_{ij}|} = \frac{\int_{\Gamma_{ij}} v ds}{|\Gamma_{ij}|}, \quad \forall \Gamma_{ij} \subset \Gamma. \tag{16}$$

The intergrid transfer operator $I_h : X_H(\Omega) \rightarrow \widetilde{X}_h(\Gamma)$ is defined by

$$\frac{\int_e I_h v ds}{|e|} = \frac{\int_e v ds}{|e|}, \quad \forall e \in \partial\Omega_{i,h}^e \ (i = 1, \ldots, N). \tag{17}$$

We define the extension operator $R_i^T : X_h(\Gamma_i) \rightarrow X_h(\Gamma)$ as

$$\frac{\int_e R_i^T v ds}{|e|} = \begin{cases} \dfrac{\int_e v ds}{|e|}, & \forall e \in \partial\Omega_{i,h}^e, \\ 0, & \forall e \in \Gamma \setminus \partial\Omega_{i,h}^e. \end{cases} \tag{18}$$

Its transpose R_i is defined by

$$(R_i w, v) = \left(w, R_i^T v\right), \qquad \forall w \in X_h(\Gamma), \ v \in X_h(\Gamma_i). \tag{19}$$

Denote $R_{\Delta_i}^T : X_\Delta(\Gamma_i) \rightarrow X_h(\Gamma)$ by $R_{\Delta_i}^T = R_i^T|_{X_\Delta(\Gamma_i)}$; the corresponding transpose is defined by

$$\left(R_{\Delta_i} w, v\right) = \left(w, R_{\Delta_i}^T v\right), \quad \forall w \in X_h(\Gamma), \ v \in X_\Delta(\Gamma_i). \tag{20}$$

To overcome the discontinuous coefficients ρ_i $(i = 1, \ldots, N)$, we define a family of weighted counting functions δ_i associated with Γ_i as follows:

$$\delta_i|_e = \frac{\sum_{j \in \mathscr{N}_e} \rho_j}{\rho_i}, \quad \forall e \in \partial\Omega_{i,h}^e \setminus \partial\Omega, \tag{21}$$

here $\mathscr{N}_e$ is the set of indices j of the subregions such that $e \in \partial\Omega_{j,h}^e$. Actually, δ_i are piecewise constants associated with Γ. Let δ_i^+ be the corresponding pseudoinverse; for any $u \in X_h(\Gamma)$, they provide a partition of unity as follows:

$$\sum_i R_i^T \delta_i^+ \left(u|_{\Omega_i}\right) \equiv u. \tag{22}$$

By an elementary argument, we can see

$$\rho_i \delta_j^{+^2} \leq \min\left\{\rho_i, \rho_j\right\}. \quad (23)$$

According to the construction of R_i^T, given the scaling factors δ_i^+ at the subdomain interface elements' edges, we can define the two scaled extension operators. $R_{D,i}^T : X_h(\Gamma_i) \rightarrow X_h(\Gamma)$

$$\frac{\int_e R_{D,i}^T v ds}{|e|} = \begin{cases} \dfrac{\delta^+ \int_e v ds}{|e|}, & \forall e \in \partial\Omega_{i,h}^e, \\ 0, & \forall e \in \Gamma \setminus \partial\Omega_{i,h}^e, \end{cases} \quad (24)$$

where $R_{D,\Delta_i}^T : X_\Delta(\Gamma_i) \rightarrow X_h(\Gamma)$, $R_{D,\Delta_i}^T = R_{D,i}^T|_{X_\Delta(\Gamma_i)}$. Following (19) the corresponding transposes are denoted by $R_{D,i}$ and R_{D,Δ_i}, respectively.

In what follows, we describe our BDDC preconditioning algorithm, by using the basic framework of additive Schwarz method (or parallel subspace correction method [10]). From the decomposition (15), we only need to define appropriate subspace solvers.

First of all, the coarse subspace solver $B_H : X_H(\Omega) \rightarrow X_H(\Omega)$ is defined by

$$\left(B_H u_H, v_H\right) = a_h\left(u_H, v_H\right), \quad \forall u_H, v_H \in X_H(\Omega). \quad (25)$$

On each subdomain, the similar solver $B_i : X_\Delta(\Gamma_i) \rightarrow X_\Delta(\Gamma_i)$ is given by

$$\left(B_i u, v\right) = a_{h,i}(u, v), \quad \forall u, v \in X_\Delta(\Gamma_i). \quad (26)$$

Remark 1. The bilinear forms on the coarse space can be different from substructure space; here we only use the exact solvers; on each subdomain, we avoid the possible singularity of local subproblem.

Now we define the BDDC preconditioner as

$$B_{bddc} = R_0^T B_H^{-1} R_0 + \sum_{i=1}^{N} R_{D,\Delta_i}^T B_i^{-1} R_{D,\Delta_i}, \quad (27)$$

where $R_0^T = \sum_{i=1}^N R_{D,i}^T I_h$, R_0 is the corresponding transpose defined by

$$\left(R_0 w, v\right) = \left(w, R_0^T v\right), \quad \forall w \in X_h(\Gamma), \ v \in X_H(\Omega). \quad (28)$$

Let P_0 be the operator from $X_h(\Gamma)$ to $X_H(\Omega)$ defined by

$$a_h\left(P_0 u, v\right) = a_h\left(u, R_0^T v\right), \quad \forall u \in X_h(\Gamma), \ v \in X_H(\Omega), \quad (29)$$

and let P_i be the operator from $X_h(\Gamma)$ to $X_\Delta(\Gamma_i)$ defined by

$$a_{h,i}\left(P_i u, v\right) = a_h\left(u, R_{\Delta,i}^T v\right), \quad \forall u \in X_h(\Gamma), \quad v \in X_\Delta(\Gamma_i). \quad (30)$$

Then the BDDC preconditioned operator P_{bddc} can be written as

$$P_{bddc} = R_0^T P_0 + \sum_{i=1}^{N} R_{D,\Delta_i}^T P_i, \qquad B_{bddc} S_h = P_{bddc}. \quad (31)$$

The next key theorem gives an estimate on P_{bddc}.

Theorem 2. *The BDDC preconditioned operators P_{bddc} satisfies*

$$a_h(u, u) \preceq a_h\left(P_{bddc} u, u\right) \preceq \left(1 + \log\frac{H}{h}\right)^2 a_h(u, u), \quad \forall u \in X_h(\Gamma), \quad (32)$$

where $H/h = \max_i(H_i/h_i)$.

4. Technical Tools

In this section we state and prove a technical lemma necessary for the proof of Theorem 2. Our theoretical analysis is based on substructuring theory of conforming element.

We assume $V^h(\Omega_i)$ to be the bilinear conforming element space associated with the partition $\mathcal{T}_h(\Omega_i)$. We split the interface $\partial\Omega_i$ into four open edges $\mathcal{E}$ and define a zero prolong operator $I_{\mathcal{E}}^0$ on $V^h(\partial\Omega_i) = V^h(\Omega_i)|_{\partial\Omega_i}$ as for any $v \in V^h(\partial\Omega_i)$

$$I_{\mathcal{E}}^0 v = \begin{cases} v, & \text{on } \mathcal{E}, \\ 0, & \text{on } \partial\Omega_i \setminus \mathcal{E}. \end{cases} \quad (33)$$

For the operator $I_{\mathcal{E}}^0$, we introduce the following result (cf. [11]).

Lemma 3. *For an edge $\mathcal{E}$ of $\partial\Omega_i$, for any $v \in V^h(\partial\Omega_i)$, one has*

$$\left\|I_{\mathcal{E}}^0 v\right\|_{H^{1/2}(\partial\Omega_i)} \preceq \left(\log\frac{H_i}{h_i}\right) \|v\|_{H^{1/2}(\partial\Omega_i)}. \quad (34)$$

Remark 4. The above lemma is related to vertex-edge-face arguments in substructuring methods, in view of characteristic for the rotated Q_1 element; here the results only concern the inequalities for faces.

Let $V^{h/2}(\Omega_i)$ be the conforming element space of bilinear continuous functions on the triangulation $\mathcal{T}_{h/2}(\Omega_i)$ which is constructed by joining the midpoints of the edges of elements of $\mathcal{T}_h(\Omega_i)$. We now introduce a local equivalence map $\mathcal{M}_i : X_h(\Omega_i) \rightarrow V^{h/2}(\Omega_i)$ as follows (cf. [12]).

Definition 5. Given $v \in X_h(\Omega_i)$, we define $\mathcal{M}_i v \in V^{h/2}(\Omega_i)$ by the values of $\mathcal{M}_i v$ at the vertices of the partition $\mathcal{T}_{h/2}(\Omega_i)$.

(i) If P is a central point of E, $E \in \mathcal{T}_h(\Omega_i)$, then

$$\left(\mathcal{M}_i v\right)(P) = \frac{1}{4}\sum_{e_i \in \partial E}\frac{1}{|e_i|}\int_{e_i} v ds. \quad (35)$$

(ii) If P is a midpoint of one edge $e \in \partial E$, $E \in \mathcal{T}_h(\Omega_i)$, then

$$\left(\mathcal{M}_i v\right)(P) = \frac{1}{|e|}\int_e v ds. \quad (36)$$

(iii) If $P \in \Omega_{i,h} \setminus \partial\Omega_{i,h}$, then

$$\left(\mathcal{M}_i v\right)(P) = \frac{1}{4}\sum_{e_i}\frac{1}{|e_i|}\int_{e_i} v ds, \quad (37)$$

where the sum is taken over all edges e_i with the common vertex P, $e_i \in \partial E_i$, $E_i \in \mathcal{T}_h(\Omega_i)$.

(iv) If $P \in \partial\Omega_{i,h}$, then

$$\begin{aligned}(\mathcal{M}_i v)(P) = {} & \frac{|e_l|}{|e_l| + |e_r|}\left(\frac{1}{|e_l|}\int_{e_l} v\,ds\right) \\ & + \frac{|e_r|}{|e_l| + |e_r|}\left(\frac{1}{|e_r|}\int_{e_r} v\,ds\right),\end{aligned} \tag{38}$$

where $e_l \in \partial E_1 \bigcap \partial\Omega_i$ and $e_r \in \partial E_2 \bigcap \partial\Omega_i$ are the left and right neighbor edges of P, $E_1, E_2 \in \mathcal{T}_h(\Omega_i)$. If P is a vertex of Ω_i, then $E_1 = E_2$.

Remark 6. For $v \in X_h^{\mathcal{E}}(\Omega_i)$, we define an operator $\mathcal{M}_i^{\mathcal{E}} : X_h^{\mathcal{E}}(\Omega_i) \to V^{h/2}(\Omega_i)$ [12, Definition 3.2] ; that is, if P is a vertex of Ω_i, let $(\mathcal{M}_i^{\mathcal{E}} v)(P) = 0$, and their stable pseudoinverse is denoted by $\mathcal{M}_i^{+}$ [12, Lemma 3.2]; here $X_h^{\mathcal{E}}(\Omega_i) = \{v \in X_h(\Omega_i) \mid \int_e v\,ds = 0, \forall e \in \partial\Omega_{i,h}^e \setminus \mathcal{E}\}$.

For the operators $\mathcal{M}_i$ and $\mathcal{M}_i^{+}$, we have the following results (see [12]):

$$\begin{aligned}\left|\mathcal{M}_i v\right|_{H^1(\Omega_i)} &\asymp |v|_{H_h^1(\Omega_i)}, \quad \forall v \in X_h(\Omega_i); \\ \left|\mathcal{M}_i^{+} v\right|_{H_h^1(\Omega_i)} &\preceq |v|_{H_h^1(\Omega_i)}, \quad \forall v \in V^{h/2}(\Omega_i).\end{aligned} \tag{39}$$

For the rotated Q_1 element, we have the following inequality.

Lemma 7. *For any $u_i \in X_\Delta(\Gamma_i)$, we can split u_i into $u_i = \sum_{\Gamma_{ij}\subset\partial\Omega_i} u_{ij}$, and one has*

$$\left|u_{ij}\right|_{H_h^1(\Omega_i)} \preceq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)|u_i|_{H_h^1(\Omega_i)}, \tag{40}$$

where $u_{ij} \in X_\Delta(\Gamma_i)$, and for any $e \in \Gamma_{ij}^e$, $\int_e u_{ij}ds/|e| = \int_e u_i ds/|e|$; for any $e \in \partial\Omega_{i,h}^e \setminus \Gamma_{ij}$, $\int_e u_{ij}ds/|e| = 0$.

Proof. By (39), Lemma 3, the inverse trace theorem, trace theorem, and Poincáre inequality, we obtain

$$\begin{aligned}\left|u_{ij}\right|_{H_h^1(\Omega_i)} &\leq \left|\mathcal{M}_i^{+}\mathcal{H}_i I_{\mathcal{E}}^0 \mathcal{M}_i u_{ij}\right|_{H_h^1(\Omega_i)} \\ &\preceq \left|\mathcal{H}_i I_{\mathcal{E}}^0 \mathcal{M}_i u_{ij}\right|_{H_h^1(\Omega_i)} \preceq \left|I_{\mathcal{E}}^0 \mathcal{M}_i u_{ij}\right|_{H^{1/2}(\partial\Omega_i)} \\ &\preceq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)\left\|\mathcal{M}_i u_i\right\|_{H^{1/2}(\partial\Omega_i)} \\ &\leq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)\left\|\mathcal{M}_i u_i\right\|_{H^1(\Omega_i)} \\ &\preceq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)\left|\mathcal{M}_i u_i\right|_{H^1(\Omega_i)} \\ &\preceq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)|u_i|_{H_h^1(\Omega_i)},\end{aligned} \tag{41}$$

where $\mathcal{H}_i$ is a piecewise bilinear harmonic operator, and we have used the minimal energy property of discrete harmonic functions. □

5. Proof of Theorem 2

In the proof of Theorem 2 we use the abstract framework of ASM methods (see [13]); we have necessary to prove three assumptions. Assumption II follows from the standard coloring argument; now we need to prove Assumption I and Assumption III.

First we show the following stable decomposition.

Lemma 8 (Assumption I). *For any $u \in X_h(\Gamma)$, there is the following decomposition:*

$$u = R_0^T u_H + \sum_{i=1}^{N} R_{D,\Delta_i}^T u_i, \quad u_H \in X_H(\Omega), \; u_i \in X_\Delta(\Gamma_i) \tag{42}$$

that satisfies

$$a_h(u_H, u_H) + \sum_{i=1}^{N} a_{h,i}(u_i, u_i) \preceq a_h(u, u). \tag{43}$$

Proof. First we show the decomposition (42). For any function $u \in X_h(\Gamma)$, let $u_H = I_H u$, and $u_\Delta = u - I_h u_H$, $u_i = u_\Delta|_{\Omega_i}$. From the definitions of I_H and I_h, we have

$$\begin{aligned}\int_{\Gamma_{ij}} u_i ds &= \int_{\Gamma_{ij}} u_\Delta ds \\ &= \int_{\Gamma_{ij}} (u - I_h u_H)\,ds = \int_{\Gamma_{ij}} (u - u_H)\,ds = 0;\end{aligned} \tag{44}$$

by (22) and the definition of R_{D,Δ_i}^T, we get

$$R_0^T u_H + \sum_{i=1}^{N} R_{D,\Delta_i}^T u_i = \sum_{i=1}^{N} R_{D,i}^T I_h u_H + \sum_{i=1}^{N} R_{D,i}^T (u - I_h u_H) = u. \tag{45}$$

Then $u_i \in X_\Delta(\Gamma_i)$ and the equality (42) holds.

Now, we prove stable decomposition (43). We assume $\overline{u}_{\Gamma_{ij}} = \int_{\Gamma_{ij}} u\,ds/|\Gamma_{ij}|$; then using Lemma 3.5 in [14], Poincaré-Friedrichs' inequality, and scaling argument, we can derive

$$\begin{aligned}\sum_{\Gamma_{ij},\Gamma_{ik}\subset\partial\Omega_i}\left|\overline{u}_{\Gamma_{ij}} - \overline{u}_{\Gamma_{ik}}\right|^2 &= \sum_{\Gamma_{ij},\Gamma_{ik}\subset\partial\Omega_i}\left(\frac{1}{\left|\Gamma_{ij}\right|}\int_{\Gamma_{ij}}\left(u - \overline{u}_{\Gamma_{ik}}\right)\right)^2 \\ &\preceq \sum_{\Gamma_{ik}\subset\partial\Omega_i}\left(\frac{1}{H_i^2}\left\|u - \overline{u}_{\Gamma_{ik}}\right\|_{L^2(\Omega_i)}^2 + |u|_{H_h^1(\Omega_i)}^2\right) \\ &\preceq |u|_{H_h^1(\Omega_i)}^2.\end{aligned} \tag{46}$$

From (46) and discrete equivalent norm, we deduce

$$\begin{aligned}a_h(u_H, u_H) &= \sum_{i=1}^{N} a_{h,i}(u_H, u_H) \\ &\asymp \sum_{i=1}^{N}\rho_i \sum_{\Gamma_{ij},\Gamma_{ik}\subset\partial\Omega_i}\left|\overline{u}_{\Gamma_{ij}} - \overline{u}_{\Gamma_{ik}}\right|^2 \\ &\preceq a_h(u, u).\end{aligned} \tag{47}$$

Meanwhile, from the fact that the harmonic function has minimal energy norm and (47), we get

$$\begin{aligned}\sum_{i=1}^{N} a_{h,i}(u_i,u_i) &= a_h(u_\Delta,u_\Delta)\\ &= a_h(u-I_hu_H,u-I_hu_H)\\ &\preceq a_h(u,u).\end{aligned} \tag{48}$$

So (47) and (48) lead to (43), and the proof is completed. □

Next we state the local stability as follows.

Lemma 9 (Assumption III). *For any $u \in X_\Delta(\Gamma_i)$, we have*

$$a_h\left(R^T_{D,\Delta_i}u, R^T_{D,\Delta_i}u\right) \preceq \left(1+\log\frac{H_i}{h_i}\right)^2 a_{h,i}(u,u). \tag{49}$$

And for any $u_H \in X_H(\Omega)$, one has

$$a_h\left(R^T_0u_H, R^T_0u_H\right) \preceq \left(1+\log\frac{H}{h}\right)^2 a_h(u_H,u_H). \tag{50}$$

Proof. To prove (49) we first define $\theta_{\Gamma_{ij}} \in X_h(\Gamma)$ associated with $\Gamma_{ij} \subset \Gamma$ which satisfies

$$\frac{1}{|e|}\int_e \theta_{\Gamma_{ij}}ds = \begin{cases}1, & \forall e \in \Gamma^e_{ij},\\ 0, & \forall e \in \Gamma\setminus\Gamma^e_{ij},\end{cases} \tag{51}$$

and define a zero prolong operator $E_i : X_h(\Omega_i) \to X_h(\Omega)$ as

$$\int_e E_ivds = \begin{cases}\int_e E_ivds, & \forall e \in \Omega^e_{i,h}\bigcup\partial\Omega^e_{i,h},\\ 0, & \text{others}.\end{cases} \tag{52}$$

Then we can decompose $R^T_{D,\Delta_i}u$ as follows:

$$\begin{aligned}R^T_{D,\Delta_i}u = R^T_{D,i}u &= \mathscr{H}\left(\sum_{\Gamma_{ij}\subset\Gamma_i}\mathscr{I}_h\left(\theta_{\Gamma_{ij}}\left(E_i\delta^+_iu\right)\right)\right)\\ &= \sum_{\Gamma_{ij}\subset\Gamma_i}\mathscr{H}\left(\mathscr{I}_h\left(\theta_{\Gamma_{ij}}\left(E_i\delta^+_iu\right)\right)\right),\end{aligned} \tag{53}$$

here $\mathscr{I}_h$ is the integral average interpolation operator on the interface Γ_{ij}, satisfying

$$\int_e \mathscr{I}_h\left(\theta_{\Gamma_{ij}}\left(E_i\delta^+_iu\right)\right)ds = \int_e\theta_{\Gamma_{ij}}ds\cdot\int_e E_i\delta^+_iuds, \quad \forall e \in \Gamma^e_{ij}. \tag{54}$$

Note that the support of $R^T_{D,i}u$ is contained in $\Omega_i\bigcup_{\Gamma_{ij}\subset\partial\Omega_i}(\Omega_j\bigcup\overline{\Gamma}_{ij})$; we denote $\widetilde{u}_j = (R^T_{D,i}u)|_{\Omega_j}$, $\widetilde{u}_i = (R^T_{D,i}u)|_{\Omega_i}$. From Lemma 7 and the definition of δ^+_i, we derive

$$\begin{aligned}\left|\widetilde{u}_i\right|^2_{H^1_\rho(\Omega_i)} &= \left|R^T_{D,i}u\right|^2_{H^1_\rho(\Omega_i)}\\ &\preceq \sum_{\Gamma_{ij}\subset\partial\Omega_i}\left|\mathscr{H}\left(\mathscr{I}_h\left(\theta_{\Gamma_{ij}}\left(E_i\delta^+_iu\right)\right)\right)\right|^2_{H^1_\rho(\Omega_i)}\\ &= \sum_{\Gamma_{ij}\subset\partial\Omega_i}\left|\mathscr{H}_i\left(\mathscr{I}_h\left(\theta_{\Gamma_{ij}}\left(E_i\delta^+_iu\right)\right)\right)\right|^2_{H^1_\rho(\Omega_i)}\\ &\preceq \left(1+\log\left(\frac{H_i}{h_i}\right)\right)^2\left|\mathscr{H}_i\left(E_i\delta^+_iu\right)\right|^2_{H^1_\rho(\Omega_i)}\\ &\preceq \left(1+\log\left(\frac{H_i}{h_i}\right)\right)^2\left|\mathscr{H}_iu\right|^2_{H^1_\rho(\Omega_i)}\\ &= \left(1+\log\left(\frac{H_i}{h_i}\right)\right)^2|u|^2_{H^1_\rho(\Omega_i)}.\end{aligned} \tag{55}$$

Moreover, since $\widetilde{u}_j$ is discrete harmonic in Ω_j with $\int_e\widetilde{u}_jds = 0$ for any $e \in \partial\Omega^e_j\setminus\Gamma_{ij}$, then from Lemma 3.3 in [12], we have

$$\left|\widetilde{u}_j\right|_{H^1_h(\Omega_j)} \preceq \left\|\mathscr{M}^{\mathscr{E}}_j\widetilde{u}_j\right\|_{H^{1/2}_{00}(\Gamma_{ij})}. \tag{56}$$

Since the meshes on subdomain Ω_i and Ω_j align across the interface Γ_{ij}, using the above inequality yields

$$\begin{aligned}\left|\widetilde{u}_j\right|^2_{H^1_\rho(\Omega_j)} &= \rho_j\left|\widetilde{u}_j\right|^2_{H^1_h(\Omega_j)}\\ &\preceq \rho_j\left\|\mathscr{M}^{\mathscr{E}}_j\widetilde{u}_j\right\|^2_{H^{1/2}_{00}(\Gamma_{ij})}\\ &= \rho_j\left\|\mathscr{M}^{\mathscr{E}}_i\widetilde{u}_{ij}\right\|^2_{H^{1/2}_{00}(\Gamma_{ij})},\end{aligned} \tag{57}$$

where $\widetilde{u}_{ij} \in X_\Delta(\Gamma_i)$, and for any $e \in \Gamma^e_{ij}$, $\int_e\widetilde{u}_{ij}ds/|e| = \int_e\widetilde{u}_ids/|e|$; for any $e \in \partial\Omega^e_{i,h}\setminus\Gamma_{ij}$, $\int_e\widetilde{u}_{ij}ds/|e| = 0$.

Using (23), the trace theorem, and Lemma 3, we obtain

$$\begin{aligned}\rho_j\left\|\mathscr{M}^{\mathscr{E}}_i\widetilde{u}_{ij}\right\|^2_{H^{1/2}_{00}(\Gamma_{ij})} &= \rho_j\left\|\mathscr{M}^{\mathscr{E}}_i\mathscr{H}_i\left(\mathscr{I}_h\left(\theta_{\Gamma_{ij}}\left(E_i\delta^+_iu\right)\right)\right)\right\|^2_{H^{1/2}_{00}(\Gamma_{ij})}\\ &\preceq \rho_i\left|\mathscr{M}^{\mathscr{E}}_iu_{ij}\right|^2_{H^{1/2}(\partial\Omega_i)}\\ &\preceq \rho_i\left|\mathscr{M}^{\mathscr{E}}_iu_{ij}\right|^2_{H^1_h(\Omega_i)}\\ &\preceq \rho_i\left|u_{ij}\right|^2_{H^1_h(\Omega_i)}\\ &\preceq \left(1+\log\left(\frac{H_i}{h_i}\right)\right)^2|u|^2_{H^1_\rho(\Omega_i)},\end{aligned} \tag{58}$$

where $u_{ij} \in X_\Delta(\Gamma_i)$, and for any $e \in \Gamma^e_{ij}$, $\int_e u_{ij}ds/|e| = \int_e uds/|e|$; for any $e \in \partial\Omega^e_{i,h}\setminus\Gamma_{ij}$, $\int_e u_{ij}ds/|e| = 0$.

Table 1: The number of iterations and condition numbers.

$M \times M$	$H/h = 4$			$H/h = 16$		
	$k = 2$	$k = 4$	$k = 6$	$k = 2$	$k = 4$	$k = 6$
4×4	9 (2.68)	9 (2.87)	9 (2.87)	11 (3.84)	11 (3.83)	11 (3.76)
8×8	10 (2.78)	10 (2.74)	10 (2.73)	13 (4.25)	13 (4.16)	13 (4.17)
16×16	10 (2.86)	11 (2.84)	12 (2.83)	13 (4.39)	14 (4.34)	14 (4.34)
32×32	10 (2.89)	11 (2.86)	12 (2.84)	13 (4.45)	13 (4.45)	14 (4.39)

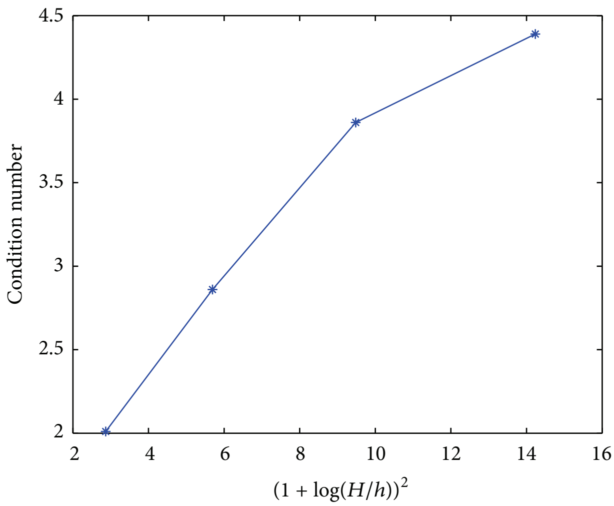

Figure 1: Plot of the condition numbers as the function of $(1 + \log(H/h))^2$.

From (57) and (58), we complete the proof of (49).

Using the similar techniques in (49) and summing over all subdomains, we can complete the proof of (50). □

6. Numerical Results

In this section, we show numerical results of our method using the model problem

$$-\operatorname{div}(\rho \nabla u) = f, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega, \tag{59}$$

where $\Omega = [0,1]^2$. The domain is composed of $M \times M$ subsquares; their mesh sizes are H, and the subsquares are divided into smaller ones with mesh sizes h. The coefficient ρ is either 1 or 10^k $(k = 2, 4, 6)$.

We use the preconditioned conjugate gradient (PCG) method with zero initial guess for the discrete system of equations. The stopping criterion for the PCG method is when the 2-norm of the residual is reduced by the factor of 10^{-6} of the initial guess. An estimate for the condition number of the corresponding system is computed by using the Lanczos algorithm.

In Table 1, we show the number of iterations and the condition numbers with different ratio H/h. In Figure 1, we plot the condition numbers as the function of $(1+\log(H/h))^2$ for 16 domains. From the results in Table 1 and Figure 1, we can see that the convergence of our method is quasioptimal since the number of iterations is independent of jumps in coefficients and almost independent of mesh sizes.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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