Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 858504, 8 pages
http://dx.doi.org/10.1155/2014/858504

Research Article

Some New Lacunary Strong Convergent Vector-Valued Sequence Spaces

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Received 29 March 2014; Accepted 28 June 2014; Published 24 July 2014

Academic Editor: Hassan Eltayeb

We introduce some vector-valued sequence spaces defined by a Musielak-Orlicz function and the concepts of lacunary convergence and strong (A)-convergence, where $A = (a_{ik})$ is an infinite matrix of complex numbers. We also make an effort to study some topological properties and some inclusion relations between these spaces.

1. Introduction and Preliminaries

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex and continuous such that $M(0) = 0$, $M(x) > 0$ for $x > 0$. Lindenstrauss and Tzafriri [1] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}, \tag{1}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. \tag{2}$$

It is shown in [1] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if, for any constant $L > 1$, there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x \eta(t)\,dt, \tag{3}$$

where η is known as the kernel of M and is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$; η is nondecreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathscr{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function; see ([2, 3]). A sequence $\mathscr{N} = (N_k)$ defined by

$$N_k(v) = \sup\left\{|v|\,u - M_k(u) : u \geq 0\right\}, \quad k = 1, 2, \ldots \tag{4}$$

is called the complementary function of a Musielak-Orlicz function $\mathscr{M}$. For a given Musielak-Orlicz function $\mathscr{M}$, the Musielak-Orlicz sequence space $t_{\mathscr{M}}$ and its subspace $h_{\mathscr{M}}$ are defined as follows:

$$\begin{aligned} t_{\mathscr{M}} &= \left\{x \in w : I_{\mathscr{M}}(cx) < \infty \text{ for some } c > 0\right\}, \\ h_{\mathscr{M}} &= \left\{x \in w : I_{\mathscr{M}}(cx) < \infty,\ \forall c > 0\right\}, \end{aligned} \tag{5}$$

where $I_{\mathscr{M}}$ is a convex modular defined by

$$I_{\mathscr{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathscr{M}}. \tag{6}$$

We consider $t_{\mathscr{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_{\mathscr{M}}\left(\frac{x}{k}\right) \leq 1\right\} \tag{7}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\left(1 + I_{\mathscr{M}}(kx)\right) : k > 0\right\}. \tag{8}$$

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants a, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k \tag{9}$$

holds for all $k \in \mathbb{N}$ and $u \in R_+$, whenever $M_k(u) \leq a$.

Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$;

(2) $p(-x) = p(x)$, for all $x \in X$;

(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$;

(4) if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [4], Theorem 10.4.2, P-183). For more details about sequence spaces, see [5–11] and references therein.

The space of lacunary strong convergence has been introduced by Freedman et al. [12]. A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ is defined by Freedman et al. [12] as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - s| = 0, \text{ for some } s \right\}. \tag{10}$$

The space $|\sigma_1|$ of strongly Cesàro summable sequences is

$$|\sigma_1| = \left\{ x = (x_k) : \text{there exists } L \text{ such that } \frac{1}{n}\sum_{i=1}^{n} |x_i - L| \longrightarrow 0, \text{ as } n \longrightarrow \infty \right\}. \tag{11}$$

In case, when $\theta = (2^r)$, $N_\theta = |\sigma_1|$. Recently, Bilgin [13] in his paper generalized the concept of lacunary convergence and introduced the space $N_0(A, f)$, as

$$N_0(A, f) = \left\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} f(|A_i(x) - s|) = 0, \text{ for some } s \right\}, \tag{12}$$

where f is a modulus function and $A = (A_i(x))$; $A_i x = \sum_{k=1}^{\infty} a_{ik} x_k$ converges for each i. Later Bilgin [14] generalized lacunary strongly A-convergent sequences with respect to a sequence of modulus function $F = (f_i)$ as follows:

$$N_0(A, F) = \left\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} f_i(|A_i(x) - s|) = 0, \text{ for some } s \right\}. \tag{13}$$

We write θ for the zero sequences.

Mursaleen and Noman [15] introduced the notion of λ-convergent and λ-bounded sequences as follows.

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity, that is,

$$0 < \lambda_0 < \lambda_1 < \cdots, \qquad \lambda_k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \tag{14}$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \to L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k. \tag{15}$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [15] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0. \tag{16}$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0, \tag{17}$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

We now introduce the concept of lacunary strongly A-convergence for sequences with the elements chosen from a Banach space $(E, \|\cdot\|)$ over the complex field $\mathbb{C}$, with respect to Musielak-Orlicz functions $\mathcal{M} = (M_i)$.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers and $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function. In the present paper we define the following sequence spaces:

$$N_\theta(E, A, \Lambda, \mathcal{M}) = \left\{ x = (x_k) : x_k \in E, \ \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} \left[M_i \left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}} \right) \right] = 0 \right.$$
$$\left. \text{for some } s = (s_1, s_2, \ldots) \in E, \ e_i \in \mathbb{C}, \ \rho^{(i)} > 0 \right\},$$

$$N_\theta^0(E,A,\Lambda,\mathscr{M}) = \Bigg\{x=(x_k): x_k \in E, \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\left[M_i\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho^{(i)}}\right)\right]=0 \text{ for some } \rho^{(i)}>0\Bigg\}. \tag{18}$$

If we take $M_i(x) = x$, for all $i \in \mathbb{N}$, we have

$$N_\theta(E,A,\Lambda) = \Bigg\{x=(x_k): x_k \in E, \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\left[\frac{\|A_i(\Lambda_i(x)-s_ie_i)\|}{\rho^{(i)}}\right]=0 \text{ for some } s=(s_1,s_2,\dots)\in E,\ e_i\in\mathbb{C},\ \rho^{(i)}>0\Bigg\},$$
$$N_\theta^0(E,A,\Lambda) = \Bigg\{x=(x_k): x_k \in E, \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\left[\frac{\|A_i(\Lambda_i(x))\|}{\rho^{(i)}}\right]=0 \text{ for some } \rho^{(i)}>0\Bigg\}, \tag{19}$$

where

$$e_i = \begin{cases}1, & \text{at the } i\text{th place},\\ 0, & \text{otherwise}.\end{cases} \tag{20}$$

A sequence x is said to be Λ-lacunary strong A-convergent with respect to $\mathscr{M}$ if there is a number $s=(s_1,s_2,\dots)\in E$, such that $x \in N_\theta(E,A,\Lambda,\mathscr{M})$.

We have generalized the strongly Cesàro-summable sequence space into Λ-strongly Cesàro-summable vector-valued sequence space as

$$|\sigma_1(E,A,\Lambda)| = \Bigg\{x=x_k: \text{there exists } L=(L_1,L_2,\dots)\in E, e_i\in\mathbb{C} \text{ such that } \frac{1}{n}\sum_{i=1}^{n}\|A_i(\Lambda_i(x)-L_ie_i)\|\longrightarrow 0\Bigg\}, \tag{21}$$

where $A=(a_{nk})$ is a Cesàro matrix, that is,

$$a_{nk} = \begin{cases}\dfrac{1}{n}, & \text{if } 1\le k\le n,\\ 0, & \text{if } k\ge n.\end{cases} \tag{22}$$

Then it can be shown that $|\sigma_1(E,A,\Lambda)|$ is a paranormed space with respect to the paranorm

$$\|x\| = \|x_1\| + \sup_n\left(\frac{1}{n}\sum_{i=1}^{n}\|A_i(\Lambda_i(x))\|\right). \tag{23}$$

2. Topological Properties of the Spaces $N_\theta(E,A,\Lambda,\mathscr{M})$ and $N_\theta^0(E,A,\Lambda,\mathscr{M})$

Theorem 1. *Let $A=(a_{ik})$ be an infinite matrix of complex numbers and let $\mathscr{M}=(M_i)$ be a Musielak-Orlicz function. Then $N_\theta(E,A,\Lambda,\mathscr{M})$ and $N_\theta^0(E,A,\Lambda,\mathscr{M})$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. It is easy to prove. □

Theorem 2. *Let $A=(a_{ik})$ be an infinite matrix of complex numbers and let $\mathscr{M}=(M_i)$ be a Musielak-Orlicz function. Then $N_\theta^0(E,A,\Lambda,\mathscr{M})$ is normal spaces, when E is normal.*

Proof. Let $x\in N_\theta^0(E,A,\Lambda,\mathscr{M})$. Let $\|y\|\le\|x\|$. Then

$$\|A_i(\Lambda_i(y))\| \le \|A_i(\Lambda_i(x))\|. \tag{24}$$

Since $\mathscr{M}=(M_i)$ is increasing,

$$\frac{1}{h_r}\sum_{i\in I_r}\left[M_i\left(\frac{\|A_i(\Lambda_i(y))\|}{\rho^{(i)}}\right)\right] \le \frac{1}{h_r}\sum_{i\in I_r}\left[M_i\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho^{(i)}}\right)\right]. \tag{25}$$

Consequently, $y\in N_\theta^0(E,A,\Lambda,\mathscr{M})$. This completes the proof of the theorem. □

Theorem 3. *The spaces $N_\theta(E,A,\Lambda,\mathscr{M})$ and $N_\theta^0(E,A,\Lambda,\mathscr{M})$ are paranormed spaces, with respect to the paranorm*

$$\|x\| = \inf\Bigg\{\rho^{(i)}>0 : M_i\left(\frac{\|a_{i0}x_1\|}{\rho^{(i)}}\right) + \sup_{r\ge1}\frac{1}{h_r}\sum_{i\in I_r}\left[M_i\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho^{(i)}}\right)\right]\le 1, \rho^{(i)}\ge 0\Bigg\}. \tag{26}$$

Proof. It is easy to prove, so we omit the details. □

3. Relation between the Spaces $N_\theta(E,A,\Lambda)$ and $N_\theta(E,A,\Lambda,\mathscr{M})$

The main purpose of this section is to study relation between $N_\theta(E,A,\Lambda)$ and $N_\theta(E,A,\Lambda,\mathscr{M})$.

Theorem 4. *Let $A=(a_{ik})$ be an infinite matrix of complex numbers and let $\mathscr{M}=(M_i)$ be a Musielak-Orlicz function* *satisfying* Δ_2*-condition. If* x *is* Λ*-lacunary strong* A*-convergent to* s*, with respect to* $\mathscr{M}$ *and* $(E, \|\cdot\|)$ *is a normal Banach space, then* $N_\theta(E, A, \Lambda) \subset N_\theta(E, A, \Lambda, \mathscr{M})$.

Proof. Let $x \in N_\theta(E, A, \Lambda)$ and $x \xrightarrow{\Lambda} s$, where $s = (s_1, s_2, \ldots) \in E, e_i \in \mathbb{C}$. Then

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho} \right) = 0 \quad \text{for some } \rho > 0. \tag{27}$$

We define two sequences y and z such that

$$\begin{aligned} &(\|A_i(\Lambda_i(y) - s_i e_i)\|) \\ &= \begin{cases} (\|A_i(\Lambda_i(x) - s_i e_i)\|), & \text{if } (\|A_i(\Lambda_k(x) - s_i e_i)\|) > 1, \\ \theta, & \text{if } (\|A_i(\Lambda_i(x) - s_i e_i)\|) \le 1, \end{cases} \\ &(\|A_i(\Lambda_i(z) - s_i e_i)\|) \\ &= \begin{cases} \theta, & \text{if } (\|A_i(\Lambda_i(x) - s_i e_i)\|) > 1, \\ (\|A_i(\Lambda_i(x) - s_i e_i)\|), & \text{if } (\|A_i(\Lambda_k(x) - s_i e_i)\|) \le 1. \end{cases} \end{aligned} \tag{28}$$

Hence,

$$\begin{aligned} (\|A_i(\Lambda_i(x) - s_i e_i)\|) = {} & (\|A_i(\Lambda_i(y) - s_i e_i)\|) \\ & + (\|A_i(\Lambda_i(z) - s_i e_i)\|). \end{aligned} \tag{29}$$

Now,

$$\begin{aligned} (\|A_i(\Lambda_i(y) - s_i e_i)\|) \le (\|A_i(\Lambda_i(x) - s_i e_i)\|), \\ (\|A_i(\Lambda_i(z) - s_i e_i)\|) \le (\|A_i(\Lambda_i(x) - s_i e_i)\|). \end{aligned} \tag{30}$$

Since $N_\theta(E,A,\Lambda)$ is normal, $y, z \in N_\theta(E, A, \Lambda)$. Let $\sup_i M_i(2) = T$. Then

$$\begin{aligned} &\frac{1}{h_r} \sum_{i\in I_r} \left[M_i \left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}} \right) \right] \\ &= \frac{1}{h_r} \sum_{i\in I_r} \left[M_i \left(\frac{\|A_i(\Lambda_i(y) - s_i e_i)\| + \|A_i(\Lambda_i(z) - s_i e_i)\|}{\rho^{(i)}} \right) \right] \\ &\le \frac{1}{h_r} \sum_{i\in I_r} \left[\frac{1}{2} M_i \left(2 \frac{\|A_i(\Lambda_i(y) - s_i e_i)\|}{\rho^{(i)}} \right) \right. \\ &\qquad \left. + \frac{1}{2} M_i \left(2 \frac{\|A_i(\Lambda_i(z) - s_i e_i)\|}{\rho^{(i)}} \right) \right] \\ &< \frac{1}{2} \frac{1}{h_r} \sum_{i\in I_r} K_1 \left(\frac{\|A_i(\Lambda_i(y) - s_i e_i)\|}{\rho^{(i)}} \right) M_i(2) \\ &\quad + \frac{1}{2} \frac{1}{h_r} \sum_{i\in I_r} K_2 \left(\frac{\|A_i(\Lambda_i(z) - s_i e_i)\|}{\rho^{(i)}} \right) M_i(2) \\ &\le \frac{1}{2} \frac{1}{h_r} \sum_{i\in I_r} K_1 \left(\frac{\|A_i(\Lambda_i(y) - s_i e_i)\|}{\rho^{(i)}} \right) \sup M_i(2) \\ &\quad + \frac{1}{2} \frac{1}{h_r} \sum_{i\in I_r} K_2 \left(\frac{\|A_i(\Lambda_i(z) - s_i e_i)\|}{\rho^{(i)}} \right) \sup M_i(2) \\ &\longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \end{aligned} \tag{31}$$

Hence $x \in N_\theta(E, A, \Lambda, \mathscr{M})$. This completes the proof of the theorem. □

Theorem 5. *Let* $A = (a_{ik})$ *be an infinite matrix of complex numbers and let* $\mathscr{M} = (M_i)$ *be a Musielak-Orlicz function satisfying* Δ_2*-condition. If*

$$\lim_{u\to\infty} \inf_i \frac{M_i(v/\rho^{(i)})}{v/\rho^{(i)}} > 0 \quad \textit{for some } \rho^{(i)} > 0, \tag{32}$$

then $N_\theta(E, A, \Lambda) = N_\theta(E, A, \Lambda, \mathscr{M})$.

Proof. If $\lim_{v\to\infty} \inf_i (M_i(v/\rho^{(i)})/v/\rho^{(i)}) > 0$ for some $\rho^{(i)} > 0$, then there exists a number $\gamma > 0$ such that

$$M_i\left(\frac{v}{\rho^{(i)}}\right) \ge \gamma \left(\frac{v}{\rho^{(i)}}\right) \quad \forall v > 0 \text{ and some } \rho^{(i)} > 0. \tag{33}$$

Let $x \in N_\theta(E, A, \Lambda, \mathscr{M})$ and $x \xrightarrow{\Lambda} s$, where $s = (s_1, s_2, \ldots) \in E$, $e_i \in \mathbb{C}$. Then clearly

$$\begin{aligned} &\frac{1}{h_r} \sum_{i\in I_r} \left[M_i \left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}} \right) \right] \\ &\qquad \ge \frac{1}{h_r} \sum_{i\in I_r} u_i \left[\gamma \left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}} \right) \right] \\ &\qquad = \gamma \frac{1}{h_r} \sum_{i\in I_r} \left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}} \right). \end{aligned} \tag{34}$$

Hence $x \in N_\theta(E, A, \Lambda)$. This completes the proof. □

4. Relation between the Spaces $|\sigma_1(E,A,\Lambda)|$ and $N_\theta(E,A,\Lambda)$

In this section of the paper we study relation between the spaces $|\sigma_1(E, A, \Lambda)|$ and $N_\theta(E, A, \Lambda)$.

Lemma 6. $|\sigma_1(E, A, \Lambda)|^0 \subset N_\theta(E, A, \Lambda)$ *if and only if* $\liminf_r q_r > 1$.

Proof. First suppose that $\liminf_r q_r > 1$. Then there exist $\delta > 0$ such that $1 + \delta \le q_r$ for all $r \ge 1$. Let $x \in |\sigma_1(E, A, \Lambda)|^0$. Then

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}\|A_i(\Lambda_i(x))\| &= \frac{1}{h_r}\sum_{i=1}^{k_r}\|A_i(\Lambda_i(x))\| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}}\|A_i(\Lambda_i(x))\| \\ &= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r}\|A_i(\Lambda_i(x))\|\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}\|A_i(\Lambda_i(x))\|\right).\end{aligned} \tag{35}$$

Now, $h_r = k_r - k_{r-1}$. So we have

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_r - 1} = 1 + \frac{1}{q_r - 1} \le 1 + \frac{1}{\delta} = \frac{\delta + 1}{\delta}. \tag{36}$$

Also

$$\frac{k_{r-1}}{h_r} = \frac{k_{r-1}}{k_r - k_{r-1}} = \frac{1}{q_r - 1} \le \frac{1}{\delta}. \tag{37}$$

Since $x \in |\sigma_1(E, A, \Lambda)|^0$, then

$$\frac{1}{k_r}\sum_{i=1}^{k_r}\|A_i(\Lambda_i(x))\| \longrightarrow 0, \qquad \frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}\|A_i(\Lambda_i(x))\| \longrightarrow 0, \tag{38}$$

and hence

$$\frac{1}{h_r}\sum_{i\in I_r}\|A_i(\Lambda_i(x))\| \longrightarrow 0; \tag{39}$$

that is, $x \in N_\theta^0(E, A, \Lambda)$. By linearity, it follows that $|\sigma_1(E, A, \Lambda)|^0 \subset N_\theta(E, A, \Lambda)$.

Next, suppose that $\liminf_r q_r = 1$. Since θ is lacunary we can select a subsequence k_{r_j} of θ such that

$$\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j}, \qquad \frac{k_{r_j-1}}{k_{r_{j-1}}} > j, \tag{40}$$

where $r_j \ge r_{j-1} + 2$. Define $x = (x_i)$ by

$$\Lambda_i(x) = \begin{cases} e_i, & \text{if } i \in I_{r_j}, \text{ for some } j = 1, 2, \ldots, \\ \theta, & \text{otherwise,}\end{cases} \tag{41}$$

where $\|e_i\| = 1$ and let $A = I$, and then for any $L = (L_1, L_2, \ldots) \in E$, $e_i \in \mathbb{C}$,

$$\begin{aligned}\frac{1}{h_{r_j}}\sum_{i\in I_r}\left(\frac{\|A_i(\Lambda_i(x) - L)\|}{\rho}\right) &= \frac{\|e_i - L_i e_i\|}{\rho} \\ &= \frac{\|1 - L_i\|}{\rho} \quad \text{for } j = 1, 2, \ldots, \\ \frac{1}{h_r}\sum_{i\in I_r}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) &= \frac{\|e_i\|}{\rho} = \frac{1}{\rho}.\end{aligned} \tag{42}$$

So, $x \notin N_\theta(E, A, \Lambda)$. But x is strongly Cesàro-summable, since if t is sufficiently large integer we can find the unique j for which $k_{r_j-1} < t \le k_{r_{j+1}-1}$ and hence

$$\begin{aligned}\frac{1}{t}\sum_{i=1}^{t}(\|A_i(\Lambda_i(x))\|) &< \frac{1}{k_{r_j-1}}\sum_{i=1}^{t} 1 \\ &\le \frac{1}{k_{r_j-1}} k_{r_j} \le \frac{k_{r_j-1} + h_{r_j}}{k_{r_j-1}} \\ &< \frac{1}{j} + \frac{1}{j} = \frac{2}{j}, \quad \text{as } t \longrightarrow \infty,\end{aligned} \tag{43}$$

and it follows that also $j \to \infty$. Hence $x \in |\sigma_1(E, A, \Lambda)|^0$. □

Lemma 7. *$N_\theta(E, A, \Lambda) \subset |\sigma_1(E, A, \Lambda)|$ if and only if $\limsup_r q_r < \infty$.*

Proof. First suppose that if $\limsup_r q_r < \infty$, there exists $M > 0$ such that $q_r < M$ for all $r \ge 1$. Let $x \in N_\theta(E, A, \Lambda)$ and $\epsilon > 0$. Then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) = 0 \quad \text{for some } \rho > 0. \tag{44}$$

Then we can find $R > 0$ and $K > 0$ such that

$$\begin{aligned}&\sup_{j\ge R}\frac{1}{h_j}\sum_{I_j}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) < \epsilon, \\ &\frac{1}{h_j}\sum_{I_j}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) < K \quad \forall i = 1, 2, \ldots\end{aligned} \tag{45}$$

Then if t is any integer with

$$k_{r-1} \le t \le k_r, \quad \text{where } r > R, \tag{46}$$

then

$$\begin{aligned}&\frac{1}{t}\sum_{j=1}^{t}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) \\ &\le \frac{1}{k_{r-1}}\sum_{i=1}^{k_r}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) \\ &= \frac{1}{k_{r-1}}\left(\sum_{I_1}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) + \sum_{I_2}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right) + \cdots + \sum_{I_r}\left(\frac{\|A_i(\Lambda_i(x))\|}{\rho}\right)\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{k_1}{k_{r-1}} \frac{1}{h_1} \sum_{I_1} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \\
&\quad + \frac{k_2 - k_1}{k_{r-1}} \frac{1}{h_2} \sum_{I_2} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \\
&\quad + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}} \frac{1}{h_R} \sum_{I_R} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \\
&\quad + \frac{k_{R+1} - k_R}{k_{r-1}} \frac{1}{h_{R+1}} \sum_{I_{R+1}} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \\
&\quad + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}} \frac{1}{h_r} \sum_{I_r} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \\
&\le \frac{k_R}{k_{r-1}} \sup_{i \ge 1} \frac{1}{h_i} \sum_{I_i} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \\
&\quad + \frac{k_r - k_R}{k_{r-1}} \frac{1}{h_r} \sum_{I_r} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \\
&< K \frac{k_R}{k_{r-1}} + \epsilon \left(q_r - \frac{k_R}{k_{r-1}} \right) \\
&< K \frac{k_R}{k_{r-1}} + \epsilon q_r \\
&< K \frac{k_R}{k_{r-1}} + \epsilon M.
\end{aligned} \tag{47}$$

Since $k_{r-1} \to \infty$ as $r \to \infty$, it follows that

$$\frac{1}{t} \sum_{j=1}^{t} \left(\frac{\|A_i(\Lambda_i(x))\|}{\rho} \right) \longrightarrow 0 \tag{48}$$

and hence $x \in |\sigma_1(E, A, \Lambda)|$.

Next, suppose that $\limsup_r q_r = \infty$. We construct a sequence in $N_\theta(E, A, \Lambda)$ that is not Cesaro Λ-summable. By the idea of Freedman et al. [12] we can construct a subsequence k_{r_j} of the lacunary sequence $\theta = (k_r)$ such that $q_{r_j} > j$, and then define a bounded difference sequence x by

$$\Lambda_i(x) = \begin{cases} e_i, & \text{if } k_{r_j-1} < i < 2k_{r_j-1}, \\ \theta, & \text{otherwise}, \end{cases} \tag{49}$$

where $\|e_i\| = 1$. Let $A = I$ and $\rho = 1$. Then,

$$\frac{1}{h_{r_j}} \sum_{I_{r_j}} (\|A_i(\Lambda_i(x))\|) = \frac{2k_{r_j-1} - k_{r_j-1}}{k_{r_j} - k_{r_j-1}} = \frac{k_{r_j-1}}{k_{r_j} - k_{r_j-1}} < \frac{1}{j-1} \tag{50}$$

and if $r \neq r_j$,

$$\frac{1}{h_{r_j}} \sum_{I_{r_j}} (\|A_i(\Lambda_i(x))\|) = 0. \tag{51}$$

Thus $x \in N_\theta(E, A, \Lambda)$. For the above sequence and for $i = 1, 2, \ldots, k_{r_j}$

$$\begin{aligned}
\frac{1}{k_{r_j}} \sum_i (\|A_i(\Lambda_i(x) - e_i)\|) &> \frac{1}{k_{r_j}} \left(2k_{r_j-1} - k_{r_j-1} \right) \\
&= 1 - \frac{2}{q_{r_j}} > 1 - \frac{2}{j},
\end{aligned} \tag{52}$$

this converges to 1, but for $i = 1, 2, \ldots, 2k_{r_j-1}$,

$$\frac{2}{k_{r_{j-1}}} \sum_i (\|A_i(\Lambda_i(x))\|) \ge \frac{k_{r_j-1}}{2k_{r_j-1}} = \frac{1}{2}. \tag{53}$$

It proves that $x \notin |\sigma_1(E, A, \Lambda)|$, since any sequence in $|\sigma_1(E, A, \Lambda)|$ consisting of θ's and e_i's has a limit only 0 or 1. □

Theorem 8. *Let θ be a lacunary sequence. Then $|\sigma_1(E, A, \Lambda)| = N_\theta(E, A, \Lambda)$ if and only if $1 \le \liminf_r q_r \le \limsup_r q_r < \infty$.*

Proof. The proof of the theorem follows from Lemmas 6 and 7. □

5. Statistical Convergence

The notion of statistical convergence was introduced by Fast [16] and Schoenberg [17] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [18], Connor [19], Šalát [20], Mursaleen and Edely [21], Isk [22], Mohiuddine and Alghamdi [23], Hazarika et al. [24], Kolk [25], Maddox [26], Alotaibi and Mursaleen [27], Mohiuddine et al. [28], Mohiuddine and Aiyub [29], and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists:

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k), \tag{54}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ k \le n : |x_k - L| \ge \epsilon \} \right| = 0. \tag{55}$$

Bilgin [14] also introduced the concept of statistical convergence in $N_0(A, F)$ and proved some inclusion relation.

Let θ be a lacunary sequence and let $A = (a_{ik})$ be an infinite matrix of complex numbers. Then a sequence $x \in N_\theta(E, A, \Lambda, \mathscr{M})$ is said to be Λ-lacunary A-statistically convergent to a number $s = (s_1, s_2, \ldots) \in E$, $e_i \in \mathbb{C}$ if for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\Lambda_k A_0(\epsilon)\right| = 0, \tag{56}$$

where

$$\Lambda A_0(\epsilon) = \left\{ i \in I_r : M_i\left(\frac{\|A_i(\Lambda_k(x) - s_i e_i)\|}{\rho^{(i)}}\right) \geq \epsilon \right\}. \tag{57}$$

We denote it as $x \xrightarrow{\Lambda\text{-stat}} s$. The vertical bar denotes the cardinality of the set. The set of all Λ-lacunary A-statistical convergent sequences is denoted by $S_\theta(E, A, \Lambda, \mathscr{M})$.

In this section we study some relation between the spaces $S_\theta(E, A, \Lambda, \mathscr{M})$ and $N_\theta(E, A, \Lambda, \mathscr{M})$.

Theorem 9. *Let $\mathscr{M} = (M_i)$ be a Musielak-Orlicz function and let (M_i) be pointwise convergent. Then $N_\theta(E, A, \Lambda, \mathscr{M}) \subset S_\theta(E, A, \Lambda, \mathscr{M})$ if and only if $\lim_i M_i(v/\rho^{(i)}) > 0$ for some $v > 0$, $\rho^{(i)} > 0$.*

Proof. Let $\epsilon > 0$ and $x \in N_\theta(E, A, \Lambda, \mathscr{M})$. Let $x \xrightarrow{\Lambda} s$, where $s = (s_1, s_2, \ldots) \in E$, $e_i \in \mathbb{C}$. Since $\lim_i M_i(v/\rho) > 0$, there exists a number $c > 0$ such that

$$M_i\left(\frac{v}{\rho}\right) \geq c \quad \text{for } v > \epsilon. \tag{58}$$

Let

$$I_r^1 = \left\{ i \in I_r : \left[M_i\left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}}\right)\right] \geq \epsilon \right\}. \tag{59}$$

Then

$$\begin{aligned} &\frac{1}{h_r}\sum_{i\in I_r}\left[M_i\left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}}\right)\right] \\ &\quad \geq \frac{1}{h_r}\sum_{i\in I_r^1}\left[M_i\left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}}\right)\right] \\ &\quad \geq c\frac{1}{h_r}\left|\Lambda_k A_0(\epsilon)\right|. \end{aligned} \tag{60}$$

Hence it follows that $x \in S_\theta(E, A, \Lambda, \mathscr{M})$.

Conversely, let us assume that the condition does not hold good. Then there is a number $v > 0$ such that $\lim_i M_i(v/\rho) = 0$ for some $\rho > 0$. Now, we select a lacunary sequence $\theta = (k_r)$ such that $M_i(v/\rho) < 2^{-r}$ for any $i > k_r$.

Let $A = I$, and define the sequence x by putting

$$\Lambda_i(x) = \begin{cases} v, & \text{if } k_{r-1} < i \leq \dfrac{k_r + k_{r-1}}{2}, \\ \theta, & \text{if } \dfrac{k_r + k_{r-1}}{2} < i \leq k_r. \end{cases} \tag{61}$$

Therefore,

$$\begin{aligned} &\frac{1}{h_r}\sum_{i\in I_r} M_i\left(\frac{\|\Lambda_i(x)\|}{\rho^{(i)}}\right) \\ &\quad = \frac{1}{h_r}\sum_{k_{r-1} < i \leq (k_r + k_{r-1})/2} M_i\left(\frac{v}{\rho^{(i)}}\right) \\ &\quad < \frac{1}{h_r}\frac{1}{2^{r-1}}\left[\frac{k_r + k_{r-1}}{2} - k_{r-1}\right] \\ &\quad = \frac{1}{2^r} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \end{aligned} \tag{62}$$

Thus, we have $x \in N_\theta^0(E, A, \Lambda, \mathscr{M})$. But

$$\begin{aligned} &\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{ i \in I_r : M_i\left(\frac{\|\Lambda_i(x)\|}{\rho^{(i)}}\right) \geq \epsilon \right\}\right| \\ &= \lim_{r\to\infty}\frac{1}{h_r}\left|\left\{ i \in \left(k_{r-1}, \frac{k_r + k_{r-1}}{2}\right) : M_i\left(\frac{v}{\rho^{(i)}}\right) \geq \epsilon \right\}\right| \\ &= \lim_{r\to\infty}\frac{1}{h_r}\frac{k_r - k_{r-1}}{2} = \frac{1}{2}. \end{aligned} \tag{63}$$

So $x \notin S_\theta(E, A, \Lambda, \mathscr{M})$. □

Theorem 10. *Let $\mathscr{M} = (M_i)$ be a Musielak-Orlicz function. Then $S_\theta(E, A, \Lambda, \mathscr{M}) \subset N_\theta(E, A, \Lambda, \mathscr{M})$ if and only if $\sup_v \sup_i M_i(v/\rho) < \infty$.*

Proof. Let $x \in S_\theta(E, A, \Lambda, \mathscr{M})$ and $x \xrightarrow{\Lambda\text{-stat}} s$. Suppose $h(v) = \sup_i M_i(v/\rho)$ and $h = \sup_v h(v)$. Let

$$I_r^2 = \left\{ i \in I_r : M_i\left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}}\right) < \epsilon \right\}. \tag{64}$$

Now, $M_i(v) \leq h$ for all i, $v > 0$. So

$$\begin{aligned} &\frac{1}{h_r}\sum_{i\in I_r}\left[M_i\left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}}\right)\right] \\ &\quad = \frac{1}{h_r}\sum_{i\in I_r^1}\left[M_i\left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}}\right)\right] \\ &\qquad + \frac{1}{h_r}\sum_{i\in I_r^2}\left[M_i\left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}}\right)\right] \\ &\quad \leq h\frac{1}{h_r}\left|\Lambda_k A_0(\epsilon)\right| + h(\epsilon). \end{aligned} \tag{65}$$

Hence, as $\epsilon \to 0$, it follows that $x \in N_\theta(E, A, \Lambda, \mathscr{M})$.

Conversely, suppose that

$$\sup_v \sup_i M_i\left(\frac{v}{\rho}\right) = \infty. \tag{66}$$

Then we have

$$0 < v_1 < v_2 < \cdots < v_{r-1} < v_r < \cdots, \tag{67}$$

so that $M_{k_r}(v_r/\rho) \geq h_r$ for $r \geq 1$. Let $A = I$. We set a sequence $x = (x_i)$ by

$$\Lambda_i(x) = \begin{cases} v_r, & \text{if } i = k_r \text{ for some } r = 1, 2, \ldots, \\ \theta, & \text{otherwise.} \end{cases} \tag{68}$$

Then

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : \left[M_i \left(\frac{\|\Lambda_i(x)\|}{\rho^{(i)}} \right) \right] \geq \epsilon \right\} \right| = \lim_{r\to\infty} \frac{1}{h_r} = 0. \tag{69}$$

Hence $x \xrightarrow{\Lambda\text{-stat}} 0$ and hence $x \in S_\theta(E, A, \Lambda, \mathscr{M})$.

But

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \left[M_i \left(\frac{\|A_i(\Lambda_i(x) - s_i e_i)\|}{\rho^{(i)}} \right) \right] \\ &= \lim_{r\to\infty} \frac{1}{h_r} \left[M_{k_r} \left(\frac{\|v_r - s_i e_i\|}{\rho^{(i)}} \right) \right] \\ &\geq \lim_{r\to\infty} \frac{1}{h_r} h_r = 1. \end{aligned} \tag{70}$$

So, $x \notin N_\theta(E, A, \Lambda, \mathscr{M})$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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