

Research Article

A Suzuki Type Coupled Fixed Point Theorem for Generalized Multivalued Mapping

Pushpendra Semwal and Ramesh Chandra Dimri

Department of Mathematics, HNB Garhwal University, Srinagar Garhwal, Pauri Garhwal, Uttarakhand 246174, India

Correspondence should be addressed to Pushpendra Semwal; psr sdm@gmail.com

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We obtain a new Suzuki type coupled fixed point theorem for a multivalued mapping T from $X \times X$ into $CB(X)$, satisfying a generalized contraction condition in a complete metric space. Our result unifies and generalizes various known comparable results in the literature. We also give an application to certain functional equations arising in dynamic programming.

1. Introduction and Preliminaries

In 2008, Suzuki [1] introduced a new type of mappings which generalize the well-known Banach contraction principle [2], and, further, Kikkawa and Suzuki [3] proved a Kannan [4] version of mappings.

Theorem 1 (see [3]). *Let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be a self-map and let $\phi : [0, 1) \rightarrow (1/2, 1]$ be defined by*

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r \leq 1. \end{cases} \quad (1)$$

Let $\alpha \in [0, 1/2)$ and $r = \alpha/(1 - \alpha) \in [0, 1)$. Suppose that

$$\phi(r) d(x, Tx) \leq d(x, y) \text{ implies} \quad (2)$$

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$. Then, T has a unique fixed point z , and $\lim_n T^n x = z$ holds for every $x \in X$.

Let (X, d) be a metric space. We denote by $CB(X)$ the family of all nonempty, closed bounded subsets of X . Let H be a Hausdorff metric; that is,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\} \quad (3)$$

for $A, B \in CB(X)$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

Nadler [5] proved multivalued extension of the Banach contraction principle as follows.

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that

$$H(Tx, Ty) \leq rd(x, y) \quad (4)$$

for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$. Many fixed point theorems have been proved by various authors as a generalization of Nadler's theorem [6–9]. One of the general fixed point theorems for a generalized multivalued mapping appears in [10].

Theorem 2 (see [11]). Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists a function ϕ from $[0, 1]$ into $(0, 1]$ defined by

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2}, \\ 1-r & \text{if } \frac{1}{2} \leq r < 1, \end{cases} \quad (5)$$

such that

$$\phi(r) d(x, Tx) \leq d(x, y) \text{ implies} \quad (6)$$

$$H(Tx, Ty) \leq r \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \quad (7)$$

for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.

Bhaskar and Lakshmikantham [12] introduced the concept of coupled fixed point for a mapping F from $X \times X$ to X and established some coupled fixed point theorems in partially ordered sets. As an application, they studied the existence and uniqueness of solution for a periodic boundary value problem associated with a first order ordinary differential equation.

Definition 3. Let (X, d) be a metric space and $F : X \times X \rightarrow 2^X$. An element $(x, y) \in X \times X$ is called a coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.

The aim of this paper is to obtain coupled fixed point for a multivalued mapping $T : X \times X \rightarrow CB(X)$ which satisfies the generalized contraction condition in complete metric spaces. Our results unify, extend, and generalize various known comparable results in the literature.

2. Main Results

Now, we shall prove our main result.

Firstly, we define a nonincreasing function ϕ from $[0,1]$ into $[0,1]$ by

$$\phi(r) = \begin{cases} r & \text{if } 0 \leq r < \frac{1}{2}, \\ 1-r & \text{if } \frac{1}{2} \leq r < 1. \end{cases} \quad (8)$$

Theorem 4. Let (X, d) be a complete metric space and let T be a mapping from $X \times X$ into $CB(X)$. Assume that there exists $r \in [0, 1]$ such that

$$\phi(r) [d(x, T(x, y)) + d(u, T(u, v))] \leq d(x, u) + d(y, v) \quad (9)$$

implies

$$H(T(x, y), T(u, v)) \leq \frac{r}{2} \max \left\{ d(x, u) + d(y, v), d(x, T(x, y)) + d(y, T(y, x)), d(u, T(u, v)) + d(v, T(v, u)), \frac{d(x, T(u, v)) + d(y, T(v, u))}{2}, \frac{d(u, T(x, y)) + d(v, T(y, x))}{2} \right\} \quad (10)$$

for all $x, y, u, v \in X$. Then, there exist $z, z' \in X$ such that $z \in T(z, z')$ and $z' \in T(z', z)$.

Proof. Let r_1 be a real number such that $0 \leq r < r_1 < 1$, and $x_1, y_1 \in X$ such that $x_2 \in T(x_1, y_1)$ and $y_2 \in T(y_1, x_1)$. Since $x_2 \in T(x_1, y_1)$ and $y_2 \in T(y_1, x_1)$, then

$$\begin{aligned} d(x_2, T(x_2, y_2)) &\leq H(T(x_1, y_1), T(x_2, y_2)), \\ d(y_2, T(y_2, x_2)) &\leq H(T(y_1, x_1), T(y_2, x_2)), \end{aligned} \quad (11)$$

and, as $\phi(r) < 1$,

$$\begin{aligned} \phi(r) (d(x_2, T(x_2, y_2)) + d(y_2, T(y_2, x_2))) &\leq d(x_2, T(x_2, y_2)) + d(y_2, T(y_2, x_2)) \\ &\leq d(x_2, x_3) + d(y_2, y_3). \end{aligned} \quad (12)$$

Thus, from assumption (10), we have

$$\begin{aligned} d(x_2, T(x_2, y_2)) &\leq H(T(x_1, y_1), T(x_2, y_2)) \\ &\leq \frac{r}{2} \max \left\{ d(x_1, x_2) + d(y_1, y_2), d(x_1, T(x_1, y_1)) + d(y_1, T(y_1, x_1)), d(x_2, T(x, y_2)) + d(y_2, T(y_2, x_2)), \frac{d(x_1, T(x_2, y_2)) + d(y_1, T(y_2, x_2))}{2}, \frac{d(x_2, T(x_1, y_1)) + d(y_2, T(y_1, x_1))}{2} \right\}, \end{aligned} \quad (13)$$

$$\begin{aligned}
 & d(x_2, x_3) \\
 & \leq \frac{r}{2} \max \left\{ d(x_1, x_2) + d(y_1, y_2), d(x_1, x_2) \right. \\
 & \quad \left. + d(y_1, y_2), d(x_2, x_3) + d(y_2, y_3), \right. \\
 & \quad \left. \frac{d(x_1, x_3) + d(y_1, y_3)}{2}, \right. \\
 & \quad \left. \frac{d(x_2, x_2) + d(y_2, y_2)}{2} \right\}, \tag{14}
 \end{aligned}$$

$$\begin{aligned}
 & d(y_2, T(y_2, x_2)) \\
 & \leq H(T(y_1, x_1), T(y_2, x_2)) \\
 & \leq \frac{r}{2} \max \left\{ d(x_1, x_2) + d(y_1, y_2), d(x_1, T(x_1, y_1)) \right. \\
 & \quad \left. + d(y_1, T(y_1, x_1)), d(x_2, T(x, y_2)) \right. \\
 & \quad \left. + d(y_2, T(y_2, x_2)), \right. \\
 & \quad \left. \frac{d(x_1, T(x_2, y_2)) + d(y_1, T(y_2, x_2))}{2}, \right. \\
 & \quad \left. \frac{d(x_2, T(x_1, y_1)) + d(y_2, T(y_1, x_1))}{2} \right\}, \tag{15}
 \end{aligned}$$

$$\begin{aligned}
 & d(y_2, y_3) \\
 & \leq \frac{r}{2} \max \left\{ d(x_1, x_2) + d(y_1, y_2), d(x_1, x_2) \right. \\
 & \quad \left. + d(y_1, y_2), d(x_2, x_3) + d(y_2, y_3), \right. \\
 & \quad \left. \frac{d(x_1, x_3) + d(y_1, y_3)}{2}, \right. \\
 & \quad \left. \frac{d(x_2, x_2) + d(y_2, y_2)}{2} \right\}. \tag{16}
 \end{aligned}$$

Adding (14) and (16), we have

$$\begin{aligned}
 & d(x_2, x_3) + d(y_2, y_3) \\
 & \leq r \max \left\{ d(x_1, x_2) + d(y_1, y_2), d(x_2, x_3) + d(y_2, y_3), \right. \\
 & \quad \left. \frac{d(x_1, x_3) + d(y_1, y_3)}{2} \right\}. \tag{17}
 \end{aligned}$$

If $\max\{d(x_1, x_2) + d(y_1, y_2), d(x_2, x_3) + d(y_2, y_3), (d(x_1, x_3) + d(y_1, y_3))/2\} = d(x_2, x_3) + d(y_2, y_3)$, then we

have $d(x_2, x_3) + d(y_2, y_3) \leq r(d(x_2, x_3) + d(y_2, y_3))$ as $r < 1$; a contradiction. Therefore,

$$\begin{aligned}
 & d(x_2, x_3) + d(y_2, y_3) \\
 & \leq r \max \left\{ d(x_1, x_2) + d(y_1, y_2), \frac{d(x_1, x_3) + d(y_1, y_3)}{2} \right\}. \tag{18}
 \end{aligned}$$

Again, if $\max\{d(x_1, x_2) + d(y_1, y_2), (d(x_1, x_3) + d(y_1, y_3))/2\} = d(x_1, x_2) + d(y_1, y_2)$, we get

$$d(x_2, x_3) + d(y_2, y_3) \leq r(d(x_1, x_2) + d(y_1, y_2)) \tag{19}$$

and if $\max\{d(x_1, x_2) + d(y_1, y_2), (d(x_1, x_3) + d(y_1, y_3))/2\} = (d(x_1, x_3) + d(y_1, y_3))/2$, we get

$$d(x_2, x_3) + d(y_2, y_3) \leq r \frac{d(x_1, x_3) + d(y_1, y_3)}{2}. \tag{20}$$

Using triangle inequality, we obtain

$$\begin{aligned}
 & d(x_2, x_3) + d(y_2, y_3) \\
 & \leq r \frac{d(x_1, x_2) + d(x_2, x_3) + d(y_1, y_2) + d(y_2, y_3)}{2}. \tag{21}
 \end{aligned}$$

This implies

$$d(x_2, x_3) + d(y_2, y_3) \leq \left(\frac{r}{2-r}\right) d(x_1, x_2) + d(y_1, y_2). \tag{22}$$

Hence, there exist $x_3, y_3 \in X$ with $x_3 \in T(x_2, y_2)$ and $y_3 \in T(y_2, x_2)$ such that

$$d(x_2, x_3) + d(y_2, y_3) \leq r(d(x_1, x_2) + d(y_1, y_2)), \tag{23}$$

$$d(x_2, x_3) + d(y_2, y_3) \leq \left(\frac{r}{2-r}\right) d(x_1, x_2) + d(y_1, y_2). \tag{24}$$

Thus, we construct such sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} \in T(x_n, y_n)$, $y_{n+1} \in T(y_n, x_n)$, and

$$\begin{aligned}
 & d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq r(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)), \\
 & \quad d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \\
 & \leq \left(\frac{r}{2-r}\right) d(x_{n-1}, x_n) + d(y_{n-1}, y_n). \tag{25}
 \end{aligned}$$

Then, we have

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \\
 & \leq \sum_{n=1}^{\infty} r^n (d(x_1, x_2) + d(y_1, y_2)) < \infty, \tag{26} \\
 & \sum_{n=1}^{\infty} (d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \\
 & \leq \sum_{n=1}^{\infty} \left(\frac{r}{2-r}\right)^n (d(x_1, x_2) + d(y_1, y_2)) < \infty.
 \end{aligned}$$

Hence, we conclude that in both cases the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, there are some points z and z' in X such that $\lim_{n \rightarrow \infty} x_n = z$ and $\lim_{n \rightarrow \infty} y_n = z'$.

Now, we shall show that

$$\begin{aligned} & d(z, T(x, y)) + d(z', T(y, x)) \\ & \leq r \max \{d(z, x) + d(z', y), \\ & \quad d(x, T(x, y)) + d(y, T(y, x))\} \end{aligned} \quad (27)$$

for all $x \in X/\{z\}$ and $y \in X/\{z'\}$. Since $x_n \rightarrow x$ and $y_n \rightarrow y$, there exists $n_0 \in N$ such that $d(z, x_n) + d(z', y_n) \leq (1/3)(d(z, x) + d(z', y))$ for all $n \geq n_0$. Therefore

$$\begin{aligned} & \phi(r) d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n)) \\ & \leq r(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)) \\ & \leq d(x_n, z) + d(z, x_{n+1}) + d(y_n, z') + d(z', y_{n+1}) \\ & \leq d(x_n, z) + d(y_n, z') + d(z, x_{n+1}) + d(z', y_{n+1}) \\ & \leq \frac{2}{3}(d(x, z) + d(y, z')). \end{aligned} \quad (28)$$

Thus,

$$\begin{aligned} & \phi(r) d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n)) \\ & \leq \frac{2}{3}(d(x, z) + d(y, z')). \end{aligned} \quad (29)$$

Since

$$\begin{aligned} & \frac{2}{3}(d(x, z) + d(y, z')) \\ & = (d(x, z) + d(y, z')) - \frac{1}{3}(d(x, z) + d(y, z')) \\ & \leq (d(x, z) + d(y, z')) - (d(x_n, z) + d(y_n, z')) \\ & \leq d(x_n, x) + d(y_n, y), \end{aligned} \quad (30)$$

from (29) we have

$$\begin{aligned} & \phi(r) d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n)) \\ & \leq d(x_n, x) + d(y_n, y). \end{aligned} \quad (31)$$

Then, from (10) we have

$$\begin{aligned} & H(T(x_n, y_n), T(x, y)) \\ & \leq \frac{r}{2} \max \left\{ d(x_n, x) + d(y_n, y), d(x_n, T(x_n, y_n)) \right. \\ & \quad + d(y_n, T(y_n, x_n)), d(x, T(x, y)) \\ & \quad + d(y, T(y, x)), \\ & \quad \frac{d(x_n, T(x, y)) + d(y_n, T(y, x))}{2}, \\ & \quad \left. \frac{d(x, T(x_n, y_n)) + d(y, T(y_n, x_n))}{2} \right\}, \end{aligned} \quad (32)$$

$$\begin{aligned} & H(T(y_n, x_n), T(y, x)) \\ & \leq \frac{r}{2} \max \left\{ d(x_n, x) + d(y_n, y), d(x_n, T(x_n, y_n)) \right. \\ & \quad + d(y_n, T(y_n, x_n)), d(x, T(x, y)) \\ & \quad + d(y, T(y, x)), \\ & \quad \frac{d(x_n, T(x, y)) + d(y_n, T(y, x))}{2}, \\ & \quad \left. \frac{d(x, T(x_n, y_n)) + d(y, T(y_n, x_n))}{2} \right\}. \end{aligned} \quad (33)$$

By adding, (32) and (33), we get

$$\begin{aligned} & d(x_{n+1}, T(x, y)) + d(y_{n+1}, T(y, x)) \\ & \leq r \max \left\{ d(x_n, x) + d(y_n, y), d(x_n, x_{n+1}) \right. \\ & \quad + d(y_n, y_{n+1}), d(x, T(x, y)) \\ & \quad + d(y, T(y, x)), \\ & \quad \frac{d(x_n, T(x, y)) + d(y_n, T(y, x))}{2}, \\ & \quad \left. \frac{d(x, x_{n+1}) + d(y, y_{n+1})}{2} \right\}. \end{aligned} \quad (34)$$

Letting n tends to ∞ , we obtain

$$\begin{aligned} & d(z, T(x, y)) + d(z', T(y, x)) \\ & \leq r \max \left\{ d(z, x) + d(z', y), d(x, T(x, y)) \right. \\ & \quad + d(y, T(y, x)), \\ & \quad \left. \frac{d(z, T(x, y)) + d(z', T(y, x))}{2} \right\}. \end{aligned} \quad (35)$$

Hence, we have:

$$\begin{aligned}
 & d(z, T(x, y)) + d(z', T(y, x)) \\
 & \leq r \max \left\{ d(z, x) + d(z', y), d(x, T(x, y)) \right. \\
 & \quad \left. + d(y, T(y, x)) \right\}. \tag{36}
 \end{aligned}$$

Now, we shall prove that $z \in T(z, z')$ and $z' \in T(z', z)$.

Case 1. First, we consider the case $0 \leq r < 1/2$. Suppose, on contrary, that $z \notin T(z, z')$ and $z' \notin T(z', z)$. Let $a \in T(z, z')$ and $b \in T(z', z)$ be such that $2r\{d(a, z) + d(b, z')\} < d(z, T(z, z')) + d(z', T(z', z))$. Since $a \in T(z, z')$ and $b \in T(z', z)$ imply $a \neq z$ and $b \neq z'$, from (10), we have

$$\begin{aligned}
 & d(z, T(a, b)) + d(z', T(b, a)) \\
 & \leq r \max \left\{ d(z, a) + d(z', b), d(a, T(a, b)) + d(b, T(b, a)) \right\}. \tag{37}
 \end{aligned}$$

On the other hand, since $\phi(r)d(z, T(z, z')) + d(z', T(z', z)) \leq d(z, T(z, z')) + d(z', T(z', z)) < d(z, a) + d(z', b)$, from (10), we get

$$\begin{aligned}
 & H(T(z, z'), T(a, b)) \\
 & \leq \frac{r}{2} \max \left\{ d(z, a) + d(z', b), d(z, T(z, z')) \right. \\
 & \quad \left. + d(z', T(z', z)), d(a, T(a, b)) \right. \\
 & \quad \left. + d(b, T(b, a)), \right. \\
 & \quad \frac{d(z, T(a, b)) + d(z', T(b, a))}{2}, \\
 & \quad \left. \frac{d(a, T(z, z')) + d(b, T(z', z))}{2} \right\}, \tag{38}
 \end{aligned}$$

$$\begin{aligned}
 & H(T(z', z), T(b, a)) \\
 & \leq \frac{r}{2} \max \left\{ d(z, a) + d(z', b), d(z, T(z, z')) \right. \\
 & \quad \left. + d(z', T(z', z)), d(a, T(a, b)) \right. \\
 & \quad \left. + d(b, T(b, a)), \right. \\
 & \quad \frac{d(z, T(a, b)) + d(z', T(b, a))}{2}, \\
 & \quad \left. \frac{d(a, T(z, z')) + d(b, T(z', z))}{2} \right\}. \tag{39}
 \end{aligned}$$

By adding, (38) and (39), we have

$$\begin{aligned}
 & d(a, T(a, b)) + d(b, T(b, a)) \\
 & \leq r \max \left\{ d(z, a) + d(z', b), d(a, T(a, b)) \right. \\
 & \quad \left. + d(b, T(b, a)), \right. \\
 & \quad \left. \frac{d(z, T(a, b)) + d(z', T(b, a))}{2} \right\}. \tag{40}
 \end{aligned}$$

This implies

$$\begin{aligned}
 & d(a, T(a, b)) + d(b, T(b, a)) \\
 & \leq r \{d(z, a) + d(z', b)\} < d(z, a) + d(z', b). \tag{41}
 \end{aligned}$$

And, from (40), we have

$$d(z, T(a, b)) + d(z', T(b, a)) \leq r \{d(z, a) + d(z', b)\}. \tag{42}$$

Therefore, we obtain

$$\begin{aligned}
 & d(z, T(z, z')) + d(z', T(z', z)) \\
 & \leq d(z, T(a, b)) + H(T(a, b), T(z, z')) \\
 & \quad + d(z', T(b, a)) + H(T(b, a), T(z', z)) \tag{43} \\
 & \leq 2r(d(z, a) + d(z', b)) \\
 & < d(z, T(z, z')) + d(z', T(z', z)).
 \end{aligned}$$

This is a contradiction. As a result, we have $z \in T(z, z')$ and $z' \in T(z', z)$.

Case 2. Now, we consider the case $1/2 \leq r < 1$. We shall first prove that

$$\begin{aligned}
 & H(T(x, y), T(z, z')) + H(T(y, x), T(z', z)) \\
 & \leq r \max \left\{ d(x, z) + d(y, z'), d(x, T(x, y)) \right. \\
 & \quad \left. + d(y, T(y, x)), d(z, T(z, z')) \right. \\
 & \quad \left. + d(z', T(z', z)), \right. \\
 & \quad \frac{d(x, T(z, z')) + d(y, T(z', z))}{2}, \\
 & \quad \left. \frac{d(z, T(x, y)) + d(z', T(y, x))}{2} \right\} \tag{44}
 \end{aligned}$$

for all $x, y \in X$. If $x = z$ and $y = z'$, then the previous equation (44) obviously holds. Hence, let us assume $x \neq z$

and $y \neq z'$. Then, for every $n \in N$, there exist sequences $t_n \in T(x, y)$ and $t'_n \in T(y, x)$ such that

$$\begin{aligned} & d(z, t_n) + d(z', t'_n) \\ & \leq r \{d(z, T(x, y)) + d(z', T(y, x))\} \\ & \quad + \frac{1}{n} d(z, x) + d(z', y). \end{aligned} \quad (45)$$

Then for all $n \in N$, we have

$$\begin{aligned} & d(x, T(x, y)) + d(y, T(y, x)) \\ & \leq d(x, t_n) + d(y, t'_n) \\ & \leq d(x, z) + d(z, t_n) + d(y, z') + d(z', t'_n) \\ & \leq d(x, z) + d(y, z') + d(z, T(x, y)) + d(z', T(y, x)) \\ & \quad + \frac{1}{n} (d(x, z) + d(y, z')) \\ & \leq d(x, z) + d(y, z') \\ & \quad + r \max \{d(x, z) + d(y, z'), d(x, T(x, y)) \\ & \quad \quad + d(y, T(y, x))\} + \frac{1}{n} (d(x, z) + d(y, z')). \end{aligned} \quad (46)$$

If $d(x, z) + d(y, z') \geq d(x, T(x, y)) + d(y, T(y, x))$, then we get

$$\begin{aligned} & d(x, T(x, y)) + d(y, T(y, x)) \\ & \leq \left(1 + r + \frac{1}{n}\right) (d(x, z) + d(y, z')). \end{aligned} \quad (47)$$

Letting $n \rightarrow \infty$, we have

$$d(x, T(x, y)) + d(y, T(y, x)) \leq (1 + r) (d(x, z) + d(y, z')). \quad (48)$$

Thus

$$\begin{aligned} & \phi(r) (d(x, T(x, y)) + d(y, T(y, x))) \\ & \leq (1 - r) (d(x, T(x, y)) + d(y, T(y, x))) \\ & \leq \frac{1}{1 + r} (d(x, T(x, y)) + d(y, T(y, x))) \\ & \leq (d(x, z) + d(y, z')), \end{aligned} \quad (49)$$

and, from (10), we obtain (44). If $d(x, z) + d(y, z') < d(x, T(x, y)) + d(y, T(y, x))$, then

$$\begin{aligned} & d(x, T(x, y)) + d(y, T(y, x)) \\ & \leq (d(x, z) + d(y, z')) \\ & \quad + r \{d(x, T(x, y)) + d(y, T(y, x))\} \\ & \quad + \frac{1}{n} (d(x, z) + d(y, z')). \end{aligned} \quad (50)$$

And, therefore

$$\begin{aligned} & (1 - r) (d(x, T(x, y)) + d(y, T(y, x))) \\ & \leq \left(1 + \frac{1}{n}\right) (d(x, z) + d(y, z')). \end{aligned} \quad (51)$$

Letting $n \rightarrow \infty$, we have

$$\begin{aligned} & \phi(r) (d(x, T(x, y)) + d(y, T(y, x))) \\ & \leq (d(x, z) + d(y, z')), \end{aligned} \quad (52)$$

and, thus from condition (56), we obtain (44).

Finally, from (44), we obtain

$$\begin{aligned} & d(z, T(z, z')) + d(z', T(z', z)) \\ & = \lim_{n \rightarrow \infty} (d(x_{n+1}, T(z, z')) + d(y_{n+1}, T(z', z))) \\ & \leq \lim_{n \rightarrow \infty} r \max \left\{ d(x_n, z) + d(y_n, z'), d(x_n, x_{n+1}) \right. \\ & \quad \left. + d(y_n, y_{n+1}), d(z, T(z, z')) \right. \\ & \quad \left. + d(z', T(z', z)), \right. \\ & \quad \left. \frac{d(x_n, T(z, z')) + d(z', T(z', z))}{2}, \right. \\ & \quad \left. \frac{d(z, x_{n+1}) + d(z', y_{n+1})}{2} \right\} \\ & \leq r \max \{d(z, z) + d(z', z'), d(z, T(z, z')) \\ & \quad + d(z', T(z', z))\} \\ & = rd(z, T(z, z')) + d(z', T(z', z)). \end{aligned} \quad (53)$$

Hence, as $r < 1$, we obtain

$$d(z, T(z, z')) + d(z', T(z', z)) = 0. \quad (54)$$

This implies that $z \in T(z, z')$ and $z' \in T(z', z)$. \square

Corollary 5. Let (X, d) be a complete metric space and let T be a mapping from $X \times X$ into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $\phi(r)[d(x, T(x, y)) + d(u, T(u, v))] \leq (d(x, u) + d(y, v))$ implies

$$\begin{aligned} & H(T(x, y), T(u, v)) \\ & \leq \frac{r}{2} \max \{d(x, u) + d(y, v), d(x, T(x, y)) \\ & \quad + d(y, T(y, x)), \\ & \quad d(u, T(u, v)) + d(v, T(v, u))\} \end{aligned} \quad (55)$$

for all $x, y, u, v \in X$, where the function ϕ is defined as in Theorem 4. Then, there exist $z, z' \in X$ such that $z \in T(z, z')$ and $z' \in T(z', z)$.

Corollary 6. Let (X, d) be a complete metric space and let T be a mapping from $X \times X$ into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $\phi(r)[d(x, T(x, y)) + d(u, T(u, v))] \leq (d(x, u) + d(y, v))$ implies

$$\begin{aligned}
 & H(T(x, y), T(u, v)) \\
 & \leq \frac{r}{2} \max \left\{ d(x, u) + d(y, v), d(x, T(x, y)) \right. \\
 & \quad \left. + d(y, T(y, x)), d(u, T(u, v)) + d(v, T(v, u)), \right. \\
 & \quad \left. \frac{d(u, T(x, y)) + d(v, T(y, x))}{2} \right\} \quad x \in W. \tag{56}
 \end{aligned}$$

for all $x, y, u, v \in X$, where the function ϕ is defined as in Theorem 4. Then, there exist $z, z' \in X$ such that $z \in T(z, z')$ and $z' \in T(z', z)$.

3. An Application

The existence and uniqueness of solutions of functional equations and system of functional equations arising in dynamic programming have been studied by using various fixed point theorems. In this paper, we shall prove the existence and uniqueness of a solution for a class of functional equations using Corollary 6.

In this section, we assume that U and V are Banach spaces. $W \subset U, D \subset V$, and \mathfrak{R} is a field of real numbers. Let $B(W)$ denote the set of all the real valued functions on W . It is known that $B(W)$ endowed with the metric

$$d_B(h, k) = \sup_{x \in W} |h(x) - k(x)|, \quad h, k \in B(W), \tag{57}$$

is a complete metric space. According to Bellman and Lee, the basic form of the functional equation of dynamic programming is given as

$$p(x) = \sup_y H(x, y, p(\tau(x, y))), \tag{58}$$

where x and y represent the state and decision vectors, respectively, $\tau : W \times D \rightarrow W$ represents the transformation of the process, and $p(x)$ represents the optimal return function with initial state x . In this section, we study the existence and uniqueness of a solution of the following functional equation:

$$p(x) = \sup_y [g(x, y) + G(x, y, p(\tau(x, y)))] \tag{59}$$

where $g : W \times D \rightarrow \mathfrak{R}$ and $G : W \times D \times \mathfrak{R} \rightarrow \mathfrak{R}$ are bounded functions. In this section, we study the existence and uniqueness of a solution of (59) in the following new

form. Let a function ϕ be defined as in Theorem 4 and let the mapping T be defined by

$$\begin{aligned}
 & T(h, k)(x) \\
 & = \sup_{y \in D} [g(x, y) + G(x, y, h(\tau(x, y)), k(\tau(x, y)))] \tag{60}
 \end{aligned}$$

Theorem 7. Suppose that there exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D, h, k, h', k' \in B(W)$, and $t, s \in W$, the inequality

$$\begin{aligned}
 & \phi(r) [d_B(h, T(h(t), k(s))) + d_B(h', T(h'(s), k'(t)))] \\
 & \leq d(h(t), h'(s)) + d(k(s), k'(t)) \tag{61}
 \end{aligned}$$

implies

$$\begin{aligned}
 & |G(x, y, h(\tau(x, y)), k(\tau(x, y))) \\
 & \quad - G(x, y, h'(\tau(x, y)), k'(\tau(x, y)))| \\
 & \leq rM(h(t), k(s), h'(s), k'(t)), \tag{62}
 \end{aligned}$$

where

$$\begin{aligned}
 & M(h(t), k(s), h'(s), k'(t)) \\
 & = \max \left\{ |h(t) - h'(s)| + |k(s) - k'(t)|, \right. \\
 & \quad |h(t) - T(h(t), k(s))| + |k(s) - T(k(s), h(t))|, \\
 & \quad |h'(s) - T(h'(s), k'(t))| \\
 & \quad \left. + |k'(t) - T(k'(t), h'(s))|, \right. \\
 & \quad \left. \frac{|h'(s), T(h(t), k(s))| + |k'(t) - T(k(s), h(t))|}{2} \right\}. \tag{63}
 \end{aligned}$$

Then, the functional equation (59) has a unique bounded solution in $B(W)$.

Proof. Note that T is a map from $B(W) \times B(W)$ onto $B(W)$ and that $(B(W), d_B)$ is a complete metric space, where d_B is the metric defined in (57). Let λ be an arbitrary positive real number, and $h, h', k, k' \in B(W)$. For arbitrary $x \in W$ and $y \in D$ so that

$$T(h, k)(x) < [g(x, y) + G(x, y, h(\tau_1), k(\tau_2))] + \lambda, \tag{64}$$

$$T(h', k')(x) < [g(x, y) + G(x, y, h'(\tau_2), k'(\tau_1))] + \lambda. \tag{65}$$

From the definition of mapping T , we have

$$T(h, k)(x) \geq [g(x, y) + G(x, y, h(\tau_2), k(\tau_1))], \quad (66)$$

$$T(h', k')(x) \geq [g(x, y) + G(x, y, h'(\tau_1), k'(\tau_2))]. \quad (67)$$

If inequality (61) holds, then from (64) and (67), we get

$$\begin{aligned} T(h, k)(x) - T(h', k')(x) &< G(x, y, h(\tau_1), k(\tau_2)) - G(x, y, h'(\tau_1), k'(\tau_2)) + \lambda \\ &\leq |G(x, y, h(\tau_1), k(\tau_2)) - G(x, y, h'(\tau_1), k'(\tau_2))| + \lambda \\ &\leq rM(h(t), k(s), h'(s), k'(t)) + \lambda. \end{aligned} \quad (68)$$

Similarly, (65) and (66) implies that

$$\begin{aligned} T(h', k')(x) - T(h, k)(x) &\leq rM(h(t), k(s), h'(s), k'(t)) + \lambda. \end{aligned} \quad (69)$$

Hence, from (68) and (69), we have

$$\begin{aligned} |T(h, k)(x) - T(h', k')(x)| &\leq rM(h(t), k(s), h'(s), k'(t)) + \lambda. \end{aligned} \quad (70)$$

Since inequality (70) is true for any $x \in W$ and arbitrary $\lambda > 0$, then (61) implies

$$\begin{aligned} d_B(T(h(t), k(s)), T(h'(s), k'(t))) &\leq r \max \left\{ |h(t) - h'(s)| + |k(s) - k'(t)|, \right. \\ &|h(t) - T(h(t), k(s))| \\ &+ |k(s) - T(k(s), h(t))|, \\ &|h'(s) - T(h'(s), k'(t))| \\ &+ |k'(t) - T(k'(t), h'(s))|, \\ &\left. \frac{|h'(s), T(h(t), k(s))| + |k'(t) - T(k(s), h(t))|}{2} \right\}. \end{aligned} \quad (71)$$

Therefore, all the conditions of Corollary 6 are met for the mapping T , and hence the functional equation (59) has a unique bounded solution. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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