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Research Article

Limit Cycle Bifurcations by Perturbing a Compound Loop with a Cusp and a Nilpotent Saddle

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We study the expansions of the first order Melnikov functions for general near-Hamiltonian systems near a compound loop with a cusp and a nilpotent saddle. We also obtain formulas for the first coefficients appearing in the expansions and then establish a bifurcation theorem on the number of limit cycles. As an application example, we give a lower bound of the maximal number of limit cycles for a polynomial system of Liénard type.

1. Introduction

Consider a planar system of the form

$$\dot{x} = H_y + \epsilon p(x, y, \delta), \qquad \dot{y} = -H_x + \epsilon q(x, y, \delta), \tag{1}$$

where ϵ is a small parameter and $H(x, y)$, $p(x, y, \delta)$, and $q(x, y, \delta)$ are C^∞ functions in $(x, y) \in \mathbb{R}^2$ and $\delta \in D \subset \mathbb{R}^m$ with D bounded. For $\epsilon = 0$, (1) becomes

$$\dot{x} = H_y, \qquad \dot{y} = -H_x, \tag{2}$$

which is a Hamiltonian system. As we know, the system (1) is said to be a near-Hamiltonian system. For (1), the main task is to study the number of limit cycles which are bifurcated from periodic orbits of the unperturbed system (2). On this aspect, the first order Melnikov function of (1) plays an important role. We can use the expansions of it near Hamiltonian values corresponding to a center or an invariant loop to find its zeros and hence the number of limit cycles. See a survey article [1]. There have been many works on this topic. For the study of general near-Hamiltonian systems, see [2–12]; and especially for the system (2) with the elliptic case, one can see [13–17] and references therein. In [2–4], the number of limit cycles of the system (1) near a homoclinic loop with a cusp of order one or two or a nilpotent saddle of order one (for the definition of an order of a cusp or nilpotent saddle, see [5]) was studied. In the heteroclinic case with two hyperbolic saddles, a hyperbolic saddle and a cusp of order one, or two cusps of order one or two, the number of limit cycles of the system (1) was studied in [5, 8, 9], respectively. In this paper, we suppose that the unperturbed system (2) has a compound loop consisting of a cusp S_1 of order one, a nilpotent saddle S_2 of order one, a homoclinic loop to S_2, and two heteroclinic orbits connecting S_1 and S_2, as shown in Figure 1. We aim to study the number of limit cycles of (1) near the loop for $\epsilon \neq 0$ small.

2. Main Results with Proof

Now consider the C^∞ systems (1) and (2). Suppose that (2) has a compound loop denoted by $L_0 = L_1 \cup L_2 \cup L_3 \cup \{S_1, S_2\}$ and defined by equation $H(x, y) = 0$, where $S_1(x_1, y_1)$ is a cusp and $S_2(x_2, y_2)$ is a nilpotent saddle both having order one, L_1, L_2 are heteroclinic orbits satisfying $\omega(L_1) = \alpha(L_2) = S_2$ and $\omega(L_2) = \alpha(L_1) = S_1$, and L_3 is a homoclinic loop to S_2. Then, the level curves of $H(x, y)$ define two families of periodic orbits L_{h1} and L_{h2} for h on one side of $h = 0$ and a family of periodic orbits L_{h3} for h on another side of $h = 0$. For the definiteness, let both L_{h1} and L_{h2} exist for $0 < -h \ll 1$ and L_{h3} exist for $0 < h \ll 1$. Thus, we have three Melnikov functions

$$M_i(h, \delta) = \oint_{L_{hi}} q\,dx - p\,dy, \quad i = 1, 2, 3. \tag{3}$$

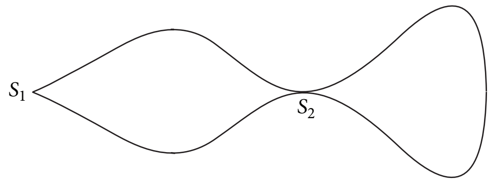

FIGURE 1: Compound loop with a cusp and a nilpotent saddle.

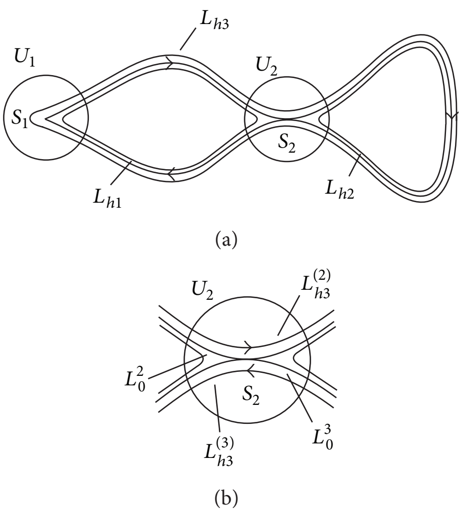

FIGURE 2

Let U_i denote a closed set with diameter $\epsilon_0 > 0$ and with center at S_i, $i = 1, 2$. See Figure 2(a). And further introduce

$$\begin{gathered} L_{h1}^{(j)} = L_{h1} \cap U_j, \quad j = 1, 2, \\ L_{h1}^{(3)} = \text{Cl.}\left(L_{h1} - \bigcup_{j=1}^{2} L_{h1}^{(j)}\right), \qquad L_{h2}^{(1)} = L_{h2} \cap U_2, \\ L_{h2}^{(2)} = L_{h2} - L_{h2}^{(1)}, \qquad L_{h3}^{(1)} = L_{h3} \cap U_1, \\ L_{h3} \cap U_2 = L_{h3}^{(2)} \cup L_{h3}^{(3)} \quad (\text{as shown in Figure 2 (b)}), \\ L_{h3}^{(4)} = \text{Cl.}\left(L_{h3} - \bigcup_{j=1}^{3} L_{h3}^{(j)}\right). \end{gathered} \tag{4}$$

Here the Cl. denotes the closure of a set. Then by (3) and (4), for ϵ_0 sufficiently small we can write

$$M_1(h,\delta) = I_{11}(h,\delta) + I_{12}(h,\delta) + I_{13}(h,\delta) \quad \text{for } 0 \le -h \ll 1, \tag{5}$$

$$M_2(h,\delta) = I_{21}(h,\delta) + I_{22}(h,\delta) \quad \text{for } 0 \le -h \ll 1, \tag{6}$$

$$M_3(h,\delta) = I_{31}(h,\delta) + I_{32}(h,\delta) + I_{33}(h,\delta) + I_{34}(h,\delta) \quad \text{for } 0 \le h \ll 1, \tag{7}$$

where

$$I_{ij}(h,\delta) = \int_{L_{hi}^{(j)}} q\,dx - p\,dy, \quad i \in \{1,2,3\}, \ j \in \{1,2,3,4\}. \tag{8}$$

By [5], there exist two transformations of the form

$$(x, y)^T = T_i(u, v)^T + S_i, \quad i = 1, 2, \tag{9}$$

where T_i is a 2×2 matrix satisfying $\det T_i = 1$ such that (1) becomes

$$\dot{u} = H_{iv} + \epsilon p_i(u,v,\delta), \qquad \dot{v} = -H_{iu} + \epsilon q_i(u,v,\delta), \tag{10}$$

where

$$\begin{gathered} H_1(u,v) = \frac{v^2}{2} + \sum_{k+j\ge 3} \tilde{h}_{kj} u^k v^j, \\ H_1(u,\varphi_1(u)) = \tilde{h}_3 u^3 + O(u^4), \quad \tilde{h}_3 < 0, \\ H_{1v}(u,\varphi_1(u)) = 0, \qquad p_1(u,v,\delta) = \sum_{i+j\ge 0} \tilde{a}_{ij} u^i v^j, \\ q_1(u,v,\delta) = \sum_{i+j\ge 0} \tilde{b}_{ij} u^i v^j, \end{gathered} \tag{11}$$

$$\begin{gathered} H_2(u,v) = \frac{v^2}{2} + \sum_{k+j\ge 3} \overline{h}_{kj} u^k v^j, \\ H_2(u,\varphi_2(u)) = \overline{h}_4 u^4 + O(u^5), \quad \overline{h}_4 < 0, \\ H_{2v}(u,\varphi_2(u)) = 0, \\ p_2(u,v,\delta) = \sum_{i+j\ge 0} \overline{a}_{ij} u^i v^j, \\ q_2(u,v,\delta) = \sum_{i+j\ge 0} \overline{b}_{ij} u^i v^j, \end{gathered} \tag{12}$$

for (u, v) near $(0, 0)$. Note that $qdx - pdy = q_1du - p_1dv$ for (x, y) near S_1 and $qdx - pdy = q_2du - p_2dv$ for (x, y) near S_2. Then we have

$$I_{i1}(h,\delta) = \int_{\widetilde{L}_{hi}^{(1)}} q_1du - p_1dv, \quad i = 1, 3, \tag{13}$$

$$I_{21}(h,\delta) = \int_{\overline{L}_{h2}^{(1)}} q_2du - p_2dv, \tag{14}$$

$$\begin{aligned} I_{12}(h,\delta) &= \int_{\overline{L}_{h1}^{(2)}} q_2du - p_2dv, \\ I_{3j}(h,\delta) &= \int_{\overline{L}_{h3}^{(j)}} q_2du - p_2dv, \quad j = 2, 3, \end{aligned} \tag{15}$$

where $\widetilde{L}_{hi}^{(1)}$ denote the image of $L_{hi}^{(1)}$ under T_1 and $\overline{L}_{h2}^{(1)}$, $\overline{L}_{h1}^{(2)}$, and $\overline{L}_{h3}^{(j)}$ denote the image of $L_{h2}^{(1)}$, $L_{h1}^{(2)}$, and $L_{h3}^{(j)}$ under T_2, respectively. Then, by using [3, 4] we can obtain the following two lemmas, respectively.

Lemma 1. *Consider system* (10) *with* $i = 1$ *and suppose* (11), (13) *hold. Then there are constants* B_{00}, B_{00}^*, B_{10}, B_{10}^* *satisfying*

$$\begin{aligned} B_{00} &= \frac{3}{5}\int_0^1 \frac{dv}{\sqrt{v(1-v^3)}} = \frac{3}{5} \times 2.4286\cdots > 0, \\ B_{00}^* &= -\frac{3}{5}\int_{-\infty}^1 \frac{dv}{\sqrt{1-v^3}} = -\frac{3}{5} \times 4.2065\cdots < 0, \\ B_{10} &= -\frac{3}{7}\left(\int_0^1 \frac{v^{3/2}dv}{\sqrt{1-v^3}\left(1+\sqrt{1-v^3}\right)} - 2\right) > 0, \\ B_{10}^* &= \frac{3}{7}\left(\int_1^{-1} \frac{v\,dv}{\sqrt{1-v^3}} - \int_0^1 \frac{v^{3/2}dv}{\sqrt{1+v^3}\left(1+\sqrt{1+v^3}\right)} - 2\right) < 0 \end{aligned} \tag{16}$$

such that

$$\begin{aligned} I_{11}(h,\delta) = {} & B_{00}c_1(S_1,\delta)|h|^{5/6} + B_{10}c_3(S_1,\delta)|h|^{7/6} \\ & - \frac{1}{11}B_{00}c_4(S_1,\delta)|h|^{11/6} + O\left(h^2\right) + N_{11}(h,\delta) \end{aligned} \tag{17}$$

for $0 < -h \ll 1$,

$$\begin{aligned} I_{31}(h,\delta) = {} & B_{00}^*c_1(S_1,\delta)h^{5/6} + B_{10}^*c_3(S_1,\delta)h^{7/6} \\ & + \frac{1}{11}B_{00}^*c_4(S_1,\delta)h^{11/6} + O\left(h^2\right) + N_{31}(h,\delta) \end{aligned} \tag{18}$$

for $0 < h \ll 1$, *where* $N_{i1}(h,\delta) \in C^\omega$ *at* $h = 0$ *with* $N_{i1}(0,\delta) = O(\epsilon_0)$, $i = 1, 3$, *and*

$$\begin{aligned} c_1(S_1,\delta) &= 2\sqrt{2}\,\widetilde{h}_3^{-1/3}\left(\widetilde{a}_{10} + \widetilde{b}_{01}\right), \\ c_3(S_1,\delta) &= 2\sqrt{2}\,\widetilde{h}_3^{-5/3} \\ &\quad \times \left[\widetilde{h}_3\left(2\widetilde{a}_{20} + \widetilde{b}_{11}\right) \right. \\ &\quad \left. + \frac{1}{3}\left(\widetilde{h}_{21}^2 - 2\widetilde{h}_4 - 3\widetilde{h}_3\widetilde{h}_{12}\right)\left(\widetilde{a}_{10} + \widetilde{b}_{01}\right)\right], \\ c_4(S_1,\delta) &= 9\mu_1^{-1}\widetilde{\alpha}_{01} \\ &\quad - 2\mu_1^{-7}\left[\left(20\mu_2^3 - 20\mu_1\mu_2\mu_3 + 4\mu_1^2\mu_4\right)\widetilde{\alpha}_{00} \right. \\ &\quad + \left(4\mu_1^2\mu_3 - 10\mu_1\mu_2^2\right)\widetilde{\alpha}_{10} \\ &\quad \left. + 4\mu_1^2\mu_2\widetilde{\alpha}_{20} - \mu_1^3\widetilde{\alpha}_{30}\right], \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mu_1 &= \widetilde{h}_3^{1/3}, \qquad \mu_2 = \frac{1}{3}\widetilde{h}_3^{-2/3}\widetilde{h}_4, \\ \mu_3 &= \frac{1}{9}\widetilde{h}_3^{-5/3}\left(3\widetilde{h}_3\widetilde{h}_5 - \widetilde{h}_4^2\right), \\ \mu_4 &= \frac{1}{81}\widetilde{h}_3^{-8/3}\left(27\widetilde{h}_3^2\widetilde{h}_6 - 18\widetilde{h}_3\widetilde{h}_4\widetilde{h}_5 + 5\widetilde{h}_4^3\right), \\ \widetilde{h}_3 &= \widetilde{h}_{30}, \qquad \widetilde{h}_4 = -\frac{1}{2}\widetilde{h}_{21}^2 + \widetilde{h}_{40}, \\ \widetilde{h}_5 &= \widetilde{h}_{12}\widetilde{h}_{21}^2 - \widetilde{h}_{21}\widetilde{h}_{31} + \widetilde{h}_{50}, \\ \widetilde{h}_6 &= -2\widetilde{h}_{12}^2\widetilde{h}_{21}^2 - \widetilde{h}_{03}\widetilde{h}_{21}^3 + \widetilde{h}_{21}^2\widetilde{h}_{22} \\ &\quad + 2\widetilde{h}_{12}\widetilde{h}_{21}\widetilde{h}_{31} - \frac{1}{2}\widetilde{h}_{31}^2 - \widetilde{h}_{21}\widetilde{h}_{41} + \widetilde{h}_{60}, \\ \widetilde{\alpha}_{00} &= 2\sqrt{2}\left(\widetilde{a}_{10} + \widetilde{b}_{01}\right), \\ \widetilde{\alpha}_{10} &= 2\sqrt{2}\left[-\widetilde{h}_{12}\left(\widetilde{a}_{10} + \widetilde{b}_{01}\right) + 2\widetilde{a}_{20} + \widetilde{b}_{11}\right], \\ \widetilde{\alpha}_{20} &= 2\sqrt{2}\left[\left(\widetilde{a}_{10} + \widetilde{b}_{01}\right)\left(3\widetilde{h}_{03}\widetilde{h}_{21} - \widetilde{h}_{22} + \frac{3}{2}\widetilde{h}_{12}^2\right) - 2\widetilde{h}_{12}\widetilde{a}_{20} \right. \\ &\quad \left. - \widetilde{h}_{12}\widetilde{b}_{11} + 3\widetilde{a}_{30} + \widetilde{b}_{21} - \widetilde{a}_{11}\widetilde{h}_{21} - 2\widetilde{b}_{02}\widetilde{h}_{21}\right], \\ \widetilde{\alpha}_{30} &= 2\sqrt{2}\left[\left(\widetilde{a}_{10} + \widetilde{b}_{01}\right)\left(3\widetilde{h}_{13}\widetilde{h}_{21} + 3\widetilde{h}_{03}\widetilde{h}_{31} + 3\widetilde{h}_{12}\widetilde{h}_{22} \right.\right. \\ &\quad \left. - 15\widetilde{h}_{12}\widetilde{h}_{03}\widetilde{h}_{21} - \frac{5}{2}\widetilde{h}_{12}^3 - \widetilde{h}_{32}\right) \\ &\quad + \left(3\widetilde{a}_{11} + 6\widetilde{b}_{02}\right)\widetilde{h}_{12}\widetilde{h}_{21} - 2\left(\widetilde{b}_{12} + \widetilde{a}_{21}\right)\widetilde{h}_{21} \\ &\quad - \left(\widetilde{a}_{11} + 2\widetilde{b}_{02}\right)\widetilde{h}_{31} + 4\widetilde{a}_{40} + \widetilde{b}_{31} \end{aligned}$$

$$+\left(3\widetilde{b}_{11}+6\widetilde{a}_{20}\right)\widetilde{h}_{03}\widetilde{h}_{21}-\left(2\widetilde{a}_{20}+\widetilde{b}_{11}\right)\widetilde{h}_{22}$$
$$+\left(3\widetilde{a}_{20}+\frac{3}{2}\widetilde{b}_{11}\right)\widetilde{h}_{12}^{2}-\left(3\widetilde{a}_{30}+\widetilde{b}_{21}\right)\widetilde{h}_{12}\Big],$$
$$\widetilde{\alpha}_{01}=2\sqrt{2}\Big[\frac{2}{3}\widetilde{a}_{12}+2\widetilde{b}_{03}-2\widetilde{h}_{03}\widetilde{a}_{11}-4\widetilde{h}_{03}\widetilde{b}_{02}+\left(\widetilde{a}_{10}+\widetilde{b}_{01}\right)\left(5\widetilde{h}_{03}^{2}-2\widetilde{h}_{04}\right)\Big]. \quad (20)$$

Lemma 2. *Consider system* (10) *with* $i = 2$ *and suppose* (12), (15) *hold. Then we have*

$$I_{12}(h,\delta)=c_1(S_2,\delta)|h|^{3/4}-c_2(S_2,\delta)h\ln|h|+c_4(S_2,\delta)|h|^{5/4}+c_5(S_2,\delta)|h|^{7/4}-c_6(S_2,\delta)h^2\ln|h|+O\left(h^2\right)+N_{12}(h,\delta) \quad (21)$$

for $0<-h\ll 1$,

$$I_{32}(h,\delta)=\frac{1}{2}c_1^{*}(S_2,\delta)h^{3/4}+\frac{1}{2}c_3^{*}(S_2,\delta)h^{5/4}+\frac{1}{2}c_4^{*}(S_2,\delta)h^{7/4}+O\left(h^2\right)+N_{32}(h,\delta),$$
$$I_{33}(h,\delta)=\frac{1}{2}c_1^{*}(S_2,\delta)h^{3/4}+\frac{1}{2}c_3^{*}(S_2,\delta)h^{5/4}+\frac{1}{2}c_4^{*}(S_2,\delta)h^{7/4}+O\left(h^2\right)+N_{33}(h,\delta) \quad (22)$$

for $0<h\ll 1$, *where* $N_{ij}(h,\delta)\in C^{\omega}$ *at* $h=0$ *with* $N_{ij}(0,\delta)=O(\epsilon_0)$, $(i,j)\in\{(1,2),(3,2),(3,3)\}$, *and*

$$c_1(S_2,\delta)=-2\sqrt{2}\left|\widetilde{A}_0\right|\left|\overline{h}_4\right|^{-1/4}\left(\overline{a}_{10}+\overline{b}_{01}\right),$$
$$c_2(S_2,\delta)=-\frac{\sqrt{2}}{4}\left|\overline{h}_4\right|^{-1/2}\left(2\overline{a}_{20}+\overline{b}_{11}\right)+O\left(\overline{a}_{10}+\overline{b}_{01}\right),$$
$$c_4(S_2,\delta)=\left|\widetilde{A}_2\right|\Big[\left(\frac{21}{32}\overline{h}_5^{2}-\frac{3}{4}\overline{h}_4\overline{h}_6\right)\left|\overline{h}_4\right|^{-11/4}\overline{\alpha}_{00}+\frac{3}{4}\left|\overline{h}_4\right|^{-7/4}\overline{h}_5\overline{\alpha}_{10}+\left|\overline{h}_4\right|^{-3/4}\overline{\alpha}_{20}\Big],$$
$$c_5(S_2,\delta)=\frac{1}{7}\left|\widetilde{A}_0\right|\Big[6d_1^{-1}\overline{\alpha}_{01}+d_1^{-9}\left(105d_1d_2^2d_3-30d_1^2d_2d_4-15d_1^2d_3^2+5d_1^3d_5-70d_2^4\right)\overline{\alpha}_{00}+d_1^{-8}\left(35d_2^3-30d_1d_2d_3+5d_1^2d_4\right)\overline{\alpha}_{10}-d_1^{-7}\left(15d_2^2-5d_1d_3\right)\overline{\alpha}_{20}+5d_1^{-6}d_2\overline{\alpha}_{30}-d_1^{-5}\overline{\alpha}_{40}\Big],$$
$$c_6(S_2,\delta)=-\frac{1}{32}\Big[-6d_1^{-3}d_2\overline{\alpha}_{01}+3d_1^{-2}\overline{\alpha}_{11}-d_1^{-11}\left(504d_1d_2^3d_3-168d_1^2d_2d_3^2+42d_1^3d_3d_4-168d_1^2d_2^2d_4+42d_1^3d_2d_5-252d_2^5-6d_1^4d_6\right)\overline{\alpha}_{00}+d_1^{-10}\left(168d_1d_2^2d_3-126d_2^4-42d_1^2d_2d_4-21d_1^2d_3^2+6d_1^3d_5\right)\overline{\alpha}_{10}-d_1^{-9}\left(42d_1d_2d_3-56d_2^3-6d_1^2d_4\right)\overline{\alpha}_{20}+d_1^{-8}\left(6d_1d_3-21d_2^2\right)\overline{\alpha}_{30}+6d_1^{-7}d_2\overline{\alpha}_{40}-d_1^{-6}\overline{\alpha}_{50}\Big],$$
$$c_1^{*}(S_2,\delta)=-D_1c_1(S_2,\delta),\qquad c_3^{*}(S_2,\delta)=-D_2c_4(S_2,\delta),$$
$$c_4^{*}(S_2,\delta)=D_1c_5(S_2,\delta), \quad (23)$$

where $D_1=2|\overline{A}_0|/|\widetilde{A}_0|$, $D_2=2|\overline{A}_1|/|\widetilde{A}_2|$, $\overline{A}_0$, $\widetilde{A}_0$, $\overline{A}_1$, *and* $\widetilde{A}_2$ *are constants, given by*

$$\overline{A}_0=\frac{2}{3}\int_0^{\infty}\frac{dv}{\sqrt{1+v^4}}\approx 1.236049785>0,$$
$$\widetilde{A}_0=-\frac{2}{3}\int_0^{1}\frac{dv}{\sqrt{1-v^4}}=-\frac{\sqrt{2}\pi^{3/2}}{6[\Gamma(3/4)]^2}\approx -0.8740191847<0,$$
$$\overline{A}_1=-\frac{2}{5}\int_0^{\infty}\frac{dv}{\sqrt{1+v^4}\left[v^2+\sqrt{1+v^4}\right]}\approx -0.3388852337<0,$$
$$\widetilde{A}_2=\frac{2}{5}\left[1-\int_0^{1}\frac{v^2dv}{\sqrt{1-v^4}\left(1+\sqrt{1-v^4}\right)}\right]\approx 0.2396280470>0,$$
$$d_1=\left|\overline{h}_4\right|^{1/4},\qquad d_2=-\frac{1}{4}\left|\overline{h}_4\right|^{-3/4}\overline{h}_5,$$
$$d_3=\frac{1}{32}\left|\overline{h}_4\right|^{-7/4}\left(8\overline{h}_4\overline{h}_6-3\overline{h}_5^{2}\right),$$
$$d_4=-\frac{1}{128}\left|\overline{h}_4\right|^{-11/4}\left(7\overline{h}_5^{3}+32\overline{h}_4^{2}\overline{h}_7-24\overline{h}_4\overline{h}_5\overline{h}_6\right),$$

$$d_5 = \frac{1}{2048}\left|\bar{h}_4\right|^{-15/4}\left(512\bar{h}_4^3\bar{h}_8 - 192\bar{h}_4^2\bar{h}_6^2 - 384\bar{h}_4^2\bar{h}_5\bar{h}_7 + 336\bar{h}_4\bar{h}_5^2\bar{h}_6 - 77\bar{h}_5^4\right),$$

$$d_6 = -\frac{1}{8192}\left|\bar{h}_4\right|^{-19/4}\left(2048\bar{h}_4^4\bar{h}_9 + 1344\bar{h}_4^2\bar{h}_5\bar{h}_6^2 - 1536\bar{h}_4^3\bar{h}_5\bar{h}_8 - 1536\bar{h}_4^3\bar{h}_6\bar{h}_7 + 1344\bar{h}_4^2\bar{h}_5^2\bar{h}_7 - 1232\bar{h}_4\bar{h}_5^3\bar{h}_6 + 231\bar{h}_5^5\right);$$

$$\bar{\alpha}_{00} = 2\sqrt{2}\left(\bar{a}_{10} + \bar{b}_{01}\right),$$

$$\bar{\alpha}_{10} = 2\sqrt{2}\left[-\bar{h}_{12}\left(\bar{a}_{10} + \bar{b}_{01}\right) + 2\bar{a}_{20} + \bar{b}_{11}\right],$$

$$\bar{\alpha}_{20} = 2\sqrt{2}\left[\left(\bar{a}_{10} + \bar{b}_{01}\right)\left(3\bar{h}_{03}\bar{h}_{21} - \bar{h}_{22} + \frac{3}{2}\bar{h}_{12}^2\right) - 2\bar{h}_{12}\bar{a}_{20} - \bar{h}_{12}\bar{b}_{11} + 3\bar{a}_{30} + \bar{b}_{21} - \bar{a}_{11}\bar{h}_{21} - 2\bar{b}_{02}\bar{h}_{21}\right],$$

$$\begin{aligned}\bar{\alpha}_{30} = 2\sqrt{2}\Big[&\left(\bar{a}_{10} + \bar{b}_{01}\right)\\ &\times\left(3\bar{h}_{13}\bar{h}_{21} + 3\bar{h}_{03}\bar{h}_{31} + 3\bar{h}_{12}\bar{h}_{22} - 15\bar{h}_{12}\bar{h}_{03}\bar{h}_{21} - \frac{5}{2}\bar{h}_{12}^3 - \bar{h}_{32}\right)\\ &+ \left(3\bar{a}_{11} + 6\bar{b}_{02}\right)\bar{h}_{12}\bar{h}_{21} - 2\left(\bar{b}_{12} + \bar{a}_{21}\right)\bar{h}_{21}\\ &- \left(\bar{a}_{11} + 2\bar{b}_{02}\right)\bar{h}_{31} + 4\bar{a}_{40} + \bar{b}_{31}\\ &+ \left(3\bar{b}_{11} + 6\bar{a}_{20}\right)\bar{h}_{03}\bar{h}_{21}\\ &- \left(2\bar{a}_{20} + \bar{b}_{11}\right)\bar{h}_{22} + \left(3\bar{a}_{20} + \frac{3}{2}\bar{b}_{11}\right)\bar{h}_{12}^2\\ &- \left(3\bar{a}_{30} + \bar{b}_{21}\right)\bar{h}_{12}\Big],\end{aligned}$$

$$\begin{aligned}\bar{\alpha}_{40} = 2\sqrt{2}\Big[&n_1\left(\bar{a}_{10} + \bar{b}_{01}\right)\\ &- \left(\frac{5}{2}\bar{h}_{12}^3 - 3\bar{h}_{12}\bar{h}_{22} + 15\bar{h}_{12}\bar{h}_{03}\bar{h}_{21} + \bar{h}_{32} - 3\bar{h}_{13}\bar{h}_{21} - 3\bar{h}_{03}\bar{h}_{31}\right)\left(2\bar{a}_{20} + \bar{b}_{11}\right)\\ &+ \left(\frac{3}{2}\bar{h}_{12}^2 + 3\bar{h}_{03}\bar{h}_{21} - \bar{h}_{22}\right)\\ &\times\left(3\bar{a}_{30} + \bar{b}_{21} - \bar{a}_{11}\bar{h}_{21} - 2\bar{b}_{02}\bar{h}_{21}\right)\\ &- \bar{h}_{12}\left(4\bar{a}_{40} + \bar{b}_{31} - 2\bar{a}_{21}\bar{h}_{21} - 2\bar{b}_{12}\bar{h}_{21} + 2\bar{a}_{11}\bar{h}_{12}\bar{h}_{21} + 4\bar{b}_{02}\bar{h}_{12}\bar{h}_{21} - \bar{a}_{11}\bar{h}_{31} - 2\bar{b}_{02}\bar{h}_{31}\right) + n_3\Big],\end{aligned}$$

$$\begin{aligned}\bar{\alpha}_{50} = 2\sqrt{2}\Big[&n_2\left(\bar{a}_{10} + \bar{b}_{01}\right) + n_1\left(2\bar{a}_{20} + \bar{b}_{11}\right)\\ &- \left(\frac{5}{2}\bar{h}_{12}^3 - 3\bar{h}_{12}\bar{h}_{22} + 15\bar{h}_{12}\bar{h}_{03}\bar{h}_{21} + \bar{h}_{32} - 3\bar{h}_{13}\bar{h}_{21} - 3\bar{h}_{03}\bar{h}_{31}\right)\\ &\times\left(3\bar{a}_{30} + \bar{b}_{21} - \bar{a}_{11}\bar{h}_{21} - 2\bar{b}_{02}\bar{h}_{21}\right)\\ &+ \left(\frac{3}{2}\bar{h}_{12}^2 + 3\bar{h}_{03}\bar{h}_{21} - \bar{h}_{22}\right)\\ &\times\left(4\bar{a}_{40} + \bar{b}_{31} - 2\bar{a}_{21}\bar{h}_{21} - 2\bar{b}_{12}\bar{h}_{21} + 2\bar{a}_{11}\bar{h}_{12}\bar{h}_{21} + 4\bar{b}_{02}\bar{h}_{12}\bar{h}_{21} - \bar{a}_{11}\bar{h}_{31} - 2\bar{b}_{02}\bar{h}_{31}\right)\\ &- \bar{h}_{12}n_3 + n_4\Big],\end{aligned}$$

$$\bar{\alpha}_{01} = 2\sqrt{2}\left[\frac{2}{3}\bar{a}_{12} + 2\bar{b}_{03} - 2\bar{h}_{03}\bar{a}_{11} - 4\bar{h}_{03}\bar{b}_{02} + \left(\bar{a}_{10} + \bar{b}_{01}\right)\left(5\bar{h}_{03}^2 - 2\bar{h}_{04}\right)\right],$$

$$\begin{aligned}\bar{\alpha}_{11} = 2\sqrt{2}\Big[&\left(\bar{a}_{10} + \bar{b}_{01}\right)\left(10\bar{h}_{04}\bar{h}_{12} - 2\bar{h}_{14} + 10\bar{h}_{03}\bar{h}_{13} - 35\bar{h}_{12}\bar{h}_{03}^2\right)\\ &+ \left(\bar{a}_{11} + 2\bar{b}_{02}\right)\left(-2\bar{h}_{13} + 10\bar{h}_{12}\bar{h}_{03}\right)\\ &- 2\bar{h}_{12}\left(\bar{a}_{12} + 3\bar{b}_{03}\right)\\ &+ \left(2\bar{a}_{20} + \bar{b}_{11}\right)\left(5\bar{h}_{03}^2 - 2\bar{h}_{04}\right)\\ &- 4\bar{h}_{03}\left(\bar{b}_{12} + \bar{a}_{21}\right) + \frac{4}{3}\bar{a}_{22} + 2\bar{b}_{13}\Big];\end{aligned}$$

$$\bar{h}_4 = -\frac{1}{2}\bar{h}_{21}^2 + \bar{h}_{40}, \qquad \bar{h}_5 = \bar{h}_{12}\bar{h}_{21}^2 - \bar{h}_{21}\bar{h}_{31} + \bar{h}_{50},$$

$$\bar{h}_6 = -2\bar{h}_{12}^2\bar{h}_{21}^2 - \bar{h}_{03}\bar{h}_{21}^3 + \bar{h}_{21}^2\bar{h}_{22} + 2\bar{h}_{12}\bar{h}_{21}\bar{h}_{31} - \frac{1}{2}\bar{h}_{31}^2 - \bar{h}_{21}\bar{h}_{41} + \bar{h}_{60},$$

$$\begin{aligned}\bar{h}_7 = &-4\bar{h}_{22}\bar{h}_{21}^2\bar{h}_{12} + \bar{h}_{70} - \bar{h}_{51}\bar{h}_{21} - \bar{h}_{13}\bar{h}_{21}^3\\ &+ \bar{h}_{32}\bar{h}_{21}^2 + 4\bar{h}_{12}^3\bar{h}_{21}^2 - \bar{h}_{31}\bar{h}_{41} + \bar{h}_{12}\bar{h}_{31}^2\\ &+ 2\bar{h}_{22}\bar{h}_{21}\bar{h}_{31} + 6\bar{h}_{03}\bar{h}_{21}^3\bar{h}_{12} - 3\bar{h}_{03}\bar{h}_{21}^2\bar{h}_{31}\\ &- 4\bar{h}_{12}^2\bar{h}_{21}\bar{h}_{31} + 2\bar{h}_{12}\bar{h}_{21}\bar{h}_{41},\end{aligned}$$

$$
\begin{aligned}
\bar h_8 ={}& \bar h_{80} - \bar h_{23}\bar h_{21}^{3} - \bar h_{61}\bar h_{21} + \bar h_{04}\bar h_{21}^{4} + \bar h_{42}\bar h_{21}^{2} \\
&- 2\bar h_{12}^{2}\bar h_{31}^{2} - 8\bar h_{12}^{4}\bar h_{21}^{2} - 2\bar h_{22}^{2}\bar h_{21}^{2} \\
&- \frac{9}{2}\bar h_{03}^{2}\bar h_{21}^{4} - \bar h_{31}\bar h_{51} + \bar h_{22}\bar h_{31}^{2} - \frac{1}{2}\bar h_{41}^{2} \\
&+ 8\bar h_{12}^{3}\bar h_{21}\bar h_{31} - 4\bar h_{12}^{2}\bar h_{21}\bar h_{41} + 12\bar h_{22}\bar h_{21}^{2}\bar h_{12}^{2} \\
&- 24\bar h_{03}\bar h_{21}^{3}\bar h_{12}^{2} + 2\bar h_{12}\bar h_{31}\bar h_{41} + 6\bar h_{12}\bar h_{21}^{3}\bar h_{13} \\
&- 4\bar h_{12}\bar h_{21}^{2}\bar h_{32} + 2\bar h_{12}\bar h_{21}\bar h_{51} + 6\bar h_{22}\bar h_{21}^{3}\bar h_{03} \\
&+ 2\bar h_{22}\bar h_{21}\bar h_{41} + 2\bar h_{32}\bar h_{21}\bar h_{31} - 3\bar h_{03}\bar h_{21}\bar h_{31}^{2} \\
&- 3\bar h_{03}\bar h_{21}^{2}\bar h_{41} - 3\bar h_{13}\bar h_{21}^{2}\bar h_{31} \\
&- 8\bar h_{12}\bar h_{22}\bar h_{21}\bar h_{31} + 18\bar h_{12}\bar h_{03}\bar h_{21}^{2}\bar h_{31}, \\
\bar h_9 ={}& \bar h_{12}\Big[\bar h_{21}^{4}\left(45\bar h_{03}^{2} - 8\bar h_{04}\right) + 6\bar h_{21}^{3}\left(\bar h_{23} - 8\bar h_{03}\bar h_{22}\right) \\
&+ 2\bar h_{21}^{2}\left(9\bar h_{13}\bar h_{31} - 2\bar h_{42} + 6\bar h_{22}^{2} + 9\bar h_{03}\bar h_{41}\right) \\
&+ 2\bar h_{21}\left(9\bar h_{03}\bar h_{31}^{2} - 4\bar h_{41}\bar h_{22} - 4\bar h_{31}\bar h_{32}\right) \\
&-4\bar h_{22}\bar h_{31}^{2} + \bar h_{41}^{2} + 2\bar h_{31}\bar h_{51} + 2\bar h_{21}\bar h_{61}\Big] \\
&+ 4\bar h_{12}^{2}\Big[\bar h_{21}\Big(3\bar h_{21}\bar h_{32} - 6\bar h_{21}^{2}\bar h_{13} - \bar h_{51} \\
&\quad + 6\bar h_{31}\bar h_{22} - 18\bar h_{31}\bar h_{03}\bar h_{21}\Big) - \bar h_{31}\bar h_{41}\Big] \\
&+ 4\bar h_{12}^{3}\left(20\bar h_{03}\bar h_{21}^{3} + 2\bar h_{41}\bar h_{21} - 8\bar h_{22}\bar h_{21}^{2} + \bar h_{31}^{2}\right) \\
&+ \bar h_{21}\Big(2\bar h_{42}\bar h_{31} - 16\bar h_{12}^{4}\bar h_{31} - 4\bar h_{22}^{2}\bar h_{31} + 2\bar h_{22}\bar h_{51} \\
&\quad +2\bar h_{32}\bar h_{41} - 3\bar h_{13}\bar h_{31}^{2} - \bar h_{71} - 6\bar h_{03}\bar h_{31}\bar h_{41}\Big) \\
&+ \bar h_{21}^{2}\Big(16\bar h_{12}^{5} - 4\bar h_{32}\bar h_{22} - 3\bar h_{31}\bar h_{23} \\
&\quad - 3\bar h_{03}\bar h_{51} + \bar h_{52} + 18\bar h_{22}\bar h_{31}\bar h_{03}\Big) \\
&+ 2\bar h_{22}\bar h_{31}\bar h_{41} + 6\bar h_{32}\bar h_{21}^{3}\bar h_{03} + 4\bar h_{04}\bar h_{21}^{3}\bar h_{31} \\
&- \bar h_{41}\bar h_{51} - \bar h_{31}\bar h_{61} + \bar h_{32}\bar h_{31}^{2} - \bar h_{03}\bar h_{31}^{3} \\
&- 9\bar h_{03}\bar h_{21}^{4}\bar h_{13} - 18\bar h_{31}\bar h_{03}^{2}\bar h_{21}^{3} + \bar h_{14}\bar h_{21}^{4} \\
&- \bar h_{33}\bar h_{21}^{3} + 6\bar h_{22}\bar h_{21}^{3}\bar h_{13} + \bar h_{90}; \\
n_1 ={}& \frac{1}{8}\Big(12\bar h_{22}^{2} + 35\bar h_{12}^{4} + 24\bar h_{12}\bar h_{32} \\
&- 60\bar h_{12}^{2}\bar h_{22} - 72\bar h_{12}\bar h_{03}\bar h_{31} + 324\bar h_{12}^{2}\bar h_{03}\bar h_{21} \\
&+108\bar h_{03}^{2}\bar h_{21}^{2} - 72\bar h_{12}\bar h_{13}\bar h_{21} - 72\bar h_{22}\bar h_{03}\bar h_{21}\Big), \\
n_2 ={}& -\frac{1}{8}\Big(-24\bar h_{22}\bar h_{32} + 60\bar h_{12}\bar h_{22}^{2} + 60\bar h_{12}^{2}\bar h_{32} \\
&- 140\bar h_{12}^{3}\bar h_{22} + 63\bar h_{12}^{5} \\
&- 180\bar h_{12}^{2}\bar h_{03}\bar h_{31} + 780\bar h_{12}^{3}\bar h_{03}\bar h_{21} \\
&+ 72\bar h_{22}\bar h_{03}\bar h_{31} - 504\bar h_{12}\bar h_{22}\bar h_{03}\bar h_{21} \\
&- 216\bar h_{03}^{2}\bar h_{21}\bar h_{31} + 972\bar h_{12}\bar h_{03}^{2}\bar h_{21}^{2} \\
&+ 72\bar h_{03}\bar h_{21}\bar h_{32} + 72\bar h_{22}\bar h_{13}\bar h_{21} \\
&- 180\bar h_{12}^{2}\bar h_{13}\bar h_{21} - 216\bar h_{03}\bar h_{21}^{2}\bar h_{13}\Big), \\
n_3 ={}& 5\bar a_{50} + \bar b_{41} - 2\bar b_{22}\bar h_{21} - \bar a_{11}\bar h_{41} \\
&+ 2\bar a_{11}\bar h_{22}\bar h_{21} - 3\bar a_{11}\bar h_{03}\bar h_{21}^{2} + 2\bar a_{11}\bar h_{12}\bar h_{31} \\
&- 4\bar a_{11}\bar h_{12}^{2}\bar h_{21} - 2\bar b_{02}\bar h_{41} + 4\bar b_{02}\bar h_{22}\bar h_{21} \\
&- 6\bar b_{02}\bar h_{03}\bar h_{21}^{2} + 4\bar b_{02}\bar h_{12}\bar h_{31} - 8\bar b_{02}\bar h_{12}^{2}\bar h_{21} \\
&- 2\bar b_{12}\bar h_{31} + 4\bar b_{12}\bar h_{12}\bar h_{21} - 2\bar a_{21}\bar h_{31} \\
&+ 4\bar a_{21}\bar h_{12}\bar h_{21} - 3\bar a_{31}\bar h_{21} + 3\bar b_{03}\bar h_{21}^{2} + \bar a_{12}\bar h_{21}^{2}, \\
n_4 ={}& 4\bar b_{12}\bar h_{22}\bar h_{21} + 4\bar b_{02}\bar h_{32}\bar h_{21} + 6\bar a_{31}\bar h_{12}\bar h_{21} \\
&+ 2\bar a_{11}\bar h_{12}\bar h_{41} + 4\bar b_{12}\bar h_{12}\bar h_{31} - 4\bar a_{11}\bar h_{12}^{2}\bar h_{31} \\
&+ 4\bar b_{02}\bar h_{12}\bar h_{41} - 12\bar b_{03}\bar h_{21}^{2}\bar h_{12} - 4\bar a_{12}\bar h_{21}^{2}\bar h_{12} \\
&+ 2\bar a_{11}\bar h_{22}\bar h_{31} + 6\bar b_{03}\bar h_{21}\bar h_{31} - 3\bar a_{11}\bar h_{13}\bar h_{21}^{2} \\
&+ 6\bar a_{60} + \bar b_{51} - 6\bar b_{12}\bar h_{03}\bar h_{21}^{2} + 2\bar a_{11}\bar h_{32}\bar h_{21} \\
&+ 4\bar b_{02}\bar h_{22}\bar h_{31} + 4\bar a_{21}\bar h_{22}\bar h_{21} + 2\bar a_{12}\bar h_{21}\bar h_{31} \\
&+ 4\bar b_{22}\bar h_{12}\bar h_{21} - 8\bar b_{12}\bar h_{12}^{2}\bar h_{21} + 16\bar b_{02}\bar h_{12}^{3}\bar h_{21} \\
&+ 8\bar a_{11}\bar h_{12}^{3}\bar h_{21} - 6\bar b_{02}\bar h_{13}\bar h_{21}^{2} - 8\bar a_{21}\bar h_{12}^{2}\bar h_{21} \\
&- 8\bar b_{02}\bar h_{12}^{2}\bar h_{31} - 4\bar a_{41}\bar h_{21} - 2\bar b_{32}\bar h_{21} \\
&- 6\bar a_{21}\bar h_{03}\bar h_{21}^{2} + 4\bar a_{21}\bar h_{12}\bar h_{31} + 3\bar b_{13}\bar h_{21}^{2} \\
&+ 2\bar a_{22}\bar h_{21}^{2} - 6\bar a_{11}\bar h_{03}\bar h_{21}\bar h_{31} + 18\bar a_{11}\bar h_{03}\bar h_{21}^{2}\bar h_{12} \\
&- 8\bar a_{11}\bar h_{12}\bar h_{22}\bar h_{21} - 12\bar b_{02}\bar h_{03}\bar h_{21}\bar h_{31} + 36\bar b_{02}\bar h_{03}\bar h_{21}^{2}\bar h_{12} \\
&- 16\bar b_{02}\bar h_{12}\bar h_{22}\bar h_{21} - 2\bar b_{22}\bar h_{31} - \bar a_{11}\bar h_{51} - 2\bar b_{12}\bar h_{41} \\
&- 2\bar a_{21}\bar h_{41} - 2\bar b_{02}\bar h_{51} - 3\bar a_{31}\bar h_{31}.
\end{aligned}
\tag{24}
$$

For convenience, let

$$L^* = L_1 \cup L_2, \qquad L_1^* = L^* \cap U_1,$$
$$L_2^* = L^* \cap U_2, \qquad L_3^* = \text{Cl.}\left(L^* - \bigcup_{i=1}^{2} L_i^*\right),$$
$$L_0^1 = L_1^*, \tag{25}$$
$$L_0 \cap U_2 = L_0^2 \cup L_0^3 \quad (\text{as shown in Figure 2 (b)}),$$
$$L_0^4 = \text{Cl.}\left(L_0 - \bigcup_{j=1}^{3} L_0^j\right).$$

Theorem 3. *Assume that system* (1) *has a compound loop* L_0 *as stated before. Then, the functions* $M_i(h,\delta)$ *given in* (3) *at* $h = 0$ *have the following expansions:*

$$\begin{aligned} M_1(h,\delta) &= c_0(\delta) + c_1(\delta)|h|^{3/4} + B_{00}c_2(\delta)|h|^{5/6} \\ &\quad - c_3(\delta)h\ln|h| + c_4(\delta)h + B_{10}c_5(\delta)|h|^{7/6} \\ &\quad + c_6(\delta)|h|^{5/4} + c_7(\delta)|h|^{7/4} \\ &\quad - \frac{1}{11}B_{00}c_8(\delta)|h|^{11/6} - c_9(\delta)h^2\ln|h| + O\left(h^2\right), \end{aligned} \tag{26}$$

$$\begin{aligned} M_2(h,\delta) &= \bar{c}_0(\delta) + c_1(\delta)|h|^{3/4} + c_3(\delta)h\ln|h| \\ &\quad + \bar{c}_3(\delta)h + c_6(\delta)|h|^{5/4} + c_7(\delta)|h|^{7/4} \\ &\quad + c_9(\delta)h^2\ln|h| + O\left(h^2\right) \end{aligned} \tag{27}$$

for $0 < -h \ll 1$, *and*

$$\begin{aligned} M_3(h,\delta) &= \tilde{c}_0(\delta) - D_1c_1(\delta)h^{3/4} + B_{00}^*c_2(\delta)h^{5/6} \\ &\quad + \tilde{c}_2(\delta)h + B_{10}^*c_5(\delta)h^{7/6} - D_2c_6(\delta)h^{5/4} \\ &\quad + D_1c_7(\delta)h^{7/4} + \frac{1}{11}B_{00}^*c_8(\delta)h^{11/6} + O\left(h^2\right) \end{aligned} \tag{28}$$

for $0 < h \ll 1$, *where*

$$\begin{aligned} c_1(\delta) &= c_1(S_2,\delta), & c_2(\delta) &= c_1(S_1,\delta), \\ c_3(\delta) &= c_2(S_2,\delta), & c_5(\delta) &= c_3(S_1,\delta), \\ c_6(\delta) &= c_4(S_2,\delta), & c_7(\delta) &= c_5(S_2,\delta), \\ c_8(\delta) &= c_4(S_1,\delta), & c_9(\delta) &= c_6(S_2,\delta), \end{aligned} \tag{29}$$

$$\begin{aligned} c_0(\delta) &= M_1(0,\delta) = \oint_{L^*} q\,dx - p\,dy = \sum_{i=1}^{2}\int_{L_i} q\,dx - p\,dy, \\ \bar{c}_0(\delta) &= M_2(0,\delta) = \oint_{L_3} q\,dx - p\,dy, \\ \tilde{c}_0(\delta) &= M_3(0,\delta) = c_0(\delta) + \bar{c}_0(\delta), \end{aligned} \tag{30}$$

$$\begin{aligned} c_4(\delta) &= \int_{L_1^*}\left(p_x + q_y - \sigma\right)dt \\ &\quad + \int_{L_2^*}\left[p_x + q_y - \eta_0 - \eta_1(x - x_2)\right]dt \\ &\quad + \int_{L_3^*}\left(p_x + q_y\right)dt + t_2c_1(\delta) + t_3c_2(\delta) + t_4c_3(\delta), \\ \bar{c}_3(\delta) &= \oint_{L_3}\left[p_x + q_y - \eta_0 - \eta_1(x - x_2)\right]dt \\ &\quad + t_0c_1(\delta) + t_1c_3(\delta), \\ \tilde{c}_2(\delta) &= \int_{L_1^*}\left(p_x + q_y - \sigma\right)dt + \int_{L_0^2\cup L_0^3}\left(p_x + q_y - \eta_0\right)dt \\ &\quad + \int_{L_0^4}\left(p_x + q_y\right)dt + t_5c_1(\delta) + t_6c_2(\delta), \end{aligned} \tag{31}$$

where $\sigma = (p_x + q_y)|_{(x_1,y_1)}$, $\eta_0 = (p_x + q_y)|_{(x_2,y_2)}$, $\eta_1 = (p_{xx} + q_{yx})|_{(x_2,y_2)}$. *In particular,*

$$\begin{aligned} c_4(\delta) &= \sum_{i=1}^{2}\int_{L_i}\left(p_x + q_y\right)dt, \qquad \bar{c}_3(\delta) = \oint_{L_3}\left(p_x + q_y\right)dt, \\ \tilde{c}_2(\delta) &= \sum_{i=1}^{3}\int_{L_i}\left(p_x + q_y\right)dt \end{aligned} \tag{32}$$

if $c_1(\delta) = c_2(\delta) = c_3(\delta) = 0$. *Here,* t_i, $i = 0, 1, \ldots, 6$, *are constants and* B_{00}, B_{00}^*, B_{10}, B_{10}^* *are given in Lemma 1.*

Proof. First, by (6), (10) with $i = 2$, (12), (14), (29), and Theorem 2.2 in [4], we directly obtain (27) with $\bar{c}_0(\delta)$, $\bar{c}_3(\delta)$ given by (30) and (31), respectively. Then we study the expansions of M_1 and M_3.

By (5), (7), (29), and Lemmas 1 and 2, we have

$$\begin{aligned} M_1(h,\delta) &= c_1(\delta)|h|^{3/4} + B_{00}c_2(\delta)|h|^{5/6} - c_3(\delta)h\ln|h| \\ &\quad + B_{10}c_5(\delta)|h|^{7/6} + c_6(\delta)|h|^{5/4} \\ &\quad + c_7(\delta)|h|^{7/4} - \frac{1}{11}B_{00}c_8(\delta)|h|^{11/6} \\ &\quad - c_9(\delta)h^2\ln|h| + O\left(h^2\right) + N(h,\delta) \end{aligned} \tag{33}$$

for $0 < -h \ll 1$, and

$$
\begin{aligned}
M_3(h,\delta) = &-D_1 c_1(\delta) h^{3/4} + B^*_{00} c_2(\delta) h^{5/6} + B^*_{10} c_5(\delta) h^{7/6} \\
&- D_2 c_6(\delta) h^{5/4} + D_1 c_7(\delta) h^{7/4} \\
&+ \frac{1}{11} B^*_{00} c_8(\delta) h^{11/6} + O\left(h^2\right) + N^*(h,\delta)
\end{aligned} \tag{34}
$$

for $0 < h \ll 1$, where

$$
\begin{aligned}
N(h,\delta) &= N_{11}(h,\delta) + N_{12}(h,\delta) + I_{13}(h,\delta), \\
N^*(h,\delta) &= N_{31}(h,\delta) + N_{32}(h,\delta) + N_{33}(h,\delta) + I_{34}(h,\delta).
\end{aligned} \tag{35}
$$

Let

$$
\begin{aligned}
N(h,\delta) &= c_0(\delta) + c_4(\delta) h + O\left(h^2\right), \\
N^*(h,\delta) &= \tilde{c}_0(\delta) + \tilde{c}_2(\delta) h + O\left(h^2\right).
\end{aligned} \tag{36}
$$

It follows further that

$$
\begin{aligned}
c_0(\delta) &= N_{11}(0,\delta) + N_{12}(0,\delta) + I_{13}(0,\delta) \\
&= \lim_{\epsilon_0 \to 0} \left[N_{11}(0,\delta) + N_{12}(0,\delta) + I_{13}(0,\delta)\right] \\
&= \lim_{\epsilon_0 \to 0} I_{13}(0,\delta) = \oint_{L_1 \cup L_2} q\,dx - p\,dy \\
&= \sum_{i=1}^{2} \int_{L_i} q\,dx - p\,dy = M_1(0,\delta), \\
\tilde{c}_0(\delta) &= N_{31}(0,\delta) + N_{32}(0,\delta) + N_{33}(0,\delta) + I_{34}(0,\delta) \\
&= \lim_{\epsilon_0 \to 0} \left[N_{31}(0,\delta) + N_{32}(0,\delta) \right.\\
&\qquad \left. + N_{33}(0,\delta) + I_{34}(0,\delta)\right] \\
&= \lim_{\epsilon_0 \to 0} I_{34}(0,\delta) = \oint_{L_0} q\,dx - p\,dy \\
&= \sum_{i=1}^{3} \int_{L_i} q\,dx - p\,dy = M_3(0,\delta), \\
c_4(\delta) + O(h) &= N_h(h,\delta) = M_{1h}(h,\delta) \\
&\quad + \frac{3}{4} c_1(\delta) |h|^{-1/4} + \frac{5}{6} B_{00} c_2(\delta) |h|^{-1/6} \\
&\quad + c_3(\delta)(\ln|h| + 1) + O\left(|h|^{1/6}\right), \\
\tilde{c}_2(\delta) + O(h) &= N^*_h(h,\delta) \\
&= M_{3h}(h,\delta) + \frac{3}{4} D_1 c_1(\delta) h^{-1/4} \\
&\quad - \frac{5}{6} B^*_{00} c_2(\delta) h^{-1/6} + O\left(h^{1/6}\right).
\end{aligned} \tag{37}
$$

Then by Lemma 3.1.2 in [5], we have

$$
\begin{aligned}
c_4(\delta) &= N_h(0,\delta) \\
&= \lim_{h \to 0^-} \left[\oint_{L_{h1}} \left(p_x + q_y\right) dt + \frac{3}{4} c_1(\delta) |h|^{-1/4} \right. \\
&\qquad \left. + \frac{5}{6} B_{00} c_2(\delta) |h|^{-1/6} + c_3(\delta)(\ln|h| + 1)\right], \\
\tilde{c}_2(\delta) &= N^*_h(0,\delta) \\
&= \lim_{h \to 0^+} \left[\oint_{L_{h3}} \left(p_x + q_y\right) dt \right. \\
&\qquad \left. + \frac{3}{4} D_1 c_1(\delta) h^{-1/4} - \frac{5}{6} B^*_{00} c_2(\delta) h^{-1/6}\right].
\end{aligned} \tag{38}
$$

It is easy to see that

$$
\begin{aligned}
\oint_{L_{h1}} \left(p_x + q_y\right) dt &= \sum_{i=1}^{3} \int_{L^{(i)}_{h1}} \left(p_x + q_y\right) dt \\
&= \int_{L^{(1)}_{h1}} \left(p_x + q_y - \sigma\right) dt \\
&\quad + \int_{L^{(2)}_{h1}} \left[p_x + q_y - \eta_0 \right.\\
&\qquad \left. - \eta_1 (x - x_2)\right] dt \\
&\quad + \int_{L^{(3)}_{h1}} \left(p_x + q_y\right) dt + \sigma \int_{L^{(1)}_{h1}} dt \\
&\quad + \eta_0 \int_{L^{(2)}_{h1}} dt + \eta_1 \int_{L^{(2)}_{h1}} (x - x_2)\, dt,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&\oint_{L_{h3}} \left(p_x + q_y\right) dt \\
&= \sum_{i=1}^{4} \int_{L^{(i)}_{h3}} \left(p_x + q_y\right) dt = \int_{L^{(1)}_{h3}} \left(p_x + q_y - \sigma\right) dt \\
&\quad + \int_{L^{(2)}_{h3} \cup L^{(3)}_{h3}} \left(p_x + q_y - \eta_0\right) dt + \int_{L^{(4)}_{h3}} \left(p_x + q_y\right) dt \\
&\quad + \sigma \int_{L^{(1)}_{h3}} dt + \eta_0 \int_{L^{(2)}_{h3} \cup L^{(3)}_{h3}} dt.
\end{aligned}
$$

Noting $\tilde{h}_3 < 0$, $\bar{h}_4 < 0$, by (19), (23), and (29), we have

$$
\begin{aligned}
c_4(\delta) = \lim_{h \to 0^-} &\left[\int_{L^{(1)}_{h1}} \left(p_x + q_y - \sigma\right) dt \right. \\
&+ \int_{L^{(2)}_{h1}} \left[p_x + q_y - \eta_0 - \eta_1 (x - x_2)\right] dt \\
&\left. + \int_{L^{(3)}_{h1}} \left(p_x + q_y\right) dt\right]
\end{aligned}
$$

$$+\lim_{h\to 0^-}\sigma\left[\int_{L_{h1}^{(1)}}dt-\frac{5\sqrt{2}}{3}B_{00}\left|\widetilde{h}_3\right|^{-1/3}|h|^{-1/6}\right]$$
$$+\lim_{h\to 0^-}\eta_0\left[\int_{L_{h1}^{(2)}}dt+\frac{3\sqrt{2}}{2}\widetilde{A}_0\left|\overline{h}_4\right|^{-1/4}|h|^{-1/4}+O(1)(\ln|h|+1)\right]$$
$$+\lim_{h\to 0^-}\eta_1\left[\int_{L_{h1}^{(2)}}(x-x_2)\,dt-\frac{\sqrt{2}}{4}\left|\overline{h}_4\right|^{-1/2}(\ln|h|+1)\right],$$
$$\widetilde{c}_2(\delta)=\lim_{h\to 0^+}\left[\int_{L_{h3}^{(1)}}\left(p_x+q_y-\sigma\right)dt+\int_{L_{h3}^{(2)}\cup L_{h3}^{(3)}}\left(p_x+q_y-\eta_0\right)dt+\int_{L_{h3}^{(4)}}\left(p_x+q_y\right)dt\right]$$
$$+\lim_{h\to 0^+}\sigma\left[\int_{L_{h3}^{(1)}}dt+\frac{5\sqrt{2}}{3}B_{00}^{*}\left|\widetilde{h}_3\right|^{-1/3}h^{-1/6}\right]$$
$$+\lim_{h\to 0^+}\eta_0\left[\int_{L_{h3}^{(2)}\cup L_{h3}^{(3)}}dt-3\sqrt{2}\,\overline{A}_0\left|\overline{h}_4\right|^{-1/4}h^{-1/4}\right].\tag{40}$$

Then by the proof of (3.13) in [3], the following equations hold:

$$\lim_{h\to 0^-}\left[\int_{L_{h1}^{(1)}}dt-\frac{5\sqrt{2}}{3}B_{00}\left|\widetilde{h}_3\right|^{-1/3}|h|^{-1/6}\right]=T_0,$$
$$\lim_{h\to 0^+}\left[\int_{L_{h3}^{(1)}}dt+\frac{5\sqrt{2}}{3}B_{00}^{*}\left|\widetilde{h}_3\right|^{-1/3}h^{-1/6}\right]=T_0^{*},$$
$$\lim_{h\to 0^-}\int_{L_{h1}^{(1)}}\left(p_x+q_y-\sigma\right)dt=\lim_{h\to 0^+}\int_{L_{h3}^{(1)}}\left(p_x+q_y-\sigma\right)dt=\int_{L_1^{*}}\left(p_x+q_y-\sigma\right)dt.\tag{41}$$

Here, T_0 and T_0^* are constants. By a similar argument used in Theorems 2.2 and 2.4 in [4], one can obtain

$$\lim_{h\to 0^-}\left[\int_{L_{h1}^{(2)}}dt+\frac{3\sqrt{2}}{2}\widetilde{A}_0\left|\overline{h}_4\right|^{-1/4}|h|^{-1/4}+O(1)(\ln|h|+1)\right]=T_1,$$
$$\lim_{h\to 0^-}\left[\int_{L_{h1}^{(2)}}(x-x_2)\,dt-\frac{\sqrt{2}}{4}\left|\overline{h}_4\right|^{-1/2}(\ln|h|+1)\right]=T_1^{*},$$
$$\lim_{h\to 0^+}\left[\int_{L_{h3}^{(i)}}dt-\frac{3\sqrt{2}}{2}\overline{A}_0\left|\overline{h}_4\right|^{-1/4}h^{-1/4}\right]=T_i,\quad i=2,3,$$
$$\lim_{h\to 0^-}\int_{L_{h1}^{(2)}}\left[p_x+q_y-\eta_0-\eta_1(x-x_2)\right]dt=\int_{L_2^{*}}\left[p_x+q_y-\eta_0-\eta_1(x-x_2)\right]dt,$$
$$\lim_{h\to 0^+}\int_{L_{h3}^{(i)}}\left(p_x+q_y-\eta_0\right)dt=\int_{L_0^{i}}\left(p_x+q_y-\eta_0\right)dt,\quad i=2,3.\tag{42}$$

Here $T_1^*, T_i, i=1,2,3$, are constants. Therefore, we can obtain (31) and (32). Thus we have proved Theorem 3. □

In the following we use Theorem 3 to study the problem of limit cycle bifurcation near L_0. For the sake of convenience, we say that (1) has a distribution $(i,j)+k$ of $i+j+k$ limit cycles if there are i and j limit cycles near the inside of L^* and L_3, respectively, and k limit cycles near the outside of L_0. Then we can prove the following theorem.

Theorem 4. *Assume that system* (1) *has a compound loop* L_0 *as stated before and* (26)–(28) *hold. Define* $c_4^*(\delta)=c_4(\delta)|_{c_1=c_2=c_3=0}$, $\overline{c}_3^*(\delta)=\overline{c}_3(\delta)|_{c_1=c_3=0}$, $\widetilde{c}_2^*(\delta)=c_4^*(\delta)+\overline{c}_3^*(\delta)$, $c_3^*(\delta)=c_3(\delta)|_{c_1=0}$. *Let there exist* $\delta_0\in\mathbb{R}^m$, *such that* $(c_0,\overline{c}_0,c_1,c_2,c_3^*,c_4^*,\overline{c}_3^*)(\delta_0)=(0,0,0,0,0,0,0)$.

(1) *If* $c_l(\delta_0)\neq 0$, $c_j(\delta_0)=0$, $j=5,\ldots,l-1$, *and*

$$\operatorname{rank}\frac{\partial\left(c_0,\overline{c}_0,c_1,c_2,c_3^*,c_4^*,\overline{c}_3^*,c_5,\ldots,c_{l-1}\right)}{\partial\left(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6,\delta_7,\delta_8,\ldots,\delta_m\right)}=l+2,\tag{43}$$

then (1) *can have* $2l+1$ *limit cycles near* L_0 *for some* (ϵ,δ) *near* $(0,\delta_0)$, *where* $l=6,7$ *or* 9.

(2) *If* $c_5(\delta_0)c_6(\delta_0)<0$, *and*

$$\operatorname{rank}\frac{\partial\left(c_0,\overline{c}_0,c_1,c_2,c_3^*,c_4^*,\overline{c}_3^*\right)}{\partial\left(\delta_1,\delta_2,\delta_3,\delta_4,\ldots,\delta_m\right)}=7,\tag{44}$$

then (1) *can have 11 limit cycles near* L_0 *for some* (ϵ,δ) *near* $(0,\delta_0)$.

(3) *If* $c_8(\delta_0)c_9(\delta_0)>0$, $c_j(\delta_0)=0$, $j=5,6,7$, *and*

$$\operatorname{rank}\frac{\partial\left(c_0,\overline{c}_0,c_1,c_2,c_3^*,c_4^*,\overline{c}_3^*,c_5,c_6,c_7\right)}{\partial\left(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6,\delta_7,\delta_8,\ldots,\delta_m\right)}=10,\tag{45}$$

then (1) *can have 18 limit cycles near* L_0 *for some* (ϵ,δ) *near* $(0,\delta_0)$.

Proof. (1) Because of the similarity in the proof, we only prove the conclusion for $l=9$ and omit the rest. By our assumptions, there exists $\delta_0\in\mathbb{R}^m$ such that $\overline{c}_0(\delta_0)=\overline{c}_3^*(\delta_0)=c_3^*(\delta_0)=c_4^*(\delta_0)=0$, $c_j(\delta_0)=0$, $j=0,1,2,5,6,7,8$, $c_9(\delta_0)\neq 0$, and

$$\operatorname{rank}\frac{\partial\left(c_0,\overline{c}_0,c_1,c_2,c_3^*,c_4^*,\overline{c}_3^*,c_5,\ldots,c_8\right)}{\partial\left(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6,\delta_7,\delta_8,\ldots,\delta_m\right)}=11.\tag{46}$$

By the implicit function theorem, we can take $\overline{c}_0$, $\overline{c}_3^*$, c_3^*, c_4^*, c_j, $j = 0, 1, 2, 5, 6, 7, 8$ as free parameters varying near zero. Obviously, for these parameters varying near zero we have $|c_9| \geq |(1/2)c_9(\delta_0)| > 0$. In the following we proceed the process by 9 steps.

Step 1. Fix $(c_0, \overline{c}_0, c_1, c_2, c_3^*, c_4^*, \overline{c}_3^*, c_5, c_6, c_7) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and vary c_8 near 0.

First, for $c_8 = 0$, we have by (26) $c_9 M_1 > 0$ for $0 < -h \ll 1$.

Let $0 < |c_8| \ll 1$. Then $c_9 M_1 < 0$ for $0 < -h \ll 1$ if $c_8 c_9 > 0$. Thus, M_1 has a zero. Hence, for $0 < |c_8| \ll 1$,

(1) the condition $c_8 c_9 > 0$ implies a distribution $(1, 0) + 0$ of one limit cycle.

Step 2. Fix $(c_0, \overline{c}_0, c_1, c_2, c_3^*, c_4^*, \overline{c}_3^*, c_5, c_6) = (0, 0, 0, 0, 0, 0, 0, 0, 0)$, $c_8 c_9 > 0$ and vary c_7 near 0.

First, for $c_7 = 0$, we have by (26), (27), and (28) $c_8 M_1 < 0$, $c_9 M_2 < 0$ for $0 < -h \ll 1$, and $c_8 M_3 < 0$ for $0 < h \ll 1$.

Let $0 < |c_7| \ll |c_8|$. Then $c_8 M_1 > 0$, $c_9 M_2 > 0$ for $0 < -h \ll 1$, and $c_8 M_3 > 0$ for $0 < h \ll 1$ if $c_7 c_8 > 0, c_7 c_9 > 0$. Thus, M_1, M_2, and M_3 each gets a zero and the zero of M_1 got in Step 1 still exists. Hence, for $0 < |c_7| \ll |c_8| \ll 1$,

(2) the conditions $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(2, 1) + 1$ of 4 limit cycles.

Step 3. Fix $(c_0, \overline{c}_0, c_1, c_2, c_3^*, c_4^*, \overline{c}_3^*, c_5) = (0, 0, 0, 0, 0, 0, 0, 0)$, $c_7 \neq 0$ and vary c_6 near 0.

First, for $c_6 = 0$, we have by (26), (27), and (28) $c_7 M_1 > 0$, $c_7 M_2 > 0$ for $0 < -h \ll 1$, and $c_7 M_3 > 0$ for $0 < h \ll 1$.

Let $0 < |c_6| \ll |c_7|$. Then $c_7 M_1 < 0$, $c_7 M_2 < 0$ for $0 < -h \ll 1$, and $c_7 M_3 > 0$ for $0 < h \ll 1$ if $c_6 c_7 < 0$. Thus, M_1 and M_2 each gets a new zero and the zeros got in above steps still exist. Hence, for $0 < |c_6| \ll |c_7| \ll |c_8| \ll 1$,

(3) the conditions $c_6 c_7 < 0$, $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(3, 2) + 1$ of 6 limit cycles.

Step 4. Fix $(c_0, \overline{c}_0, c_1, c_2, c_3^*, c_4^*, \overline{c}_3^*) = (0, 0, 0, 0, 0, 0, 0)$, $c_6 \neq 0$ and vary c_5 near 0.

First, for $c_5 = 0$, we have by (26) and (28) $c_6 M_1 > 0$ for $0 < -h \ll 1$, and $c_6 M_3 < 0$ for $0 < h \ll 1$.

Let $0 < |c_5| \ll |c_6|$. Then $c_6 M_1 < 0$ for $0 < -h \ll 1$, and $c_6 M_3 > 0$ for $0 < h \ll 1$ if $c_5 c_6 < 0$. Thus, M_1 and M_3 each has a new zero and the zeros got in above steps still exist. Hence, for $0 < |c_5| \ll |c_6| \ll |c_7| \ll |c_8| \ll 1$,

(4) the conditions $c_5 c_6 < 0$, $c_6 c_7 < 0$, $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(4, 2) + 2$ of 8 limit cycles.

Step 5. Fix $(c_0, \overline{c}_0, c_1, c_2, c_3^*) = (0, 0, 0, 0, 0)$, $c_5 c_6 < 0$ and vary $(\overline{c}_3^*, c_4^*)$ near $(0, 0)$ with $\overline{c}_2^* (= \overline{c}_3^* + c_4^*) \neq 0$.

First, for $(\overline{c}_3^*, c_4^*) = (0, 0)$, we have by (26), (27), and (28) $c_5 M_1 > 0, c_6 M_2 > 0$ for $0 < -h \ll 1$, and $c_5 M_3 < 0$ for $0 < h \ll 1$.

Let $0 < |\overline{c}_3^*, c_4^*| \ll |c_5|$. Then

$$\begin{gathered} c_5 M_1 < 0, \qquad c_6 M_2 < 0 \quad \text{for } 0 < -h \ll 1, \\ c_5 M_3 > 0 \quad \text{for } 0 < h \ll 1 \end{gathered} \tag{47}$$

if $c_4^* c_5 > 0$, $\overline{c}_3^* c_6 > 0$, $\overline{c}_2^* c_5 > 0$, and

$$\begin{gathered} c_5 M_1 < 0, \qquad c_6 M_2 < 0 \quad \text{for } 0 < -h \ll 1, \\ c_5 M_3 < 0 \quad \text{for } 0 < h \ll 1 \end{gathered} \tag{48}$$

if $c_4^* c_5 > 0$, $\overline{c}_3^* c_6 > 0$, $\overline{c}_2^* c_5 < 0$. Thus, M_1, M_2, and M_3 each has one more zero in the first case and M_1 and M_2 each has a new zero in the second case. And the zeros got in above steps still exist. Hence, for $0 < |\overline{c}_3^*, c_4^*| \ll |c_5| \ll |c_6| \ll |c_7| \ll |c_8| \ll 1$,

(5i) the conditions $\overline{c}_3^* c_4^* < 0$, $\overline{c}_3^* \overline{c}_2^* < 0$, $c_4^* c_5 > 0$, $c_5 c_6 < 0$, $c_6 c_7 < 0$, $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(5, 3) + 3$ of 11 limit cycles, and

(5ii) the conditions $\overline{c}_3^* c_4^* < 0$, $\overline{c}_3^* \overline{c}_2^* > 0$, $c_4^* c_5 > 0$, $c_5 c_6 < 0$, $c_6 c_7 < 0$, $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(5, 3) + 2$ of 10 limit cycles.

Step 6. Fix $(c_0, \overline{c}_0, c_1, c_2) = (0, 0, 0, 0)$, $c_4^* \overline{c}_3^* < 0$ with $\overline{c}_2^* \neq 0$ and vary c_3^* near 0.

First, for $c_3^* = 0$, we have by (26) and (27) $c_4^* M_1 < 0$ and $\overline{c}_3^* M_2 < 0$ for $0 < -h \ll 1$.

Let $0 < |c_3^*| \ll |c_4^*, \overline{c}_3^*|$. Then $c_4^* M_1 > 0$ and $\overline{c}_3^* M_2 > 0$ for $0 < -h \ll 1$, if $c_3^* c_4^* < 0$, $\overline{c}_3^* c_3^* > 0$. Thus, M_1 and M_2 each gets a new zero and the zeros got in above steps still exist. Hence, for $0 < |c_3^*| \ll |c_4^*, \overline{c}_3^*| \ll |c_5| \ll |c_6| \ll |c_7| \ll |c_8| \ll 1$,

(6i) the conditions $c_3^* c_4^* < 0$, $\overline{c}_3^* c_4^* < 0$, $\overline{c}_3^* \overline{c}_2^* < 0$, $c_4^* c_5 > 0$, $c_5 c_6 < 0$, $c_6 c_7 < 0$, $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(6, 4) + 3$ of 13 limit cycles,

(6ii) the conditions $c_3^* c_4^* < 0$, $\overline{c}_3^* c_4^* < 0$, $\overline{c}_3^* \overline{c}_2^* > 0$, $c_4^* c_5 > 0$, $c_5 c_6 < 0$, $c_6 c_7 < 0$, $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(6, 4) + 2$ of 12 limit cycles.

Step 7. Fix $(c_0, \overline{c}_0, c_1) = (0, 0, 0)$, $c_3^* \overline{c}_2^* \neq 0$ and vary c_2 near 0.

First, for $c_2 = 0$, we have by (26) and (28) $c_3^* M_1 < 0$ for $0 < -h \ll 1$ and $\overline{c}_2^* M_3 > 0$ for $0 < h \ll 1$.

Let $0 < |c_2| \ll |c_3^*| \ll |c_4^*, \overline{c}_3^*|$. Then

$$\begin{gathered} c_3^* M_1 > 0 \quad \text{for } 0 < -h \ll 1, \\ \overline{c}_2^* M_3 < 0 \quad \text{for } 0 < h \ll 1 \end{gathered} \tag{49}$$

if $c_2 c_3^* > 0$, $\overline{c}_2^* c_2 > 0$, and

$$\begin{gathered} c_3^* M_1 > 0 \quad \text{for } 0 < -h \ll 1, \\ \overline{c}_2^* M_3 > 0 \quad \text{for } 0 < h \ll 1 \end{gathered} \tag{50}$$

if $c_2 c_3^* > 0$, $\overline{c}_2^* c_2 < 0$. Thus, M_1 and M_3 each gets one more zero in the first case and only M_1 has a new zero in the second case. And the zeros got in above steps still exist. Hence, for $0 < |c_2| \ll |c_3^*| \ll |c_4^*, \overline{c}_3^*| \ll |c_5| \ll |c_6| \ll |c_7| \ll |c_8| \ll 1$,

(7i) the conditions $c_2 c_3^* > 0$, $c_3^* c_4^* < 0$, $\overline{c}_3^* c_4^* < 0$, $\overline{c}_3^* \overline{c}_2^* > 0$, $c_4^* c_5 > 0$, $c_5 c_6 < 0$, $c_6 c_7 < 0$, $c_7 c_8 > 0$, $c_8 c_9 > 0$ imply a distribution $(7, 4) + 3$ of 14 limit cycles, (7ii) the conditions $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* < 0$, $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$, $c_8c_9 > 0$ imply a distribution $(7,4) + 3$ of 14 limit cycles.

Step 8. Fix $(c_0, \bar{c}_0) = (0,0)$, $c_2c_3^* > 0$ and vary c_1 near 0.

First, for $c_1 = 0$, we have by (26), (27), and (28) $c_2M_1 > 0, c_3^*M_2 > 0$ for $0 < -h \ll 1$, and $c_2M_3 < 0$ for $0 < h \ll 1$.

Let $0 < |c_1| \ll |c_2|$. Then $c_2M_1 < 0$, $c_3^*M_2 < 0$ for $0 < -h \ll 1$, and $c_2M_3 > 0$ for $0 < h \ll 1$ if $c_1c_2 < 0$. Thus, M_1, M_2, and M_3 each gets a new zero and the zeros got in above steps still exist. Hence, for $0 < |c_1| \ll |c_2| \ll |c_3^*| \ll |c_4^*, \bar{c}_3^*| \ll |c_5| \ll |c_6| \ll |c_7| \ll |c_8| \ll 1$,

(8) the conditions $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$, $c_8c_9 > 0$ imply a distribution $(8,5) + 4$ of 17 limit cycles.

Step 9. Fix $c_1 \neq 0$ and vary $(c_0, \bar{c}_0)$ near $(0,0)$.

First, for $(c_0, \bar{c}_0) = (0,0)$, we have by (26), (27), and (28) $c_1M_1 > 0, c_1M_2 > 0$, for $0 < -h \ll 1$, and $c_1M_3 < 0$ for $0 < h \ll 1$.

Let $0 < |c_0, \bar{c}_0| \ll |c_1|$. Then

$$c_1M_1 < 0, \qquad c_1M_2 < 0, \quad \text{for } 0 < -h \ll 1, \qquad c_1M_3 < 0 \quad \text{for } 0 < h \ll 1 \tag{51}$$

if $c_0c_1 < 0$, $\bar{c}_0c_1 < 0$,

$$c_1M_1 < 0, \qquad c_1M_2 > 0, \quad \text{for } 0 < -h \ll 1, \qquad c_1M_3 > 0 \quad \text{for } 0 < h \ll 1 \tag{52}$$

if $c_0c_1 < 0$, $\bar{c}_0c_1 > 0$ and $(c_0 + \bar{c}_0)c_1 > 0$, and

$$c_1M_1 > 0, \qquad c_1M_2 < 0, \quad \text{for } 0 < -h \ll 1, \qquad c_1M_3 > 0 \quad \text{for } 0 < h \ll 1 \tag{53}$$

if $c_0c_1 > 0$, $\bar{c}_0c_1 < 0$ and $(c_0 + \bar{c}_0)c_1 > 0$. Thus, we have correspondingly (a) M_1 and M_2 each has a new zero, (b) M_1 and M_3 each has a new zero, or (c) M_2 and M_3 each has a new zero. And the zeros got in above steps still exist.

Hence, for $0 < |c_0, \bar{c}_0| \ll |c_1| \ll |c_2| \ll |c_3^*| \ll |c_4^*, \bar{c}_3^*| \ll |c_5| \ll |c_6| \ll |c_7| \ll |c_8| \ll 1$,

(9i) the conditions $c_0\bar{c}_0 > 0$, $c_0c_1 < 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$, $c_8c_9 > 0$ imply a distribution $(9,6) + 4$ of 19 limit cycles,

(9ii) the conditions $c_0\bar{c}_0 < 0$, $c_0c_1 < 0$, $(c_0 + \bar{c}_0)c_1 > 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$, $c_8c_9 > 0$ imply a distribution $(9,5) + 5$ of 19 limit cycles,

(9iii) the conditions $c_0\bar{c}_0 < 0$, $c_0c_1 > 0$, $(c_0 + \bar{c}_0)c_1 > 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$, $c_8c_9 > 0$ imply a distribution $(8,6) + 5$ of 19 limit cycles.

Thus we get the conclusion for $l = 9$.

(2) By our assumptions in case (2) and the implicit function theorem we can take c_0, $\bar{c}_0$, c_1, c_2, c_3^*, c_4^*, $\bar{c}_3^*$ as free parameters varying near zero. Obviously, for these parameters varying near zero we have $c_5c_6 < 0$. By a similar argument in the above proof, we can prove that for $0 < |c_0, \bar{c}_0| \ll |c_1| \ll |c_2| \ll |c_3^*| \ll |c_4^*, \bar{c}_3^*| \ll 1$,

(i) the conditions $c_0\bar{c}_0 > 0$, $c_0c_1 < 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$ imply a distribution $(5,4) + 2$ of 11 limit cycles,

(ii) the conditions $c_0\bar{c}_0 < 0$, $c_0c_1 < 0$, $(c_0 + \bar{c}_0)c_1 > 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$ imply a distribution $(5,3) + 3$ of 11 limit cycles,

(iii) the conditions $c_0\bar{c}_0 < 0$, $c_0c_1 > 0$, $(c_0 + \bar{c}_0)c_1 > 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$ imply a distribution $(4,4) + 3$ of 11 limit cycles.

(3) By our assumptions in case (3) and the implicit function theorem we can take c_0, $\bar{c}_0$, c_1, c_2, c_3^*, c_4^*, $\bar{c}_3^*$, c_5, c_6, and c_7 as free parameters varying near zero. Obviously, for these parameters varying near zero we have $c_8c_9 > 0$. By a similar argument used in proving case (1), we can prove that for $0 < |c_0, \bar{c}_0| \ll |c_1| \ll |c_2| \ll |c_3^*| \ll |c_4^*, \bar{c}_3^*| \ll |c_5| \ll |c_6| \ll |c_7| \ll 1$,

(i) the conditions $c_0\bar{c}_0 > 0$, $c_0c_1 < 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$ imply a distribution $(8,6) + 4$ of 18 limit cycles,

(ii) the conditions $c_0\bar{c}_0 < 0$, $c_0c_1 < 0$, $(c_0 + \bar{c}_0)c_1 > 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$ imply a distribution $(8,5) + 5$ of 18 limit cycles,

(iii) the conditions $c_0\bar{c}_0 < 0$, $c_0c_1 > 0$, $(c_0 + \bar{c}_0)c_1 > 0$, $c_1c_2 < 0$, $c_2c_3^* > 0$, $c_3^*c_4^* < 0$, $\bar{c}_3^*c_4^* < 0$, $\bar{c}_3^*\bar{c}_2^* > 0$ (or $\bar{c}_3^*\bar{c}_2^* < 0$), $c_4^*c_5 > 0$, $c_5c_6 < 0$, $c_6c_7 < 0$, $c_7c_8 > 0$ imply a distribution $(7,6) + 5$ of 18 limit cycles.

This completes the proof. □

3. An Application

Consider a Liénard system of the form

$$\dot{x} = y, \qquad \dot{y} = -(x+1)^2x^3\left(x^2 - \frac{1}{4}x - \frac{1}{2}\right) - \epsilon f(x,\delta)\, y, \tag{54}$$

where

$$f(x,\delta) = \sum_{j=0}^{n} a_j x^j, \qquad \delta = (a_0, a_1, \ldots, a_n), \qquad 8 \le n \le 12. \tag{55}$$

System (54) $|_{\epsilon=0}$ is Hamiltonian with

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{8}(x+1)^3x^4(x-1). \tag{56}$$

We have the following theorem.

Theorem 5. *Let $C(n)$ denote the maximal number of limit cycles of the system* (54) *for ϵ small and all δ. Then we have* $C(11) \geq 18$, $C(n) \geq 2n-5$ $(n = 8, 9, 10, 12)$.

Proof. It is easy to verify that the unperturbed system has a compound loop $L_0 = L_1 \cup L_2 \cup L_3 \cup \{S_1, S_2\}$ with a cusp $S_1(-1,0)$ of order one and a nilpotent saddle $S_2(0,0)$ of order one, L_1, L_2 are heteroclinic orbits satisfying $\omega(L_1) = \alpha(L_2) = S_2$ and $\omega(L_2) = \alpha(L_1) = S_1$, and L_3 is a homoclinic loop to S_2. Inside $L^* = L_1 + L_2$ (L_3, resp.), there is a center $C_1((1/8)(1-\sqrt{33}), 0)$ ($C_2((1/8)(1+\sqrt{33}), 0)$, resp.).

Because of the similarity in the proof, here we only prove the case for $n = 11$ and omit the rest of the proof.

Let $n = 11$. By Theorem 3, we obtain

$$\begin{aligned} c_0(\delta) &= M_1(0,\delta) = -\oint_{L^*} f(x,\delta)\, y\, dx = -\sum_{j=0}^{11} a_j I_1^j, \\ \bar{c}_0(\delta) &= M_2(0,\delta) = -\oint_{L_3} f(x,\delta)\, y\, dx = -\sum_{j=0}^{11} a_j I_2^j, \end{aligned} \tag{57}$$

where

$$\begin{aligned} I_1^j &= \oint_{L^*} x^j y\, dx = 2\int_{-1}^{0} x^j y\, dx \\ &= \int_{-1}^{0} x^{j+2}(x+1)\sqrt{1-x^2}\,dx, \quad j = 0, 1, \ldots, 11, \\ I_2^j &= \oint_{L_3} x^j y\, dx = 2\int_{0}^{1} x^j y\, dx \\ &= \int_{0}^{1} x^{j+2}(x+1)\sqrt{1-x^2}\,dx, \quad j = 0, 1, \ldots, 11. \end{aligned} \tag{58}$$

Therefore,

$$\begin{aligned} c_0(\delta) = {} & -a_0\left(\frac{\pi}{16} - \frac{2}{15}\right) - a_1\left(\frac{\pi}{32} - \frac{2}{15}\right) - a_2\left(\frac{\pi}{32} - \frac{8}{105}\right) \\ & - a_3\left(\frac{5\pi}{256} - \frac{8}{105}\right) - a_4\left(\frac{5\pi}{256} - \frac{16}{315}\right) \\ & - a_5\left(\frac{7\pi}{512} - \frac{16}{315}\right) - a_6\left(\frac{7\pi}{512} - \frac{128}{3465}\right) \\ & - a_7\left(\frac{21\pi}{2048} - \frac{128}{3465}\right) - a_8\left(\frac{21\pi}{2048} - \frac{256}{9009}\right) \\ & - a_9\left(\frac{33\pi}{4096} - \frac{256}{9009}\right) - a_{10}\left(\frac{33\pi}{4096} - \frac{1024}{45045}\right) \\ & - a_{11}\left(\frac{429\pi}{65536} - \frac{1024}{45045}\right), \end{aligned}$$

$$\begin{aligned} \bar{c}_0(\delta) = {} & -a_0\left(\frac{\pi}{16} + \frac{2}{15}\right) - a_1\left(\frac{\pi}{32} + \frac{2}{15}\right) - a_2\left(\frac{\pi}{32} + \frac{8}{105}\right) \\ & - a_3\left(\frac{5\pi}{256} + \frac{8}{105}\right) - a_4\left(\frac{5\pi}{256} + \frac{16}{315}\right) \\ & - a_5\left(\frac{7\pi}{512} + \frac{16}{315}\right) - a_6\left(\frac{7\pi}{512} + \frac{128}{3465}\right) \\ & - a_7\left(\frac{21\pi}{2048} + \frac{128}{3465}\right) - a_8\left(\frac{21\pi}{2048} + \frac{256}{9009}\right) \\ & - a_9\left(\frac{33\pi}{4096} + \frac{256}{9009}\right) - a_{10}\left(\frac{33\pi}{4096} + \frac{1024}{45045}\right) \\ & - a_{11}\left(\frac{429\pi}{65536} + \frac{1024}{45045}\right). \end{aligned} \tag{59}$$

Note that S_2 is a nilpotent saddle of order one and $\bar{h}_4 = -1/8$. By (23), we have

$$\begin{aligned} c_1(\delta) &= c_1(S_2,\delta) = 2^{9/4}\left|\widetilde{A}_0\right| a_0, \\ c_3(\delta) &= c_2(S_2,\delta) = a_1 + O_1(a_0), \\ c_6(\delta) &= c_4(S_2,\delta) = 2^{3/4}\left|\widetilde{A}_2\right|(-21a_0 + 12a_1 - 8a_2), \\ c_7(\delta) &= c_5(S_2,\delta) \\ &= 2^{1/4}\frac{\left|\widetilde{A}_0\right|}{7}\left(\frac{2035}{4}a_0 - 310a_1 + 180a_2 - 80a_3 + 32a_4\right), \\ c_9(\delta) &= c_6(S_2,\delta) \\ &= \frac{333}{4}a_0 - \frac{207}{4}a_1 + 29a_2 - 15a_3 + 6a_4 - 2a_5. \end{aligned} \tag{60}$$

Making the transformation $x = u - 1$, $y = v$, system (54) becomes

$$\dot{u} = v, \qquad \dot{v} = -u^2(u-1)^3\left(u^2 - \frac{9}{4}u + \frac{3}{4}\right) - \epsilon\tilde{f}(u,\delta)\, v. \tag{61}$$

Then we have

$$\begin{aligned} H(u,v) &= \frac{v^2}{2} + \frac{1}{8}\left(u^8 - 6u^7 + 14u^6 - 16u^5 + 9u^4 - 2u^3\right), \\ \tilde{h}_3 &= -\frac{1}{4}, \\ \tilde{f}(u,\delta) &= \sum_{j=0}^{11} a_j (u-1)^j = \sum_{j=0}^{11}(-1)^j a_j \\ & + \sum_{j=1}^{11}(-1)^{(j-1)} j a_j u + \sum_{j=2}^{11}(-1)^{(j-2)} C_j^2 a_j u^2 \\ & + \sum_{j=3}^{11}(-1)^{(j-3)} C_j^3 a_j u^3 + \sum_{j=4}^{11}(-1)^{(j-4)} C_j^4 a_j u^4 \end{aligned}$$

$$+\sum_{j=5}^{11}(-1)^{(j-5)}C_j^5a_ju^5+\sum_{j=6}^{11}(-1)^{(j-6)}C_j^6a_ju^6$$
$$+\sum_{j=7}^{11}(-1)^{(j-7)}C_j^7a_ju^7+\sum_{j=8}^{11}(-1)^{(j-8)}C_j^8a_ju^8$$
$$+\sum_{j=9}^{11}(-1)^{(j-9)}C_j^9a_ju^9+(a_{10}-11a_{11})u^{10}+a_{11}u^{11}. \quad (62)$$

By (19), we have

$$c_2(\delta)=c_1(S_1,\delta)=2^{13/6}\sum_{j=0}^{11}(-1)^ja_j,$$
$$c_5(\delta)=c_3(S_1,\delta)=2^{17/6}\left[\sum_{j=1}^{11}(-1)^j(j-3)a_j-3a_0\right],$$
$$c_8(\delta)=c_4(S_1,\delta)=2^{13/6}\left(-\frac{1316}{3}a_0+272a_1-\frac{460}{3}a_2+\frac{224}{3}a_3-28a_4+\frac{16}{3}a_5+\frac{4}{3}a_6-\frac{4}{3}a_8-\frac{16}{3}a_9+28a_{10}-\frac{224}{3}a_{11}\right). \quad (63)$$

Note that

$$c_1(\delta)=c_1(S_2,\delta)=2^{9/4}\left|\widetilde{A}_0\right|a_0,$$
$$c_2(\delta)=c_1(S_1,\delta)=2^{13/6}\sum_{j=0}^{11}(-1)^ja_j, \quad (64)$$
$$c_3(\delta)=c_2(S_2,\delta)=a_1+O_1(a_0).$$

We have $c_1(\delta)=c_2(\delta)=c_3(\delta)=0$ if and only if $a_0=a_1=0$, and $a_{11}=\sum_{i=2}^{10}(-1)^ia_i$. It implies further that

$$c_4(\delta)=\oint_{L^*}(p_x+q_y)\,dt=-\oint_{L^*}f(x,\delta)\,dt$$
$$=-\oint_{L^*}\frac{f(x,\delta)}{y}dx=-2\int_{-1}^{0}\frac{f(x,\delta)}{y}dx$$
$$=-2\int_{-1}^{0}\frac{1}{y}\left(\sum_{i=0}^{11}a_ix^i\right)dx$$
$$=-4\int_{-1}^{0}\frac{1}{x^2(x+1)\sqrt{1-x^2}}\times\left(\sum_{i=2}^{10}a_ix^i+\sum_{i=2}^{10}(-1)^ia_ix^{11}\right)dx \quad (65)$$
$$=-4\sum_{i=2}^{10}a_i\int_{-1}^{0}f_i(x)\,dx,$$

where

$$f_i(x)=\frac{x^{i-2}\left[1+(-1)^ix^{(11-i)}\right]}{(x+1)\sqrt{1-x^2}},\quad i=2,3,\ldots,10. \quad (66)$$

Similarly,

$$\overline{c}_3(\delta)=\oint_{L_3}(p_x+q_y)\,dt=-\oint_{L_3}f(x,\delta)\,dt$$
$$=-\oint_{L_3}\frac{f(x,\delta)}{y}dx=-2\int_{0}^{1}\frac{f(x,\delta)}{y}dx \quad (67)$$
$$=-4\sum_{i=2}^{10}a_i\int_{0}^{1}f_i(x)\,dx.$$

Therefore,

$$c_4(\delta)=-4a_2\left(\frac{315\pi}{256}+\frac{93}{35}\right)+4a_3\left(\frac{187\pi}{256}+\frac{93}{35}\right)-4a_4\left(\frac{187\pi}{256}+\frac{58}{35}\right)+4a_5\left(\frac{123\pi}{256}+\frac{58}{35}\right)-4a_6\left(\frac{123\pi}{256}+\frac{104}{105}\right)+4a_7\left(\frac{75\pi}{256}+\frac{104}{105}\right)-4a_8\left(\frac{75\pi}{256}+\frac{16}{35}\right)+4a_9\left(\frac{35\pi}{256}+\frac{16}{35}\right)-4a_{10}\left(\frac{35\pi}{256}\right),$$
$$\overline{c}_3(\delta)=-4a_2\left(\frac{315\pi}{256}-\frac{93}{35}\right)+4a_3\left(\frac{187\pi}{256}-\frac{93}{35}\right)-4a_4\left(\frac{187\pi}{256}-\frac{58}{35}\right)+4a_5\left(\frac{123\pi}{256}-\frac{58}{35}\right)-4a_6\left(\frac{123\pi}{256}-\frac{104}{105}\right)+4a_7\left(\frac{75\pi}{256}-\frac{104}{105}\right)-4a_8\left(\frac{75\pi}{256}-\frac{16}{35}\right)+4a_9\left(\frac{35\pi}{256}-\frac{16}{35}\right)-4a_{10}\left(\frac{35\pi}{256}\right). \quad (68)$$

Let $c_4^*(\delta)=c_4(\delta)|_{c_1=c_2=c_3=0}, \overline{c}_3^*(\delta)=\overline{c}_3(\delta)_{c_1=c_3=0}$, $c_3^*(\delta)=c_3(\delta)|_{c_1(\delta)=0}$. Furthermore, one sees that equations $c_0(\delta)=\overline{c}_0(\delta)=c_1(\delta)=c_2(\delta)=c_3^*(\delta)=c_4^*(\delta)=\overline{c}_3^*(\delta)=c_5(\delta)=c_6(\delta)=c_7(\delta)=0$ have the solution $a_0=a_1=a_2=0, a_3=(13/74)a_9-(1/2)a_7, a_4=(65/148)a_9-(5/4)a_7, a_5=-(16/37)a_9, a_6=(7/4)a_7-(255/148)a_9, a_8=(96/37)a_9, a_{10}=-(53/37)a_9, a_{11}=-(32/37)a_9$, which gives $c_8(\delta)=(55/37)2^{31/6}a_9$, $c_9(\delta)=(32/37)a_9$. And further, $c_8(\delta)c_9(\delta)>0$ if $a_9\neq 0$. Thus, fix $a_9\neq0$ and take $\delta_0=(0,0,0,(13/74)a_9-(1/2)a_7,(65/148)a_9-(5/4)a_7,-(16/37)a_9,(7/4)a_7-(255/148)a_9,a_7,(96/37)a_9,a_9,-(53/37)a_9,-(32/37)a_9)$.

Then we have

$$\operatorname{rank}\frac{\partial(c_0,\overline{c}_0,c_1,c_2,c_3^*,c_4^*,\overline{c}_3^*,c_5,c_6,c_7)}{\partial(a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11})}=10. \quad (69)$$

Hence by Theorem 4(3), we know that there are 18 limit cycles near L_0 for some δ near δ_0. This ends the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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