Research Article

On Eventually Positive Solutions of Quasilinear Second-Order Neutral Differential Equations

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We study the second-order neutral delay differential equation $[r(t)\Phi_\gamma(z'(t))]' + q(t)\Phi_\beta(x(\sigma(t))) = 0$, where $\Phi_\alpha(t) = |t|^{\alpha-1}t$, $\alpha \geq 1$ and $z(t) = x(t) + p(t)x(\tau(t))$. Based on the conversion into a certain first-order delay differential equation we provide sufficient conditions for nonexistence of eventually positive solutions of two different types. We cover both cases of convergent and divergent integral $\int^\infty r^{-1/\gamma}(t)\mathrm{d}t$. A suitable combination of our results yields new oscillation criteria for this equation. Examples are shown to exhibit that our results improve related results published recently by several authors. The results are new even in the linear case.

1. Introduction

In the paper we study the equation

$$\left[r(t)\Phi_\gamma\left(z'(t)\right)\right]' + q(t)\Phi_\beta\left(x\left(\sigma(t)\right)\right) = 0,$$
$$z(t) = x(t) + p(t)x(\tau(t)),$$

(1)

where $\Phi_\alpha(t) = |t|^{\alpha-1}t$, $\alpha \geq 1$, is the power type nonlinearity. The coefficients r and p are subject to the usual conditions $r \in C^1([t_0, \infty), \mathbb{R}^+)$, $p \in C^1([t_0, \infty), \mathbb{R}_0^+)$ and the coefficient q is positive, $q \in C([t_0, \infty), \mathbb{R}^+)$.

We assume that $\lim_{t \to \infty} \tau(t) = \infty = \lim_{t \to \infty} \sigma(t)$,

$$\sigma(\tau(t)) = \tau(\sigma(t)),$$

(2)

and there exist numbers $p_0 \geq 0$ and $\tau_0 > 0$ such that $p(t) \leq p_0$ and $\tau'(t) \geq \tau_0$.

Under the solution of (1) we understand any differentiable function $x(t)$ which does not identically equal zero eventually, such that $r(t)\Phi_\gamma(z'(t))$ is differentiable and (1) holds for large t.

Following the widely accepted terminology, the solution of (1) is said to be oscillatory if it has infinitely many zeros tending to infinity. Equation (1) is said to be oscillatory if all its solutions are oscillatory. In the opposite case, that is, if there

exists an eventually positive solution of (1), (1) is said to be nonoscillatory.

In the paper we study nonoscillatory solutions of (1). Since $x(t)$ is a solution of (1) if and only if $-x(t)$ is a solution of (1), we can focus our attention on positive solutions.

The paper is organized as follows. In the remaining part of the current section we summarize selected important facts related to (1) and trends in the oscillation theory of this equation. In Section 2 we summarize tools like inequalities and oscillation criteria used in the proofs of main results. The main results are presented in the next three sections. Results on eventually positive solutions are separated into Sections 3 and 4 according to different asymptotic behavior: $z'(t) > 0$ in Section 3 and $z'(t) < 0$ in Section 4. In both cases we provide an efficient condition which ensures that solutions of this type do not exist. Note that under some additional conditions (namely, divergence of integral (3) below) the results from Section 3 immediately yield also oscillation criteria. If (3) fails, we can formulate oscillation criteria using a suitable combination of results from Sections 3 and 4, as shown in Section 5. The results of the paper improve several recently published results even in the linear case. We discuss these improvements in detail in remarks and examples accompanying the main theorems.

Neutral differential equation (1) as well as other related equations have been studied frequently in the literature. There are two main methods in the oscillation theory of (1). One of them is based on a modification of the classical Riccati substitution which is known to be a powerful tool in theory of second-order linear differential equations. Following this method, neutral equation (1) is in some sense considered as a perturbation of some second-order ordinary differential equation. An alternative approach, used for example, in a series of papers by Baculíková et al. [1–4] and Li [5] is based on the fact that it is possible to derive neutral first-order differential inequality for quasiderivative from (1) and the resulting inequality can be studied in the scope of theory elaborated for first-order delay differential inequalities. In this paper we use the later approach. The resulting theorems are sometimes referred to as comparison theorems for neutral differential equations.

Two main approaches are used to put the shift $\tau(t)$ in the differential term under the control. If $p(t) < 1$, then (1) can be "majorized" (in the sense of the classical Sturm comparison theory, which however has no extension to delay equations) by a delay equation of the form (1) with $p(t) = 0$. Oscillation criteria for second-order delay differential equations can be then used to conclude results for neutral equation (1) (see, e.g., [6–8]). An alternative approach deals with a suitable combination of (1) and the same equation with independent variable shifted from t to $\tau(t)$. This approach, which is used also in our paper, does not require $p(t) < 1$ but yields other restrictions, such as commutativity of the composition of delays (2).

Neglecting which method is used to study the oscillation of (1), it turns out that it is necessary to distinguish two cases: either

$$\int^{\infty} \frac{1}{r^{1/\gamma}(t)} dt = \infty \qquad (3)$$

or

$$\int^{\infty} \frac{1}{r^{1/\gamma}(t)} dt < \infty. \qquad (4)$$

The absolute majority of oscillation results in the literature concerns case (3), since in this case the positive solutions of (1) exhibit simpler behavior than in case (4); see Lemma 5 below. Case (4) has been studied, for example, in [9–16]. Note that for this case it is typical that the oscillation criterion consists of two relatively independent conditions. One of them is used to eliminate positive solutions with $z'(t) < 0$; the other one to eliminate positive solutions with $z'(t) > 0$. There are also results which treat both cases $z'(t) > 0$ and $z'(t) < 0$ in one unified approach. However, following this approach a typical conclusion is weaker: the equation is proved to be almost oscillatory (all nonoscillatory solutions, if exist any, tend to zero). Note also that the paper [16] does not satisfy these rules (makes use of unified approach to both cases but concludes oscillation), but there are several inaccuracies in this paper; see [10, 17] for corrected version of [16].

In this paper we essentially use the method from [1, 2] with a modification for case (4) presented in [12]. However, to keep the influence of each condition as transparent as possible we used different organization of the paper, as we explained above. The main improvement with respect to these papers is that we replace inequalities and estimates used in these papers by suitable parametrized versions depending on parameters l and φ (see below). This yields criteria with some degree of freedom and optimization with respect to the parameters which yields sharper results, as we carefully explain on examples of equations with proportional delay. A similar method where we use parameters l and φ to refine the widely used inequalities has been used in the recent paper [18].

Finally, note that [12] in fact deals with linear equations and the extension to nonlinear equations is suggested in Remark 11 at the end of the paper [12]. However here we use an advanced technique rather than the method suggested in [12].

2. Preliminary Results

In the paper we derive results related to the existence or nonexistence of certain equations and inequalities in terms of several parameters. The following two lemmas allow to find the values of the parameters, which yield sharpest results.

The function h introduced in the following lemma plays a role in a formulation of oscillation criteria in the case $\beta \geq 1$.

Lemma 1. *Let $\beta \geq 1$. The function*

$$h(x, y) = x^{\beta-1} + y \left(\frac{x}{x-1} \right)^{\beta-1} \qquad (5)$$

satisfies

$$h(x, y) \geq h\left(1 + y^{1/\beta}, y\right) = \left(1 + y^{1/\beta}\right)^{\beta}, \qquad (6)$$

for every $x > 1$ and $y > 0$.

Proof. It follows from the fact that

$$\frac{\partial}{\partial x} h(x, y) = (\beta - 1) x^{\beta-2} \left[1 - \frac{y}{(x-1)^{\beta}} \right] \qquad (7)$$

and h as the function of x on $(1, \infty)$ attains its minimal value at the point $x = 1 + y^{1/\beta}$ and

$$h\left(1 + y^{1/\beta}, y\right) = \left(1 + y^{1/\beta}\right)^{\beta-1} + y^{1/\beta}\left(1 + y^{1/\beta}\right)^{\beta-1}$$

$$= \left(1 + y^{1/\beta}\right)^{\beta}. \qquad (8)$$

□

The following functions appear in the examples and allow to find the optimal values of the parameters which yield the sharpest result.

Lemma 2. *Let c_1, c_2 be positive numbers.*

(i) *The function $f(x) = x^{c_1} \ln(c_2/x)$ is increasing on $(0, c_2 e^{-1/c_1})$, decreasing on $(c_2 e^{-1/c_1}, \infty)$, and satisfies*

$$f(x) \le \frac{1}{c_1 e} c_2^{c_1} \qquad (9)$$

on $(0, \infty)$ with the equality, if and only if $x = c_2 e^{-1/c_1}$.

(ii) *The function $g(x) = x^{-c_1} \ln(x/c_2)$ is increasing on $(0, c_2 e^{1/c_1})$, decreasing on $(c_2 e^{1/c_1}, \infty)$, and satisfies*

$$g(x) \le \frac{1}{c_1 e} c_2^{-c_1} \qquad (10)$$

on $(0, \infty)$ with the equality, if and only if $x = c_2 e^{1/c_1}$.

Proof. By a direct computation

$$f'(x) = x^{c_1-1}\left(c_1 \ln \frac{c_2}{x} - 1\right). \qquad (11)$$

Hence $f(x)$ has a local maximum at the point $x_0 = c_2 e^{-1/c_1}$ and the value of this local maximum is $c_2^{c_1}/c_1 e$. Similarly,

$$g'(x) = x^{-1-c_1}\left(1 - c_1 \ln \frac{x}{c_2}\right), \qquad (12)$$

and hence $g(x)$ has a local maximum at the point $x_1 = c_2 e^{1/c_1}$ and the maximal value is $(1/c_1 e)c_2^{-c_1}$. □

Lemma 3. *Let $A \ge 0$, $B \ge 0$, $\beta \ge 1$, $l > 1$, $l^* = l/(l-1)$. Then*

$$(A + B)^\beta \le l^{\beta-1} A^\beta + (l^*)^{\beta-1} B^\beta. \qquad (13)$$

Proof. From the fact that the function x^β is a convex function for $\beta \ge 1$ we have

$$\left(\frac{1}{l} a + \frac{1}{l^*} b\right)^\beta \le \frac{1}{l} a^\beta + \frac{1}{l^*} b^\beta \qquad (14)$$

for nonnegative a and b. From here we obtain the desired inequality for $A = a/l$ and $B = b/l^*$. □

Lemma 4. *Let $\beta \ge 1$. The inequality*

$$l^{\beta-1} x^\beta(\sigma(t)) + (l^*)^{\beta-1} p^\beta(\sigma(t)) x^\beta(\sigma(\tau(t))) \ge z^\beta(\sigma(t)) \qquad (15)$$

holds for positive mutually conjugate numbers l, l^* and every t which satisfies $x(\sigma(t)) \ge 0$ and $x(\sigma(\tau(t))) \ge 0$.

Proof. It follows from Lemma 3, from the definition of $z(t)$ and from condition (2). □

The following lemma is well known in theory of neutral differential equations. It states (among others) that if x is an eventually positive solution, then z' is eventually of one sign and the negative sign of z' is excluded if (3) holds.

Lemma 5. *Let $x(t)$ be an eventually positive solution of (1). The corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ satisfies*

$$z(t) > 0, \qquad z'(t) > 0, \qquad \left(r(t)\Phi_\gamma\left(z'(t)\right)\right)' < 0 \qquad (16)$$

eventually if (3) holds and either (16) or

$$z(t) > 0, \qquad z'(t) < 0, \qquad \left(r(t)\Phi_\gamma\left(z'(t)\right)\right)' < 0 \qquad (17)$$

eventually if (4) holds.

Proof. It follows from [7, Lemma 10] and from the proof of that lemma. □

In the following lemma we summarize effective oscillation criteria for delay and advanced first-order equation which appear in the analysis of (1). Note that (iii) is sharper version of the related condition from [2, Lemma 4].

Lemma 6. *Let $q(t) \ge 0$.*

(i) *If $\sigma(t) < t$ and*

$$\liminf_{t \to \infty} \int_{\sigma(t)}^{t} q(s)\, ds > \frac{1}{e}, \qquad (18)$$

then

$$y'(t) + q(t) y(\sigma(t)) \le 0 \qquad (19)$$

has no eventually positive solution.

(ii) *If $\sigma(t) > t$ and*

$$\liminf_{t \to \infty} \int_{t}^{\sigma(t)} q(s)\, ds > \frac{1}{e}, \qquad (20)$$

then

$$y'(t) - q(t) y(\sigma(t)) \ge 0 \qquad (21)$$

has no eventually positive solution.

(iii) *Let $\sigma(t) < t$, $\alpha \in (0, 1)$. If*

$$\int_{t_0}^{\infty} q(s)\, ds = \infty, \qquad (22)$$

then

$$y'(t) + q(t) y^\alpha(\sigma(t)) \le 0 \qquad (23)$$

has no eventually positive solution.

(iv) *Let $\sigma(t) > t$, $\alpha \in (1, \infty)$. If*

$$\int_{t_0}^{\infty} q(s)\, ds = \infty, \qquad (24)$$

then

$$y'(t) - q(t) y^\alpha(\sigma(t)) \ge 0 \qquad (25)$$

has no eventually positive solution.

Proof. See [9, Lemmas 2.1–2.4] and [19, Lemma 2.2.9]. Note that the original proof of condition (i) is due to [20] and the proofs of conditions (iii) and (iv) for equations are due to [21]. □

3. Positive Solutions with $z'(t) > 0$ Eventually

In this section we give sufficient conditions which exclude the possibility that the equation possesses an eventually positive solution $x(t)$ such that the corresponding function $z(t)$ is eventually increasing. Note that Lemma 5 excludes other types of eventually positive solutions if (3) holds. Hence if (3) holds as well, then the criteria from this section guarantee oscillation of (1).

Denote

$$Q(t; \varphi) = \min \{q(t), \varphi q(\tau(t))\}, \tag{26}$$

$$Q_\eta^*(t; \varphi, t_1) = Q(t; \varphi) \left[\int_{t_1}^{\eta(t)} r^{-1/\gamma}(s) \, ds \right]^\beta. \tag{27}$$

The following theorem allows us to relate positive solutions of (1) with a certain first-order neutral equation. This neutral equation can be further compared with a certain nonneutral differential equation. The form of this nonneutral differential equation depends on the fact whether the deviating argument $\tau(t)$ in the differential term is delay or advanced argument. If $\varphi = 1$, $\eta \equiv \sigma$, and $l = 2$, then Theorem 7 reduces to [2, Theorems 4, 5 and 6].

Theorem 7. Let $\beta \geq 1$, $\varphi > 0$, $l > 1$ and $\eta(t)$ a function which satisfies $\eta(t) \leq \sigma(t)$ and $\lim_{t \to \infty} \eta(t) = \infty$. Suppose that there exists a number $T > t_0$ and a solution $x(t)$ of (1) which satisfy

$$x(t) > 0, \quad z'(t) > 0 \quad \text{for } t \geq T. \tag{28}$$

Let $t_1 > T$ be such that

$$\min \{\eta(\tau(t)), \eta(t)\} > T \tag{29}$$

for every $t \geq t_1$ and let $t_2 \geq t_1$ be such that $\eta(t) \geq t_1$ for $t \geq t_2$. Then the following statements are true.

(i) The inequality

$$\left[l^{\beta-1} w(t) + \frac{p_0^\beta \varphi}{\tau_0} (l^*)^{\beta-1} w(\tau(t)) \right]'$$
$$+ Q_\eta^*(t; \varphi, t_1) w^{\beta/\gamma}(\eta(t)) \leq 0 \tag{30}$$

has a positive decreasing solution on (t_2, ∞).

(ii) If $\tau(t) \geq t$, then

$$y' + Q_\eta^*(t; \varphi, t_1) h^{-\beta/\gamma} \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right) y^{\beta/\gamma}(\eta(t)) \leq 0 \tag{31}$$

has a positive solution on (t_2, ∞).

(iii) If $\tau(t) \leq t$, then

$$y' + Q_\eta^*(t; \varphi, t_1) h^{-\beta/\gamma} \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right) y^{\beta/\gamma}(\tau^{-1}(\eta(t))) \leq 0 \tag{32}$$

has a positive solution on (t_2, ∞).

Proof. Let $x(t)$ be a solution of (1) which satisfies $x(t) > 0$ and $z'(t) > 0$ for $t \geq T$. Inequalities (15), $p(t) \leq p_0$, and $\eta(t) \leq \sigma(t)$ imply

$$l^{\beta-1} x^\beta(\sigma(t)) + (l^*)^{\beta-1} p_0^\beta x^\beta(\sigma(\tau(t))) \geq z^\beta(\eta(t)) \tag{33}$$

for $t \geq t_1$.

We shift (1) from t to $\tau(t)$ and get

$$0 = \frac{1}{\tau'(t)} \left[r(\tau(t)) \Phi_\gamma \left(z'(\tau(t)) \right) \right]'$$
$$+ q(\tau(t)) \Phi_\beta(x(\sigma(\tau(t))))$$
$$\geq \frac{1}{\tau_0} \left[r(\tau(t)) \Phi_\gamma \left(z'(\tau(t)) \right) \right]' + q(\tau(t)) \Phi_\beta(x(\sigma(\tau(t)))). \tag{34}$$

Substituting $\Phi_\beta(x(\sigma(\tau(t))))$ from this inequality and $\Phi_\beta(x(\sigma(t)))$ from (1) to (33) and using (26) we obtain

$$0 \geq l^{\beta-1} \left[r(t) \Phi_\gamma \left(z'(t) \right) \right]' + \frac{(l^*)^{\beta-1} p_0^\beta \varphi}{\tau_0}$$
$$\times \left[r(\tau(t)) \Phi_\gamma \left(z'(\tau(t)) \right) \right]' + Q(t; \varphi) z^\beta(\eta(t)), \tag{35}$$

for $t \geq t_1$. Denoting $w(t) = r(t) \Phi_\gamma(z'(t))$ and using the obvious fact that w is positive and decreasing on (t_1, ∞) we have

$$z(t) = \int_{t_1}^t w^{1/\gamma}(s) r^{-1/\gamma}(s) \, ds \geq w^{1/\gamma}(t) \int_{t_1}^t r^{-1/\gamma}(s) \, ds \tag{36}$$

for $t \geq t_1$. Thus w is an eventually positive and eventually decreasing solution of (30) and claim (i) is proved.

Denote

$$y(t) = l^{\beta-1} w(t) + \frac{p_0^\beta \varphi}{\tau_0} (l^*)^{\beta-1} w(\tau(t)). \tag{37}$$

Since w is a positive decreasing function, we have $w(t) \geq w(\tau(t))$ for $t \leq \tau(t)$ and $w(t) \leq w(\tau(t))$ for $t \geq \tau(t)$. Hence if $t \leq \tau(t)$ we have

$$y(t) \leq w(t) \left(l^{\beta-1} + \frac{p_0^\beta \varphi}{\tau_0} (l^*)^{\beta-1} \right) = w(t) h \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right), \tag{38}$$

where the function h is defined by (5), and, if $t \geq \tau(t)$, then similarly

$$y(t) \leq w(\tau(t)) h \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right). \tag{39}$$

Hence we have

$$w^{\beta/\gamma}(\eta(t)) \geq h^{-\beta/\gamma} \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right) y^{\beta/\gamma}(\eta(t)), \tag{40}$$

if $t \leq \tau(t)$, and

$$w^{\beta/\gamma}\left(\eta(t)\right) \geq h^{-\beta/\gamma}\left(l, \frac{p_0^\beta \varphi}{\tau_0}\right) y^{\beta/\gamma}\left(\tau^{-1}\left(\eta(t)\right)\right) \qquad (41)$$

if $t \geq \tau(t)$. This and claim (i) prove claims (ii) and (iii) since in each case we have found an eventually positive solution y of the corresponding inequality. □

Remark 8. Note that in the proof of Theorem 7 we constructed the solutions of the inequalities (30), (31), and (32).

In the following corollary we give an efficient condition for nonexistence of the solutions mentioned in the points (ii) and (iii) of Theorem 7. According to Lemma 6 we distinguish the cases $\gamma = \beta$ and $\gamma \neq \beta$.

Corollary 9. *Let $\gamma \geq \beta \geq 1$. Equation (1) has no solution $x(t)$ which satisfies*

$$x(t) > 0, \quad z'(t) > 0 \quad \text{eventually} \qquad (42)$$

if there exists number $\varphi > 0$ and a function $\eta(t)$ satisfying $\eta(t) \leq \sigma(t)$ and $\lim_{t \to \infty} \eta(t) = \infty$ such that one of the following conditions holds.

(i) $\eta(t) < t \leq \tau(t)$ *and for every T there exists $t_1 > T$ such that*

$$\liminf_{t \to \infty} \int_{\eta(t)}^{t} Q_\eta^*\left(s; \varphi, t_1\right) ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \qquad (43)$$

if $\beta = \gamma$, and

$$\int_{t_0}^{\infty} Q_\eta^*\left(t; \varphi, t_1\right) dt = \infty \qquad (44)$$

if $\beta < \gamma$.

(ii) $\eta(t) < \tau(t) \leq t$ *and for every T there exists $t_1 > T$ such that*

$$\liminf_{t \to \infty} \int_{\tau^{-1}(\eta(t))}^{t} Q_\eta^*\left(s; \varphi, t_1\right) ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \qquad (45)$$

if $\beta = \gamma$, and (44) holds if $\beta < \gamma$.

Proof. It follows from Theorem 7 and Lemmas 6 and 1. □

In the following example we compare our results with the results of [1]. Note that in this example (3) holds and hence the results of this section ensure oscillation of the equation.

Example 10 (linear equation). Baculíková and Džurina studied in [1] differential equation

$$\left(\sqrt{t}[x(t) + p_0 x(\alpha t)]'\right)' + \frac{a}{t^{3/2}} x(\mu t) = 0, \qquad (46)$$

where $0 < \mu < 1, \alpha > 0, a > 0$, and obtained that the equation is oscillatory if either

$$\alpha \geq 1, \qquad 2a\sqrt{\mu}\ln\frac{1}{\mu} > \frac{1}{e}\left(\alpha^{3/2} + p_0\sqrt{\alpha}\right) \qquad (47)$$

or

$$0 < \mu < \alpha \leq 1, \qquad 2a\sqrt{\mu}\ln\frac{\alpha}{\mu} > \frac{\alpha + p_0}{\alpha e}. \qquad (48)$$

We have $\gamma = \beta = 1, \tau(t) = \alpha t, \sigma(t) = \mu t, r(t) = \sqrt{t}, q(t) = at^{-3/2}, q(\tau(t)) = a(\alpha t)^{-3/2}$, and

$$\int r^{-1/\gamma}(t)\, dt = \int t^{-1/2}\, dt = 2\sqrt{t}, \qquad (49)$$

and hence (3) holds. Using Corollary 9 with $\eta(t) = \lambda t, \lambda \leq \mu$ and $\varphi = \alpha^{3/2}$ we have $Q(t) = q(t)$ and consequently,

$$Q_\eta^*\left(t; \varphi, t_1\right) = \frac{a}{t^{3/2}}\left[2\sqrt{\lambda}\sqrt{t} - 2\sqrt{t_1}\right],$$

$$\liminf_{t \to \infty} \int_{\eta(t)}^{t} Q_\eta^*\left(s; \varphi, t_1\right) ds = 2a\sqrt{\lambda}\ln\frac{1}{\lambda}, \qquad (50)$$

$$\liminf_{t \to \infty} \int_{\tau^{-1}(\eta(t))}^{t} Q_\eta^*\left(s; \varphi, t_1\right) ds = 2a\sqrt{\lambda}\ln\frac{\alpha}{\lambda}.$$

Hence, (46) is oscillatory if either

$$\alpha \geq 1, \qquad 2a\sqrt{\lambda}\ln\frac{1}{\lambda} > \frac{1}{e}\left(1 + \sqrt{\alpha}p_0\right) \qquad (51)$$

or

$$0 < \lambda < \alpha \leq 1, \qquad 2a\sqrt{\lambda}\ln\frac{\alpha}{\lambda} > \frac{1}{e}\left(1 + \sqrt{\alpha}p_0\right). \qquad (52)$$

Even in case $\mu = \lambda$, if we view the right hand sides as linear functions of p_0, we easily see from the slope and y-intercept of these lines that these conditions are sharper than those from [1]. Note that the fact that the equation is linear causes that the parameter l does not have any influence on the sharpenes of these conditions, since the function h from Lemma 1 is a constant function with respect to the first variable for $\beta = 1$. Hence, the improvement with respect to the results from [1] is purely in the presence of the parameter φ in the definition of the function Q. Figure 1 reveals also different asymptotic behavior of the right hand sides of (51) and (52) with respect to the corresponding constants from (47) and (48). Based on this fact we see that the improvement is significant especially if α is sufficiently far from 1.

When looking for optimal conditions for oscillation of (46) it is easy to ensure that the case $\lambda = \mu$ is not optimal for every μ. Really, if we replace inequality signs in (51) and (52) by equality signs and view the resulting equality as a formula which defines a as a function of λ, we get U-shaped function with one local minimum (see Figure 2). Since λ can be any positive number smaller than μ, it turns out that the optimal choice for λ is $\lambda = \mu$ on the decreasing part and $\lambda = \lambda_{\max}$ on the increasing part, where $\lambda_{\max}$ is the point where
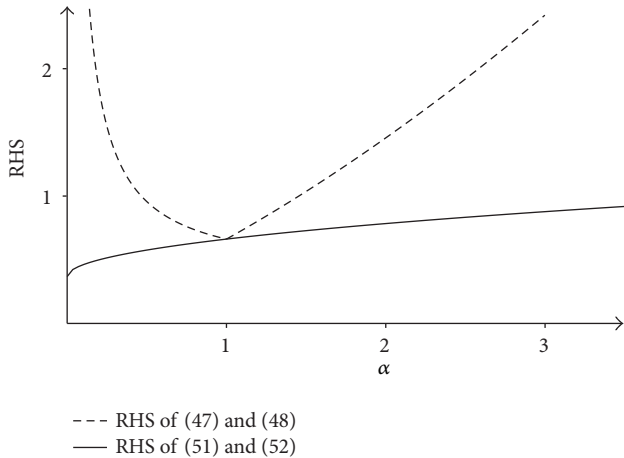
FIGURE 1: The graph of right hand sides (RHS) of (47) and (48) for $p_0 = 0.8$ compared to (51) and (52) as functions of α.

--- RHS of (47) and (48)
—— RHS of (51) and (52)

FIGURE 2: A comparison of the lower bounds for the coefficient a which guarantee oscillation of (46) for different values of μ. Parameters used for the graphs are $\alpha = 0.6$ and $p_0 = 0.8$.

the function $\sqrt{\lambda}\ln(\alpha/\lambda)$ attains its global maximum. Using Lemma 2, we have $\lambda_{\max} = \alpha/e^2$ and $\sqrt{\lambda}\ln(\alpha/\lambda) \leq (2\sqrt{\alpha}/e)$. Hence we get that the equation is oscillatory if

$$a > a_{\text{crit.}}$$

$$:= \begin{cases} \dfrac{1}{4}\left(1 + \sqrt{\alpha}p_0\right) & \text{for } \alpha \geq 1,\ \mu \geq \dfrac{1}{e^2}, \\[2ex] \dfrac{1}{4\sqrt{\alpha}}\left(1 + \sqrt{\alpha}p_0\right) & \text{for } \alpha \leq 1,\ \mu \geq \dfrac{\alpha}{e^2}, \\[2ex] \dfrac{1}{2e\sqrt{\mu}\ln(1/\mu)}\left(1 + \sqrt{\alpha}p_0\right) & \text{for } \alpha \geq 1,\ \mu \leq \dfrac{1}{e^2}, \\[2ex] \dfrac{1}{2e\sqrt{\mu}\ln(\alpha/\mu)}\left(1 + \sqrt{\alpha}p_0\right) & \text{for } \alpha \leq 1,\ \mu \leq \dfrac{\alpha}{e^2}. \end{cases}$$

$$(53)$$

From the graphical point of view these conditions arise from (51) and (52) by isolating a and replacing the increasing part of the resulting curve by a constant function; see Figure 2 for more details and for comparison with the oscillation constant resulting from (48).

Example 11 (half-linear equation). Consider the differential equation

$$\left(t^2\left[\left(x(t) + (3 + \sin t)\,x(\alpha t)\right)'\right]^3\right)' + \dfrac{b}{t^2}x^3(\mu t) = 0,$$

$$(54)$$

$$\mu < 1, \quad a > 0.$$

We have $\beta = \gamma = 3$, $\tau(t) = \alpha t$, $\sigma(t) = \mu t$, $p_0 = 4$, $\tau_0 = \alpha$, $r(t) = t^2$, $q(t) = b/t^2$, $q(\tau(t)) = b/(\alpha t)^2$ and,

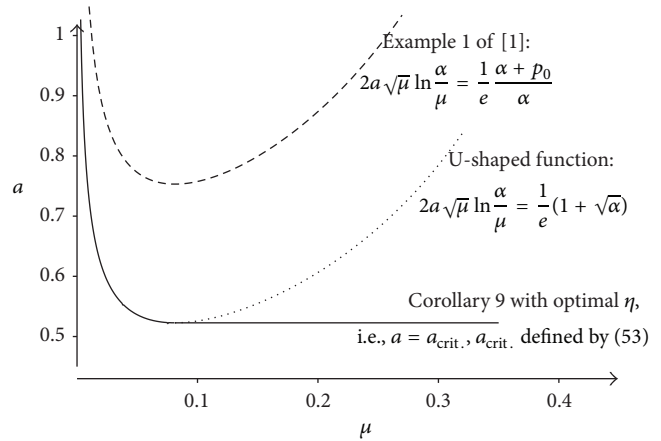$$\int r^{-1/\gamma}(t)\,dt = \int t^{-2/3}dt = 3t^{1/3},$$

$$(55)$$

and hence condition (3) holds. Using results of Baculíková and Džurina [2, Corollaries 3 and 4] we obtain that (54) is oscillatory if

$$\alpha \geq 1, \qquad 27eb\mu\ln\dfrac{1}{\mu} > 4\alpha^2\left(1 + \dfrac{4^3}{\alpha}\right)$$

$$(56)$$

or

$$0 < \mu < \alpha \leq 1, \qquad 27eb\mu\ln\dfrac{\alpha}{\mu} > 4\left(1 + \dfrac{4^3}{\alpha}\right).$$

$$(57)$$

Using Corollary 9 with $\eta(t) = \lambda t$, $\lambda \leq \mu$ and $\varphi = \alpha^2$ we have $Q(t;\varphi) = q(t)$ and consequently,

$$Q_\eta^*(t;\varphi,t_1) = \dfrac{27b}{t^2}\left[\lambda^{1/3}t^{1/3} - t_1^{1/3}\right]^3.$$

$$(58)$$

Hence,

$$\liminf_{t\to\infty}\int_{\eta(t)}^t Q_\eta^*(s;\varphi,t_1)\,ds = 27b\lambda\ln\dfrac{1}{\lambda},$$

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\eta(t))}^t Q_\eta^*(s;\varphi,t_1)\,ds = 27b\lambda\ln\dfrac{\alpha}{\lambda}$$

$$(59)$$

and (54) is oscillatory if

$$\alpha \geq 1, \quad 27eb\lambda\ln\dfrac{1}{\lambda} > \left(1 + 4\alpha^{1/3}\right)^3$$

$$(60)$$

or

$$0 < \lambda < \alpha \leq 1, \quad 27eb\lambda\ln\dfrac{\alpha}{\lambda} > \left(1 + 4\alpha^{1/3}\right)^3.$$

$$(61)$$

If $\lambda = \mu$, the comparison of our lower bound $(1 + 4\alpha^{1/3})^3$ and the upper bound $4(1 + 4^3/\alpha)$ if $\alpha < 1$ and $4\alpha^2(1 + 4^3/\alpha)$ if $\alpha > 1$ is on Figure 3. Note that in contrast to the linear case, both curves do not intersect at $\alpha = 1$.
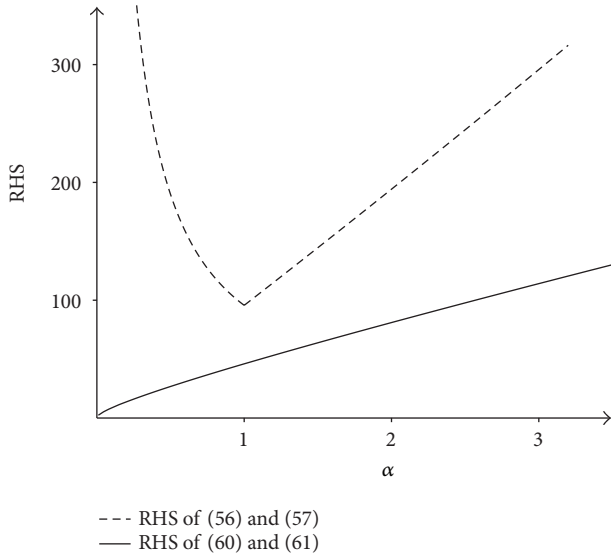
FIGURE 3: The graph of right hand sides (RHS) of (56) and (57) for $p_0 = 4$ compared to (60) and (61) as functions of α.

FIGURE 4: A comparison of the lower bounds for the coefficient b which guarantee oscillation of (54) for different values of μ. Parameters are $\alpha = 0.4$ and $p_0 = 4$.

The function $f(\lambda) = \lambda \ln(\alpha/\lambda)$ is positive on the interval $(0, \alpha)$ and it follows from Lemma 1 that the maximal value of $f(\lambda)$ is α/e at the point $\lambda = \alpha/e$. Hence, (54) is oscillatory if

$$b > b_{\text{crit}}.$$

$$:= \begin{cases} \dfrac{1}{27}\left(1 + 4\alpha^{1/3}\right)^3 & \text{for } \alpha \geq 1, \ \mu \geq \dfrac{1}{e}, \\[2mm] \dfrac{1}{27\alpha}\left(1 + 4\alpha^{1/3}\right)^3 & \text{for } \alpha \leq 1, \ \mu \geq \dfrac{\alpha}{e}, \\[2mm] \dfrac{1}{27e\mu \ln(1/\mu)}\left(1 + 4\alpha^{1/3}\right)^3 & \text{for } \alpha \geq 1, \ \mu \leq \dfrac{1}{e}, \\[2mm] \dfrac{1}{27e\mu \ln(\alpha/\mu)}\left(1 + 4\alpha^{1/3}\right)^3 & \text{for } \alpha \leq 1, \ \mu \leq \dfrac{\alpha}{e}. \end{cases} \quad (62)$$

Figure 4 shows how the critical constant which ensures oscillation of (54) is improved with respect to the results of [2] (dashed line) for various values of the delay μ. For readers convenience we graphed also a dotted curve which is only partial improvement of [2]: the values of l and η are chosen as in [2] and the value of φ (which plays role in Q) is choosen to equal to $q(t)/q(\tau(t))$.

The following theorem and corollary are variants of Theorem 7 and Corollary 9 for sublinear case $\beta \leq 1$.

Theorem 12. *Let $\beta \leq 1$, $\varphi > 0$, and $\eta(t)$ a function which satisfies $\eta(t) \leq \sigma(t)$ and $\lim_{t \to \infty}\eta(t) = \infty$. Suppose that there exists a number $T > t_0$ and a solution $x(t)$ of (1) which satisfy*

$$x(t) > 0, \quad z'(t) > 0 \quad \text{for } t \geq T. \quad (63)$$

Let $t_1 > T$ be such that

$$\min\{\eta(\tau(t)), \eta(t)\} > T \quad (64)$$

for every $t \geq t_1$ and let $t_2 \geq t_1$ be such that $\eta(t) \geq t_1$ for $t \geq t_2$. Then the following statements are true.
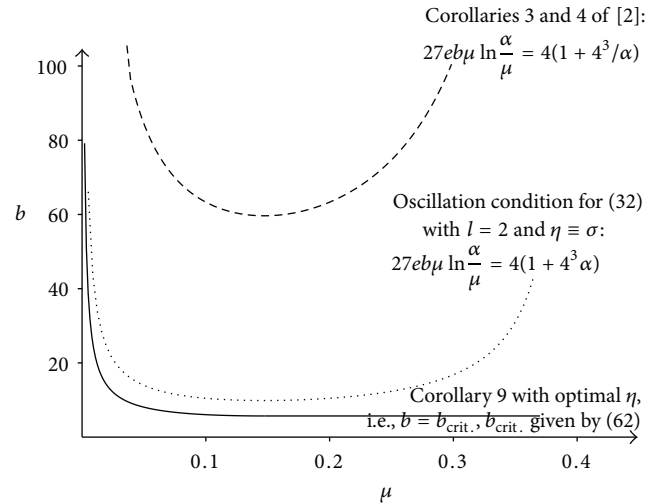
(i) *The inequality*

$$\left[w(t) + \frac{p_0^\beta \varphi}{\tau_0} w(\tau(t))\right]' + Q_\eta^*(t; \varphi, t_1) w^{\beta/\gamma}(\eta(t)) \leq 0 \quad (65)$$

has a positive decreasing solution on (t_2, ∞).

(ii) *If $\tau(t) \geq t$, then*

$$y' + Q_\eta^*(t; \varphi, t_1)\left(\frac{\tau_0}{\tau_0 + p_0^\beta \varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}(\eta(t)) \leq 0 \quad (66)$$

has a positive solution on (t_2, ∞).

(iii) *If $\tau(t) \leq t$, then*

$$y' + Q_\eta^*(t; \varphi, t_1)\left(\frac{\tau_0}{\tau_0 + p_0^\beta \varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}(\tau^{-1}(\eta(t))) \leq 0 \quad (67)$$

has a positive solution on (t_2, ∞).

Proof. The proof is the same as the proof of Theorem 7 where we formally put $l = l^* = 1$ and use

$$(A + B)^\beta \leq A^\beta + B^\beta \quad (68)$$

instead of Lemma 3. □

Corollary 13. *Let $\beta \leq 1$, $\gamma \geq \beta$. Equation (1) has no solution $x(t)$ which satisfies*

$$x(t) > 0, \quad z'(t) > 0 \quad \text{eventually} \quad (69)$$

if there exists $\varphi > 0$ and a function $\eta(t)$ satisfying $\eta(t) \leq \sigma(t)$ and $\lim_{t \to \infty}\eta(t) = \infty$ such that one of the following conditions holds.

(i) $\eta(t) < t \leq \tau(t)$ and for every T there exists $t_1 > T$ such that

$$\liminf_{t \to \infty} \int_{\eta(t)}^{t} Q_\eta^* (s; \varphi, t_1) \, ds > \frac{1}{e} \left(\frac{\tau_0 + p_0^\beta \varphi}{\tau_0} \right)^{\beta/\gamma} \quad (70)$$

if $\beta = \gamma$, and (44) holds if $\beta < \gamma$.

(ii) $\eta(t) < \tau(t) \leq t$ and for every T there exists $t_1 > T$ such that

$$\liminf_{t \to \infty} \int_{\tau^{-1}(\eta(t))}^{t} Q_\eta^* (s; \varphi, t_1) \, ds > \frac{1}{e} \left(\frac{\tau_0 + p_0^\beta \varphi}{\tau_0} \right)^{\beta/\gamma} \quad (71)$$

if $\beta = \gamma$, and (44) holds if $\beta < \gamma$.

Proof. It follows from Theorem 12 and Lemmas 6 and 1. □

Remark 14. If $\varphi = 1$ and $\eta(t) = \sigma(t)$, then Theorem 12 reduces to [2, Theorems 1, 2, and 3] and Corollary 13 reduces to [2, Corollaries 1 and 2].

4. Positive Solutions with $z'(t) < 0$ Eventually

In this section we modify the methods from previous section for positive solutions $x(t)$ which satisfy $z'(t) < 0$ eventually. Troughout this section we will suppose that (4) holds, since if it fails, then evetually positive solutions with $z'(t) < 0$ eventually do not exist.

The function Q_ζ^* defined by the relation

$$Q_\zeta^* (t; \varphi) = Q(t; \varphi) \left[\int_{\zeta(t)}^{\infty} r^{-1/\gamma}(s) \, ds \right]^\beta \quad (72)$$

and the following Theorem 15 are the corresponding modifications of the function Q_η^* and Theorem 7.

Theorem 15. *Let $\beta \geq 1$, $\varphi > 0$, $l > 1$, and let $\zeta(t)$ be a function which satisfies $\zeta(t) \geq \sigma(t)$. Suppose that there exists a number $T > t_0$ and a solution $x(t)$ of (1) which satisfy*

$$x(t) > 0, \quad z'(t) < 0 \quad \text{for } t \geq T. \quad (73)$$

Let $t_1 > T$ be such that

$$\min \{ \sigma(t), \sigma(\tau(t)) \} > T \quad (74)$$

for every $t \geq t_1$. Then the following statements are true.

(i) *The inequality*

$$\left[l^{\beta-1} u(t) + \frac{p_0^\beta \varphi}{\tau_0} (l^*)^{\beta-1} u(\tau(t)) \right]' \quad (75)$$
$$- Q_\zeta^* (t; \varphi) u^{\beta/\gamma} (\zeta(t)) \geq 0$$

has a positive increasing solution on (t_1, ∞).

(ii) *If $\tau(t) \leq t$, then*

$$y' - Q_\zeta^* (t; \varphi) h^{-\beta/\gamma} \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right) y^{\beta/\gamma} (\zeta(t)) \geq 0 \quad (76)$$

has a positive solution on (t_1, ∞).

(iii) *If $\tau(t) \geq t$, then*

$$y' - Q_\zeta^* (t; \varphi) h^{-\beta/\gamma} \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right) y^{\beta/\gamma} (\tau^{-1} (\zeta(t))) \geq 0 \quad (77)$$

has a positive solution on (t_1, ∞).

Proof. Let x be a solution of (1) which satisfies $x(t) > 0$ and $z'(t) < 0$ for $t > T$. Inequalities (15), $p(t) \leq p_0$, and $\zeta(t) \geq \sigma(t)$ imply

$$l^{\beta-1} x^\beta (\sigma(t)) + (l^*)^{\beta-1} p_0^\beta x^\beta (\sigma(\tau(t))) \geq z^\beta (\zeta(t)). \quad (78)$$

Combining this inequality with (1) and (1) shifted from t to $\tau(t)$, similarly as in the proof of Theorem 7, we obtain

$$0 \geq l^{\beta-1} \left[r(t) \Phi_\gamma (z'(t)) \right]' + \frac{(l^*)^{\beta-1} p_0^\beta \varphi}{\tau_0} \quad (79)$$
$$\times \left[r(\tau(t)) \Phi_\gamma (z'(\tau(t))) \right]' + Q(t; \varphi) z^\beta (\zeta(t))$$

for $t \geq t_1$. The function w defined by $w(t) = r(t) \Phi_\gamma (z'(t))$ is negative and decreasing. Hence for $s \geq t$ we have

$$r(s) \Phi_\gamma (z'(s)) \leq r(t) \Phi_\gamma (z'(t)),$$
$$z'(s) \leq z'(t) \Phi_\gamma^{-1} \left(\frac{r(t)}{r(s)} \right) \quad (80)$$

and hence

$$z(l) - z(t) \leq \Phi_\gamma^{-1} (r(t)) z'(t) \int_t^l r^{-1/\gamma}(s) \, ds. \quad (81)$$

Since $\lim_{l \to \infty} z(l) \geq 0$ we have

$$-z(t) \leq \Phi_\gamma^{-1} (r(t)) z'(t) \int_t^\infty r^{-1/\gamma}(s) \, ds, \quad (82)$$

which implies

$$z^\beta (\zeta(t)) \geq (-w(\zeta(t)))^{\beta/\gamma} \left[\int_{\zeta(t)}^\infty r^{-1/\gamma}(s) \, ds \right]^\beta. \quad (83)$$

Combining this inequality with (79) and multiplying by -1 we find that $u(t) = -w(t)$ is a positive and increasing solution of (75). Claim (i) is proved. Denote

$$y(t) = l^{\beta-1} w(t) + \frac{p_0^\beta \varphi}{\tau_0} (l^*)^{\beta-1} w(\tau(t)). \quad (84)$$

Since $u(t)$ is positive and increasing, we have $u(\tau(t)) \leq u(t)$ for $\tau(t) \leq t$ and $u(\tau(t)) \geq u(t)$ for $\tau(t) \geq t$. Hence, if $\tau(t) \leq t$, we have

$$y(t) \leq u(t) h \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right), \quad (85)$$

which implies

$$u^{\beta/\gamma} (\zeta(t)) \geq h^{-\beta/\gamma} \left(l, \frac{p_0^\beta \varphi}{\tau_0} \right) y^{\beta/\gamma} (\zeta(t)). \quad (86)$$

Analogously, if $\tau(t) \geq t$,

$$y(t) \leq u(\tau(t)) h\left(l, \frac{p_0^\beta \varphi}{\tau_0}\right), \qquad (87)$$

which implies

$$u^{\beta/\gamma}(\zeta(t)) \geq h^{-\beta/\gamma}\left(l, \frac{p_0^\beta \varphi}{\tau_0}\right) y^{\beta/\gamma}\left(\tau^{-1}(\zeta(t))\right). \qquad (88)$$

Claims (ii) and (iii) then follow from (i) and positivity of y. $\qquad \square$

Corollary 16. *Let* $\beta \geq 1$ *and* $\beta \geq \gamma$. *Equation* (1) *has no solution* $x(t)$ *which satisfies*

$$x(t) > 0, \quad z'(t) < 0 \quad \text{eventually} \qquad (89)$$

if there exists $\varphi > 0$ *and a function* $\zeta(t)$ *satisfying* $\zeta(t) \geq \sigma(t)$ *such that one of the following conditions holds.*

(i) $\tau(t) \leq t < \zeta(t)$ *and*

$$\liminf_{t\to\infty} \int_t^{\zeta(t)} Q_\zeta^\star(s;\varphi)\, ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \qquad (90)$$

if $\beta = \gamma$, *and*

$$\int_{t_0}^\infty Q_\zeta^\star(t;\varphi)\, dt = \infty \qquad (91)$$

if $\beta > \gamma$.

(ii) $t \leq \tau(t) < \zeta(t)$ *and*

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(\zeta(t))} Q_\zeta^\star(s;\varphi)\, ds > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{1/\beta} p_0\right)^{\beta^2/\gamma} \qquad (92)$$

if $\beta = \gamma$, *and* (91) *holds if* $\beta > \gamma$.

Proof. It follows from Theorem 15 and Lemmas 6 and 1. $\quad\square$

In a similar way as in Theorem 12 and Corollary 13, if we suppose $\beta \leq 1$ in Theorem 15 and Corollary 16 and use inequality (68) instead of inequality from Lemma 3, we get the following statements.

Theorem 17. *Let* $\beta \leq 1$, $\varphi > 0$, *and* $\zeta(t)$ *a function which satisfies* $\zeta(t) \geq \sigma(t)$. *Suppose that there exists a number* $T > t_0$ *and a solution* $x(t)$ *of* (1) *which satisfy*

$$x(t) > 0, \quad z'(t) < 0 \quad \text{for } t \geq T. \qquad (93)$$

Let $t_1 > T$ *be such that*

$$\min\{\sigma(t), \sigma(\tau(t))\} > T \qquad (94)$$

for every $t \geq t_1$. *Then the following statements are true.*

(i) *The inequality*

$$\left[u(t) + \frac{p_0^\beta \varphi}{\tau_0} u(\tau(t))\right]' - Q_\zeta^\star(t;\varphi)\, u^{\beta/\gamma}(\zeta(t)) \geq 0 \qquad (95)$$

has a positive increasing solution on (t_1, ∞).

(ii) *If* $\tau(t) \leq t$, *then*

$$y' - Q_\zeta^\star(t;\varphi)\left(\frac{\tau_0}{\tau_0 + p_0^\beta \varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}(\zeta(t)) \geq 0 \qquad (96)$$

has a positive solution on (t_1, ∞).

(iii) *If* $\tau(t) \geq t$, *then*

$$y' - Q_\zeta^\star(t;\varphi)\left(\frac{\tau_0}{\tau_0 + p_0^\beta \varphi}\right)^{\beta/\gamma} y^{\beta/\gamma}\left(\tau^{-1}(\zeta(t))\right) \geq 0 \qquad (97)$$

has a positive solution on (t_1, ∞).

Corollary 18. *Let* $\beta \leq 1$, $\gamma \leq \beta$. *Equation* (1) *has no solution* $x(t)$ *which satisfies*

$$x(t) > 0 \quad z'(t) < 0 \quad \text{eventually} \qquad (98)$$

if there exists $\varphi > 0$ *and a function* $\zeta(t)$ *satisfying* $\eta(t) \geq \sigma(t)$ *such that one of the following conditions holds.*

(i) $\tau(t) \leq t < \zeta(t)$ *and*

$$\liminf_{t\to\infty} \int_t^{\zeta(t)} Q_\zeta^\star(s;\varphi)\, ds > \frac{1}{e}\left(\frac{\tau_0 + p_0^\beta \varphi}{\tau_0}\right)^{\beta/\gamma} \qquad (99)$$

if $\beta = \gamma$, *and* (91) *holds if* $\beta > \gamma$.

(ii) $t \leq \tau(t) < \zeta(t)$ *and*

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(\zeta(t))} Q_\zeta^\star(s;\varphi)\, ds > \frac{1}{e}\left(\frac{\tau_0 + p_0^\beta \varphi}{\tau_0}\right)^{\beta/\gamma} \qquad (100)$$

if $\beta = \gamma$, *and* (91) *holds if* $\beta > \gamma$.

5. Oscillation Criteria If (4) Holds

As we explained before, it follows from Lemma 5 that if (3) holds, then the criteria from Section 3 are in fact oscillation criteria. If (3) fails and (4) holds, then the set of all possible eventually positive solutions is more comprehensive and may contain also solution which satisfy $z'(t) < 0$ eventually. Hence to ensure oscillation of (1) in the case (4) we have to eliminate both cases; criteria from both Sections 3 and 4 apply. For example, in the half-linear case $\beta = \gamma$, (1) is oscillatory if either conditions

$$(4), (43), \text{ and } (92) \text{ hold if } \tau(t) \geq t, \qquad (101)$$

or

$$(4), (45) \text{ and } (90) \text{ hold if } \tau(t) \leq t; \qquad (102)$$

see the example below.

Example 19. Consider the equation

$$\left(t^{3/2}\big(x(t) + p_0 x(\alpha t)\big)'\right)' + \frac{a}{\sqrt{t}} x(\mu t) = 0, \quad a > 0. \quad (103)$$

We have $\gamma = \beta = 1$, $\tau(t) = \alpha t$, $\sigma(t) = \mu t$, $r(t) = t^{3/2}$, $q(t) = at^{-1/2}$, $q(\tau(t)) = a(\alpha t)^{-1/2}$. We will apply Corollaries 9 and 16 with $\eta(t) = \lambda_1 t$, $\zeta(t) = \lambda_2 t$, $\lambda_1 \le \mu \le \lambda_2$ and $\varphi = \alpha^{1/2}$. We have $Q(t) = q(t)$. Since

$$\int r^{-1/\gamma}(t)\,\mathrm{d}t = \int t^{-3/2}\,\mathrm{d}t = -2t^{-1/2}, \quad (104)$$

condition (4) holds. Next we have

$$Q_\eta^*(t;\varphi,t_1) = -2a\left(\sqrt{\lambda_1}t^{-1} - \sqrt{t_1}t^{-1/2}\right),$$

$$Q_\zeta^\star(t;\varphi) = -2at^{-1/2}\left(\lim_{t\to\infty} t^{-1/2} - (\lambda_2 t)^{-1/2}\right) \quad (105)$$

$$= 2a\lambda^{-1/2}t^{-1}.$$

Consequently, if $\lambda_1 < 1 \le \alpha < \lambda_2$, then (43) and (92) give

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} Q_\eta^*(s;\varphi,t_1)\,\mathrm{d}s$$

$$= -2a\sqrt{\lambda_1}\ln\frac{1}{\lambda_1} + 2a\sqrt{t_1}\left(1 - \sqrt{\lambda_1}\right)\lim_{t\to\infty}\sqrt{t} \quad (106)$$

$$= \infty,$$

$$\liminf_{t\to\infty} \int_{t}^{\tau^{-1}(\zeta(t))} Q_\zeta^\star(s;\varphi)\,\mathrm{d}s = \frac{2a}{\sqrt{\lambda_2}}\ln\frac{\lambda_2}{\alpha}.$$

If $\lambda_1 < \alpha \le 1 < \lambda_2$, then (45) and (90) give

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\eta(t))}^{t} Q_\eta^*(s;\varphi,t_1)\,\mathrm{d}s$$

$$= -2a\sqrt{\lambda_1}\ln\frac{\alpha}{\lambda_1} + 2a\sqrt{t_1}\left(1 - \sqrt{\frac{\lambda_1}{\alpha}}\right)\lim_{t\to\infty}\sqrt{t} \quad (107)$$

$$= \infty,$$

$$\liminf_{t\to\infty} \int_{t}^{\zeta(t)} Q_\zeta^\star(s;\varphi)\,\mathrm{d}s = \frac{2a}{\sqrt{\lambda_2}}\ln\lambda_2.$$

Hence, (103) is oscillatory if

$$\lambda_1 < 1 \le \alpha < \lambda_2, \qquad \frac{2a}{\sqrt{\lambda_2}}\ln\frac{\lambda_2}{\alpha} > \frac{1}{e}\left(1 + \frac{p_0}{\sqrt{\alpha}}\right) \quad (108)$$

or

$$\lambda_1 < \alpha \le 1 < \lambda_2, \qquad \frac{2a}{\sqrt{\lambda_2}}\ln\lambda_2 > \frac{1}{e}\left(1 + \frac{p_0}{\sqrt{\alpha}}\right). \quad (109)$$

The function $g(\lambda_2) = \lambda_2^{-1/2}\ln(\lambda_2/\alpha)$ is positive on the interval (α, ∞) and it follows from Lemma 1 that the maximal

value of $g(\lambda_2)$ is $2/(e\sqrt{\alpha})$ at the point αe^2. Hence, (103) is oscillatory if

$$a > a_{\text{crit.}}$$

$$:= \begin{cases} \dfrac{\sqrt{\alpha}}{4}\left(1 + \dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha \ge 1,\ \mu \le \alpha e^2, \\[2ex] \dfrac{1}{4}\left(1 + \dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha \le 1,\ \mu \le e^2, \\[2ex] \dfrac{\sqrt{\mu}}{2e\ln\left(\dfrac{\mu}{\alpha}\right)}\left(1 + \dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha \ge 1,\ \mu \ge \alpha e^2, \\[2ex] \dfrac{\sqrt{\mu}}{2e\ln\mu}\left(1 + \dfrac{p_0}{\sqrt{\alpha}}\right) & \text{for } \alpha \le 1,\ \mu \ge e^2. \end{cases} \quad (110)$$

6. Conclusion

In the paper we derived asymptotic results for neutral quasi-linear equation (1). Note that this equation covers several types of second-order differential equations studied in the literature, namely, the linear and half-linear second-order differential equations.

Using the comparison method we derived sufficient conditions for nonexistence of eventually positive solutions with various asymptotic behaviors. Additional assumptions (such as (3)) or suitable combinations of the results yield oscillation criteria for this equation. The novelty of the presented results is in the fact that we used parametrized versions of inequalities used typically in comparison theory of neutral differential equations. Despite the fact that we introduced three parameters (l, φ, and η), the results remain simple and effective. We have shown on several examples that effective oscillation criteria can be formulated for particular equations by establishing the optimal values for these parameters.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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