

Research Article

Strong Convergence Theorems for Common Fixed Points of an Infinite Family of Asymptotically Nonexpansive Mappings

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In the framework of a real Banach space with uniformly Gateaux differentiable norm, some new viscosity iterative sequences $\{x_n\}$ are introduced for an infinite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ in this paper. Under some appropriate conditions, we prove that the iterative sequences $\{x_n\}$ converge strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^{\infty}$, which is also a solution of a variational inequality. Our results extend and improve some recent results of other authors.

1. Introduction

Let E be a real Banach space, C a nonempty subset of E , and $T : C \rightarrow C$ a nonlinear mapping. Denote by $F(T)$ the set of fixed points of T . Recall that T is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1)$$

T is said to be contraction, if there exists a constant $\alpha \in (0, 1)$, such that

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C. \quad (2)$$

We use \prod_C to denote the collection of all contractions on C .

T is said to be asymptotically nonexpansive, if there exists a sequence $h_n \subseteq [1, \infty)$ with $h_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T^n x - T^n y\| \leq h_n \|x - y\|, \quad \forall x, y \in C. \quad (3)$$

$\{T_i\}_{i=1}^{\infty}$ is said to be uniform Lipschitzian with the coefficient L , if for any $i = 1, 2, \dots$, there holds

$$\|T_i^n x - T_i^n y\| \leq L \|x - y\|, \quad \forall x, y \in C. \quad (4)$$

It is clear that the class of contraction mappings must be included in the class of nonexpansive mappings and the

class of nonexpansive mappings in that of asymptotically nonexpansive mappings.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E , then every asymptotically nonexpansive self-mapping T has a fixed point in C . Further, the set $F(T)$ of fixed points of T is closed and convex.

In 2001, Khan and Takahasi [2] used the modified Ishikawa process to approximate common fixed points of two asymptotically nonexpansive mappings. In 2003, Sun [3] studied an implicit iterative scheme initiated by Xu and Ori [4] for a finite family of asymptotically quasi-nonexpansive mappings. Shahzad and Udomene [5], in 2006, proved some convergence theorems for the modified Ishikawa iterative process of two asymptotically quasi-nonexpansive mappings to a common fixed point. Shahzad and Zegeye [6] introduced a new concept of generalized asymptotically nonexpansive mappings and proved some strong convergence theorems for fixed points of finite family of this class. In 2008, Khan et al. [7] introduced an iterative sequence for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. Meanwhile, Zhao [8] proved the following conclusion.

Let E be a uniformly smooth Banach space, $f \in \prod_C$, and let T_1, T_2, \dots, T_N be a finite family of nonexpansive mappings

from C into itself, such that the set $\bigcap_{i=1}^N F(T_i)$ is nonempty. Under some sufficient conditions, the iterative sequence $\{x_n\}$ defined by (5) converges strongly to a common fixed point of T_1, T_2, \dots, T_N . Consider

$$\begin{aligned} y_n &= \beta_{n+1}x_n + (1 - \beta_{n+1})T_{n+1}x_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_{n+1}f(x_n) + (1 - \alpha_{n+1})T_{n+1}y_n, \quad n \geq 0. \end{aligned} \tag{5}$$

Common fixed points of nonlinear mappings play an important role in solving systems of equations and inequalities. Many researchers [9–19] are interested in studying approximation method for finding common fixed points of nonlinear mapping in recent years.

Motivated and inspired by the above results, in this paper, for an infinite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^\infty$ from C into itself, we introduce a new viscosity iterative process $\{x_n\}$ defined by

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)T_i^n x_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad n \geq 0. \end{aligned} \tag{6}$$

Under appropriate conditions on $C, T_i, \{\alpha_n\}$, and $\{\beta_n\}$ in $(0, 1)$, we prove that the $\{x_n\}$ converges strongly to $\hat{x} \in F$, which is also a solution of the following variational inequality:

$$\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0, \quad \forall p \in F. \tag{7}$$

Our results extend and improve some results of other authors (e.g., see [8, 17, 19]) from nonexpansive mappings to the more general class of asymptotically nonexpansive mappings and from finite family mappings to infinite family mappings.

2. Preliminaries

In order to prove our results, we need the following definitions and lemmas.

Assume that E is a real Banach space and E^* is the dual space of E .

$J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E. \tag{8}$$

The space E is said to be with Gateaux differentiable norm, if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{9}$$

exists for each y and any x in its unit sphere $U = \{x \in E : \|x\| = 1\}$.

In a Banach space E whose norm is uniformly Gateaux differentiable, the duality mapping J is single-valued and norm-to-weak* uniformly continuous on any bounded sets of E .

Lemma 1 (see [18]). *In a Banach space E , there holds the inequality*

$$\begin{aligned} \|x + y\|^2 &\leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \quad \forall x, y \in C, \\ &\text{where } j(x + y) \in J(x + y). \end{aligned} \tag{10}$$

Lemma 2 (see [20]). *Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\delta_n, \quad n \geq 0, \tag{11}$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ are two sequences such that

$$\begin{aligned} (1) \quad \lim_{n \rightarrow \infty} \gamma_n &= 0, \quad \sum \gamma_n = \infty; \\ (2) \quad \limsup_{n \rightarrow \infty} \delta_n &\leq \left(\text{or } \sum |\mu_n \delta_n| < \infty\right). \end{aligned} \tag{12}$$

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

In order to prove the main results of this paper, the following lemmas should be used.

Lemma 3 (see [21]). *Let $\{x_n\}, \{y_n\}$ be bounded sequences in a Banach space X , $\{\alpha_n\} \subset [0, 1]$, satisfying $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$, and suppose*

$$\begin{aligned} (1) \quad x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)y_n; \\ (2) \quad \limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) &\leq 0. \end{aligned} \tag{13}$$

Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

3. Main Results

Now, we are ready to give the main results.

Theorem 4. *Let K be a nonempty closed convex subset of a real Banach space E with uniformly Gateaux differentiable norm. Let $\{T_i\}_{i=1}^\infty$ be an infinite family of asymptotically nonexpansive mappings with the coefficients $h_{in} : \lim_{n \rightarrow \infty} h_{in} = 1$ and uniform Lipschitzian with the coefficient L from K into itself, $F = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and $f \in \prod_K$ with the coefficient α . Assume that the sequences $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ satisfy the following conditions:*

- (C1) $\sum_{n=1}^\infty \alpha_n = \infty, \lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C3) $h_{in} - 1 = o(\alpha_n)$.

Then the sequence $\{x_n\}$ defined by (6) converges strongly to $\hat{x} \in F$ if and only if, for any $i, \lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0$ holds. And \hat{x} is a solution of the following variational inequality:

$$\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0, \quad \forall p \in F. \tag{14}$$

Proof.

Sufficiency. The sufficient proof is divided into five steps.

Step 1. We observe that $\{x_n\}$ is bounded. Indeed, let $a_n = \beta_n + (1 - \beta_n)h_{in}$, $a = \sup\{a_n\}$, $b = \inf\{a_n\} = 1$. Taking a fixed point p of F , we have

$$\begin{aligned} & \|y_n - p\| \\ & \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|T_i^n x_n - p\| \\ & \leq a_n \|x_n - p\|, \\ & \|x_{n+1} - p\| \\ & \leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\ & \quad + (1 - \alpha_n) \|y_n - p\| \\ & \leq \alpha_n \alpha \|x_n - p\| + (1 - \alpha_n) a_n \|x_n - p\| + \alpha_n \|f(p) - p\| \\ & = (a_n - (a_n - \alpha) \alpha_n) \|x_n - p\| + \alpha_n (a_n - \alpha) \frac{\|f(p) - p\|}{a_n - \alpha} \\ & \leq a_n \max \left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{a_n - \alpha} \right\}. \end{aligned} \tag{15}$$

Using an introduction, we have

$$\|x_n - p\| \leq a \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{b - \alpha} \right\}. \tag{16}$$

Hence, $\{x_n\}$ is bounded so are the sets $\{y_n\}$, $\{f(x_n)\}$ and $\{T_i^n x_n\}$.

Step 2. We claim that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Setting $l_n = (1 - \alpha_n)\beta_n$, $n \geq 1$, it follows from (C1) and (C2) that

$$0 < \liminf_{n \rightarrow \infty} l_n \leq \limsup_{n \rightarrow \infty} l_n < 1. \tag{17}$$

Define $x_{n+1} = l_n x_n + (1 - l_n)z_n$, and observe that

$$\begin{aligned} & z_{n+1} - z_n \\ & = \frac{x_{n+2} - l_{n+1}x_{n+1}}{1 - l_{n+1}} - \frac{x_{n+1} - l_n x_n}{1 - l_n} \\ & = \left[\left(\alpha_{n+1} f(x_{n+1}) \right. \right. \\ & \quad \left. \left. + (1 - \alpha_{n+1}) (\beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) T_i^{n+1} x_{n+1}) \right) \right. \\ & \quad \left. \times (1 - l_{n+1})^{-1} \right. \\ & \quad \left. - \frac{l_{n+1} x_{n+1}}{1 - l_{n+1}} \right] \\ & \quad - \frac{\alpha_n f(x_n) + (1 - \alpha_n) (\beta_n x_n + (1 - \beta_n) T_i^n x_n) - l_n x_n}{1 - l_n} \end{aligned}$$

$$\begin{aligned} & = \frac{\alpha_{n+1} f(x_{n+1})}{1 - l_{n+1}} - \frac{\alpha_n f(x_n)}{1 - l_n} \\ & \quad + \frac{(1 - \alpha_{n+1})(1 - \beta_{n+1}) T_i^{n+1} x_{n+1}}{1 - l_{n+1}} \\ & \quad - \frac{(1 - \alpha_n)(1 - \beta_n) T_i^n x_n}{1 - l_n} \\ & = \frac{\alpha_{n+1}}{1 - l_{n+1}} (f(x_{n+1}) - T_i^{n+1} x_{n+1}) \\ & \quad - \frac{\alpha_n}{1 - l_n} (f(x_n) - T_i^n x_n) \\ & \quad + (T_i^{n+1} x_{n+1} - T_i^n x_n). \end{aligned} \tag{18}$$

We have

$$\begin{aligned} & \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ & \leq \frac{\alpha_{n+1}}{1 - l_{n+1}} \|f(x_{n+1}) - T_i^{n+1} x_{n+1}\| \\ & \quad + \frac{\alpha_n}{1 - l_n} \|f(x_n) - T_i^n x_n\| \\ & \quad + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i^n x_n\| \\ & \quad - \|x_{n+1} - x_n\| \\ & = \frac{\alpha_{n+1}}{1 - l_{n+1}} (\|f(x_{n+1})\| + \|T_i^{n+1} x_{n+1}\|) + \frac{\alpha_n}{1 - l_n} \\ & \quad \times (\|f(x_n)\| + \|T_i^n x_n\|) + (h_{n+1} - 1) \|x_{n+1} - x_n\| \\ & \quad + h_n \|T_i x_n - x_n\|. \end{aligned} \tag{19}$$

Using the conclusion of step 1, by (C1), $\lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0$ and $\lim_{n \rightarrow \infty} h_{in} = 1$, and we obtain that $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$.

Hence, by Lemma 3, we have $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$. Consequently,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - l_n) \|z_n - x_n\| = 0. \tag{20}$$

Step 3. We prove $\lim_{n \rightarrow \infty} \|x_n - T_i^n x_n\| = 0$. From (6), we arrive at

$$\begin{aligned} & \|y_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ & = \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - y_n\|. \end{aligned} \tag{21}$$

Since $\{f(x_n)\}$ and $\{y_n\}$ are bounded, by (C1) and $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, we get $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$. As $y_n = \beta_n x_n + (1 - \beta_n) T_i^n x_n$, then $\lim_{n \rightarrow \infty} \|x_n - T_i^n x_n\| = 0$.

Step 4. We show that $\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0$. Same as [15], let $\hat{x} = \lim_{t \rightarrow 0} x_t$ with x_t being the fixed point of the contraction

$x \rightarrow tf(x) + (1-t)T_i^n x$, where $t \in (0, 1)$. That is, $x_t = tf(x_t) + (1-t)T_i^n x_t$. Thanks to Lemma 1, we have

$$\begin{aligned} & \|x_t - x_n\|^2 \\ &= \|(1-t)(T_i^n x_t - x_n) + t(f(x_t) - x_n)\|^2 \\ &\leq (1-t)^2 \|T_i^n x_t - x_n\|^2 + 2t \langle f(x_t) - x_n, J(x_t - x_n) \rangle \\ &\leq (1-t)^2 h_{in} \|x_t - x_n\|^2 + g_n(t) \\ &\quad + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle \\ &\quad + 2t \langle x_t - x_n, J(x_t - x_n) \rangle, \end{aligned} \tag{22}$$

where $g_n(t) = (2h_{in}\|x_t - x_n\| + \|T_i^n x_n - x_n\|)\|T_i^n x_n - x_n\|$. It follows from step 2 that $\lim_{n \rightarrow \infty} g_n(t) = 0$. Then from

$$\begin{aligned} & \langle x_t - f(x_t), J(x_t) - x_n \rangle \\ & \leq \frac{(1-t)^2 h_n - 1 + 2t}{2t} \|x_t - x_n\| + \frac{1}{2t} g_n(t), \end{aligned} \tag{23}$$

we see that

$$\limsup_{n \rightarrow \infty} \langle x_t - f(x_t), J(x_t) - x_n \rangle \leq \frac{t}{2} M_1, \tag{24}$$

where $M_1 \geq 0$, such that

$$M_1 \geq \|x_t - x_n\|^2, \quad \forall t \in (0, 1), \quad n \geq 1. \tag{25}$$

Then

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle x_t - f(x_t), J(x_t) - x_n \rangle \leq 0. \tag{26}$$

So for any $\epsilon > 0$, there exists $\delta_1 > 0$. When $t \in (0, \delta_1)$, we get

$$\limsup_{n \rightarrow \infty} \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle \leq \frac{\epsilon}{2}. \tag{27}$$

On the other hand, because $x_t \rightarrow \hat{x}$ and J is norm-to-norm uniformly continuous on bounded subsets of K , there exists $\delta_2 > 0$, such that when $t \in (0, \delta_2)$, we have

$$\begin{aligned} & |\langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) \rangle - \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle| \\ & \leq |\langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) \rangle - \langle f(\hat{x}) - \hat{x}, J(x_n - x_t) \rangle| \\ & \quad + |\langle f(\hat{x}) - \hat{x}, J(x_n - x_t) \rangle \\ & \quad - \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle| \\ & \leq |\langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) - J(x_n - x_t) \rangle| \\ & \quad + \langle x_t - \hat{x}, J(x_n - x_t) \rangle \leq \frac{\epsilon}{2}. \end{aligned} \tag{28}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$, $\forall t \in (0, \delta)$, we have

$$\begin{aligned} & \langle u - \hat{x}, J(x_n - \hat{x}) \rangle \leq \langle x_t - u, J(x_t - x_n) \rangle + \frac{\epsilon}{2}, \\ & \limsup_{n \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) \rangle \\ & \leq \limsup_{n \rightarrow \infty} \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle + \frac{\epsilon}{2} \leq \epsilon. \end{aligned} \tag{29}$$

Since ϵ is chosen arbitrarily, we get

$$\limsup_{n \rightarrow \infty} \langle u - \hat{x}, J(x_n - \hat{x}) \rangle \leq 0. \tag{30}$$

Hence, we have

$$\begin{aligned} & \langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0, \\ & \limsup_{n \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle \leq 0. \end{aligned} \tag{31}$$

Step 5. We prove that $\lim_{n \rightarrow \infty} \|x_n - \hat{x}\| = 0$. Setting $\xi_{n+1} = \max\{\langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle\}$, we have $\limsup_{n \rightarrow \infty} \xi_{n+1} \leq 0$

$$\begin{aligned} & \|x_{n+1} - \hat{x}\|^2 \\ &= \|\alpha_n (f(x_n) - \hat{x}) + (1 - \alpha_n) \\ & \quad \times (\beta_n (x_n - \hat{x}) + (1 - \beta_n) (T_i^n x_n - \hat{x}))\|^2 \\ &\leq (1 - \alpha_n)^2 [\beta_n \|x_n - \hat{x}\| + (1 - \beta_n) h_n \|x_n - \hat{x}\|]^2 \\ & \quad + 2\alpha_n \langle f(x_n) - f(\hat{x}), J(x_{n+1} - \hat{x}) \rangle \\ & \quad + 2\alpha_n \langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle \\ &\leq (1 - \alpha_n)^2 h_n^2 \|x_n - \hat{x}\|^2 + 2\alpha_n \alpha \|x_n - \hat{x}\| \|x_{n+1} - \hat{x}\| \\ & \quad + 2\alpha_n \langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - \hat{x}\|^2 + (1 - \alpha_n)^2 (h_n^2 - 1) \|x_n - \hat{x}\|^2 \\ & \quad + \alpha_n \alpha (\|x_n - \hat{x}\|^2 + \|x_{n+1} - \hat{x}\|^2) \\ & \quad + 2\alpha_n \langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle. \end{aligned} \tag{32}$$

So,

$$\begin{aligned} & (1 - \alpha_n \alpha) \|x_{n+1} - \hat{x}\|^2 \\ & \leq (1 - 2\alpha_n + \alpha_n^2) \|x_n - \hat{x}\|^2 + (h_n^2 - 1) \|x_n - \hat{x}\|^2 \\ & \quad + \alpha_n \alpha \|x_n - \hat{x}\|^2 + 2\alpha_n \langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle \\ & \leq (1 - 2\alpha_n + \alpha_n \alpha) \|x_n - \hat{x}\|^2 \\ & \quad + (\alpha_n^2 + h_n^2 - 1) \|x_n - \hat{x}\|^2 + 2\alpha_n \xi_{n+1}, \end{aligned} \tag{33}$$

which implies that

$$\begin{aligned} & \|x_{n+1} - \hat{x}\|^2 \\ & \leq \left(1 - \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\right) \|x_n - \hat{x}\|^2 + \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha} \\ & \quad \times \left(\frac{\alpha_n + (h_n - 1)/\alpha_n}{2(1-\alpha)} M_2 + \frac{1}{1-\alpha} \xi_{n+1}\right) \\ & = (1 - \gamma_n) \|x_n - \hat{x}\|^2 + \gamma_n \delta_n, \end{aligned} \tag{34}$$

where

$$M_2 = \sup (h_n + 1) \|x_n - \hat{x}\|^2, \quad \gamma_n = \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}, \quad (35)$$

$$\delta_n = \frac{\alpha_n + (h_n - 1)/\alpha_n}{2(1-\alpha)} M_2 + \frac{1}{1-\alpha} \xi_{n+1}.$$

Since $\{x_n\}$ is bounded, by (C1), (C3), and step 3, we have

$$\lim_{n \rightarrow \infty} \gamma_n = 0, \quad \sum \gamma_n = \infty, \quad \limsup_{n \rightarrow \infty} \delta_n \leq 0. \quad (36)$$

According to Lemma 2, we deduce that $\lim_{n \rightarrow \infty} \|x_n - \hat{x}\| = 0$.

Necessity. Since $\hat{x} \in F$, $\lim_{n \rightarrow \infty} \|x_n - \hat{x}\| = 0$, T_i is uniform Lipschitzian,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|T_i x_n - x_n\| \\ & \leq \lim_{n \rightarrow \infty} [\|T_i x_n - T_i \hat{x}\| + \|T_i \hat{x} - \hat{x}\| + \|\hat{x} - x_n\|] \\ & \leq \lim_{n \rightarrow \infty} \|T_i \hat{x} - \hat{x}\| + \lim_{n \rightarrow \infty} (L + 1) \|\hat{x} - x_n\| \leq 0. \end{aligned} \quad (37)$$

Hence, the proof of Theorem 4 is completed. □

Theorem 5. *Under the same conditions as those in Theorem 4, then the sequence $\{x_n\}$ defined by*

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) T_n^n x_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad n \geq 0, \end{aligned} \quad (38)$$

converges strongly to $\hat{x} \in F$, if and only if $\lim_{n \rightarrow \infty} \|T_n x_n - x_n\| = 0$ holds. And \hat{x} is a solution of the following variational inequality:

$$\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0, \quad \forall p \in F. \quad (39)$$

Proof. The proof is the same as that of Theorem 4. We can just use T_n^n , h_{nn} to take the place of T_i^n , h_{in} . □

Theorem 6. *Let $\{T_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive mappings with the coefficients $h_{in} : \lim_{n \rightarrow \infty} h_{in} = 1$ and $\{T_n\} = \{T_{n(\text{mod } N)}\}$. The rest of the conditions are the same as those in Theorem 4. Then the sequence $\{x_n\}$ defined by (38) converges strongly to $\hat{x} \in F$, if and only if $\lim_{n \rightarrow \infty} \|T_n x_n - x_n\| = 0$ holds. And \hat{x} is a solution of the following variational inequality:*

$$\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0, \quad \forall p \in F. \quad (40)$$

Proof. This is a special case of Theorem 5. □

Remark 7. If $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings in Theorem 6, then it is the main result in reference [8]. So our results in this paper extend and improve the recent results of many other authors (e.g., see [1, 8, 17, 19]) in different ways, such as from nonexpansive mappings to the more general class of asymptotically nonexpansive mappings and from finite family mappings to infinite family mappings. And the methods here are different from the previous ones.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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