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Research Article

Tripled Coincidence and Common Fixed Point Results for Two Pairs of Hybrid Mappings

Marwan Amin Kutbi,[1] Jamshaid Ahmad,[2] Mujahid Abbas,[3] and Muhammad Arshad[4]

[1] *Department of Mathematics, King Abdul Aziz University, Jeddah, Saudi Arabia*
[2] *Department of Mathematics, COMSATS Institute of Information Technology, Chack Shahzad, Islamabad 44000, Pakistan*
[3] *Department of Mathematics & Applied Mathematics, University of Pretoria, Pretoria 002, South Africa*
[4] *Department of Mathematics, International Islamic University, H-10, Islamabad 44000, Pakistan*

Correspondence should be addressed to Muhammad Arshad; marshad_zia@yahoo.com

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The tripled fixed point is a generalization of the well-known concept of "coupled fixed point." In this paper, tripled coincidence and common fixed point results for two hybrid pairs consisting of multivalued and single valued mappings on a metric space are proved. We give examples to illustrate our results. In the process, several comparable coincidence and fixed point results in the existing literature are improved, unified, and generalized.

1. Introduction and Preliminaries

The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler Jr. [1]. After this, fixed point theory has been developed further and applied to many disciplines to solve functional equations. Banach contraction principle has been extended in different directions. Some authors used generalized contractions for multivalued mappings and hybrid pairs of single and multi-valued mappings, while others used more general spaces. Dhage [2, 3] established hybrid fixed point theorems and obtained some applications of presented results. Gnana Bhaskar and Lakshmikantham [4] introduced the notion of a coupled fixed point and proved some coupled fixed point results under certain contractive conditions in a complete metric space endowed with a partial order. They applied their results to study the existence and uniqueness of solution for a periodic boundary value problem associated with a first-order ordinary differential equation. Later, Lakshmikantham and Ćirić [5] established the existence of coupled coincidence point results to generalize the results of Gnana Bhaskar and Lakshmikantham [4]; Karapınar [6] generalized these results on a complete cone metric space endowed with a partial order. Recently, Berinde and Borcut [7, 8] introduced the concept of a tripled fixed point for nonlinear contractive mappings in partially ordered complete metric spaces and obtained tripled coincidence and fixed point results for commuting maps. Hussain et al. [9, 10] obtained some coupled and tripled coincidence results without compatibility. Ilić et al. [11] obtained coupled coincidence and common fixed point theorems for a hybrid pair of mappings. For other related results in this direction, we refer to [12–16] and references mentioned therein. The purpose of this paper is to obtain tripled coincidence and common fixed point results for two hybrid pairs consisting of multivalued and single valued mappings.

Let us recall some definitions and well known results needed in the sequel.

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $d(x, A) = \inf\{d(x, y) : y \in A\}$. The set of all nonempty bounded and closed subsets of X is denoted by $CB(X)$. Let H be the Hausdorff metric induced by the metric d on X; that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \qquad (1)$$

for every $A, B \in CB(X)$.

Lemma 1 (see [1]). *Let $A, B \in CB(X)$ and $\alpha > 1$. Then, for every $a \in A$, there exists $b \in B$ such that*

$$d(a,b) \leq \alpha H(A,B). \tag{2}$$

Lemma 2 (see [1]). *Let $A, B \in CB(X)$ and $0 < \alpha \in \mathbb{R}$. Then, for every $a \in A$, there exists $b \in B$ such that*

$$d(a,b) \leq H(A,B) + \alpha. \tag{3}$$

Lemma 3 (see [1]). *Let $A, B \in CB(X)$. If $a \in A$, then $d(a,B) \leq H(A,B)$.*

Definition 4. Let X be a nonempty set, $F : X \times X \times X \to 2^X$ (collection of all nonempty subsets of X) and $g : X \to X$. An element $(x, y, z) \in X \times X \times X$ is called (i) a tripled fixed point of F if $x \in F(x, y, z)$, $y \in F(y, z, x)$, and $z \in F(z, x, y)$ (ii) tripled coincidence point of a hybrid pair (F, g) if $g(x) \in F(x, y, z)$, $g(y) \in F(y, z, x)$ and $g(z) \in F(z, x, y)$ (iii) tripled common fixed point of a hybrid pair (F, g) if $x = g(x) \in F(x, y, z)$, $y = g(y) \in F(y, z, x)$, and $z = g(z) \in F(z, x, y)$.

We denote the set of tripled coincidence point of a hybrid pair (F, g) by $\Upsilon(F, g)$. Note that if $(x, y, z) \in \Upsilon(F, g)$, then (y, z, x) and (z, x, y) are also in $\Upsilon(F, g)$.

Definition 5. Let $F : X\times X\times X \to 2^X$ and $g : X \to X$. Then the hybrid pair (F, g) is called w-compatible if $g(F(x, y, z)) \subseteq F(gx, gy, gz)$ whenever $(x, y, z) \in \Upsilon(F, g)$.

Definition 6. Let $F : X \times X \times X \to 2^X$ and $g : X \to X$. The mapping g is called F-idempotent at some point $(x, y, z) \in X \times X \times X$ if $g^2(x) \in F(gx, gy, gz)$, $g^2(y) \in F(gy, gz \cdot gx)$, and $g^2(z) \in F(gz, gx \cdot gy)$.

2. Main Result

Theorem 7. *Let (X, d) be a metric space, $S, T : X \times X \times X \to CB(X)$ and let $g : X \to X$ be mappings such that*

$$\begin{aligned} &H(S(x,y,z), T(u,v,w)) \\ &\quad \leq a_1 d(gx,gu) + a_2 d(gy,gv) + a_3 d(gz,gw) \\ &\qquad + a_4 d(S(x,y,z),gx) + a_5 d(T(u,v,w),gu) \\ &\qquad + a_6 d(S(x,y,z),gu) + a_7 d(T(u,v,z),gx), \end{aligned} \tag{4}$$

for all $x, y, z, u, v, w \in X$, where $a_i = a_i(x, y, z, u, v, w)$, $i = 1, 2, \ldots, 7$, are nonnegative real such that

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \leq h < 1. \tag{5}$$

If $S(X \times X \times X) \cup T(X \times X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then (S, g) and (T, g) have tripled coincidence point. Moreover (S, g) and (T, g) have tripled common fixed point if one of the following conditions holds:

(i) (S, g) and (T, g) are w-compatible, $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for some $(x, y, z) \in \Upsilon(S, g) \cap \Upsilon(T, g)$, $u, v, w \in X$ and g is continuous at u, v, w;

(ii) if $g^2x = gx$, $g^2y = gy$, and $g^2z = gz$ and g is S, T-idempotent for $(x, y, z) \in \Upsilon(S, g) \cap \Upsilon(T, g)$;

(iii) g is continuous at x, y, z for some $(x, y, z) \in \Upsilon(g, S) \cap \Upsilon(g, T)$ and for some $u, v, w \in X$; $\lim_{n\to\infty} g^n u = x$; and $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n z = w$.

Proof. Let $x_0, y_0, z_0 \in X$ be arbitrary. Choose $x_1, y_1, z_1 \in X$ such that $gx_1 \in S(x_0, y_0, z_0)$, $gy_1 \in S(y_0, z_0, x_0)$ and $gz_1 \in S(z_0, x_0, y_0)$. Choose $x_2, y_2, z_2 \in X$ such that $gx_2 \in T(x_1, y_1, z_1)$, $gy_2 \in T(y_1, z_1, x_1)$, and $gz_2 \in T(z_1, x_1, y_1)$. This can be done because $S(X \times X \times X) \cup T(X \times X \times X) \subseteq g(X)$. If

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0, \tag{6}$$

then

$$\begin{aligned} d(gx_1, T(x_1, y_1, z_1)) &\leq H(S(x_0, y_0, z_0), T(x_1, y_1, z_1)) = 0, \\ d(gx_2, S(x_2, y_2, z_2)) &\leq H(T(x_1, y_1, z_1), S(x_2, y_2, z_2)) = 0. \end{aligned} \tag{7}$$

Imply that

$$d(gx_1, T(x_1, y_1, z_1)) = 0, \qquad d(gx_2, S(x_2, y_2, z_2)) = 0. \tag{8}$$

As $T(x_1, y_1, z_1)$ and $S(x_2, y_2, z_2)$ are closed,

$$gx_1 \in T(x_1, y_1, z_1), \qquad gx_2 \in S(x_2, y_2, z_2). \tag{9}$$

Similarly

$$\begin{aligned} gy_1 &\in T(y_1, z_1, x_1), \qquad gy_2 \in S(y_2, z_2, x_2), \\ gz_1 &\in T(z_1, x_1, y_1), \qquad gz_2 \in S(z_2, x_2, y_2). \end{aligned} \tag{10}$$

Hence (x_1, y_1, z_1) and (x_2, y_2, z_2) are tripled coincidence points of pairs (T, g) and (S, g), respectively. Now assume that $a_i > 0$, for some $i = 1, 2, \ldots, 7$ which gives that $h > 0$; therefore, there exist

$$\begin{aligned} t_1 &\in T(x_1, y_1, z_1), \qquad t_2 \in T(y_1, z_1, x_1), \\ t_3 &\in T(z_1, x_1, y_1), \qquad t_4 \in S(x_2, y_2, z_2), \\ t_5 &\in S(y_2, z_2, x_2), \qquad t_6 \in S(z_2, x_2, y_2), \end{aligned} \tag{11}$$

such that

$$\begin{aligned} d(gx_1, t_1) &\leq H(S(x_0, y_0, z_0), T(x_1, y_1, z_1)) + \frac{h}{6}, \\ d(gy_1, t_2) &\leq H(S(y_0, z_0, x_0), T(y_1, z_1, x_1)) + \frac{h}{6}, \\ d(gz_1, t_3) &\leq H(S(z_0, x_0, y_0), T(z_1, x_1, y_1)) + \frac{h}{6}, \\ d(gx_2, t_4) &\leq H(T(x_1, y_1, z_1), S(x_2, y_2, z_2)) + \frac{h}{6}, \\ d(gy_2, t_5) &\leq H(T(y_1, z_1, x_1), S(y_2, z_2, x_2)) + \frac{h}{6}, \\ d(gz_2, t_6) &\leq H(T(z_1, x_1, y_1), S(z_2, x_2, y_2)) + \frac{h}{6}. \end{aligned} \tag{12}$$

Since $S(X\times X\times X)\cup T(X\times X\times X)\subseteq g(X)$, there exist x_2, y_2, z_2, x_3, y_3, and z_3 in X such that $t_1=gx_2$, $t_2=gy_2$, $t_3=gz_2$, $t_4=gx_3$, $t_5=gy_3$, and $t_6=gz_3$. Thus

$$\begin{aligned}
d(gx_1,gx_2)&\le H(S(x_0,y_0,z_0),T(x_1,y_1,z_1))+\frac{h}{6},\\
d(gy_1,gy_2)&\le H(S(y_0,z_0,x_0),T(y_1,z_1,x_1))+\frac{h}{6},\\
d(gz_1,gz_2)&\le H(S(z_0,x_0,y_0),T(z_1,x_1,y_1))+\frac{h}{6},\\
d(gx_2,gx_3)&\le H(T(x_1,y_1,z_1),S(x_2,y_2,z_2))+\frac{h}{6},\\
d(gy_2,gy_3)&\le H(T(y_1,z_1,x_1),S(y_2,z_2,x_2))+\frac{h}{6},\\
d(gz_2,gz_3)&\le H(T(z_1,x_1,y_1),S(z_2,x_2,y_2))+\frac{h}{6}.
\end{aligned}\tag{13}$$

Continuing this process, we obtain three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in X such that

$$\begin{gathered}
gx_{2n+1}\in S(x_{2n},y_{2n},z_{2n}),\qquad gy_{2n+1}\in S(y_{2n},z_{2n},x_{2n}),\\
gz_{2n+1}\in S(z_{2n},x_{2n},y_{2n}),\\
gx_{2n+2}\in T(x_{2n+1},y_{2n+1},z_{2n+1}),\\
gy_{2n+2}\in T(y_{2n+1},z_{2n+1},x_{2n+1}),\\
gz_{2n+2}\in T(z_{2n+1},x_{2n+1},y_{2n+1}),
\end{gathered}\tag{14}$$

with

$$\begin{aligned}
&d(gx_{2n+1},gx_{2n+2})\\
&\quad\le H(S(x_{2n},y_{2n},z_{2n}),T(x_{2n+1},y_{2n+1},z_{2n+1}))+\frac{h^n}{6},\\
&d(gy_{2n+1},gy_{2n+2})\\
&\quad\le H(S(y_{2n},z_{2n},x_{2n}),T(y_{2n+1},z_{2n+1},x_{2n+1}))+\frac{h^n}{6},\\
&d(gz_{2n+1},gz_{2n+2})\\
&\quad\le H(S(z_{2n},x_{2n},y_{2n}),T(z_{2n+1},x_{2n+1},y_{2n+1}))+\frac{h^n}{6},\\
&d(gx_{2n+2},gx_{2n+3})\\
&\quad\le H(T(x_{2n+1},y_{2n+1},z_{2n+1}),S(x_{2n+2},y_{2n+2},z_{2n+2}))+\frac{h^n}{6},\\
&d(gy_{2n+2},gy_{2n+3})\\
&\quad\le H(T(y_{2n+1},z_{2n+1},x_{2n+1}),S(y_{2n+2},z_{2n+2},x_{2n+2}))+\frac{h^n}{6},\\
&d(gz_{2n+2},gz_{2n+3})\\
&\quad\le H(T(z_{2n+1},x_{2n+1},y_{2n+1}),S(z_{2n+2},x_{2n+2},y_{2n+2}))+\frac{h^n}{6}.
\end{aligned}\tag{15}$$

By (4),

$$\begin{aligned}
&d(gx_{2n+1},gx_{2n+2})\\
&\quad\le H(S(x_{2n},y_{2n},z_{2n}),T(x_{2n+1},y_{2n+1},z_{2n+1}))+\frac{h^n}{6}\\
&\quad\le a_1d(gx_{2n},gx_{2n+1})+a_2d(gy_{2n},gy_{2n+1})\\
&\qquad+a_3d(gz_{2n},gz_{2n+1})+a_4d(S(x_{2n},y_{2n},z_{2n}),gx_{2n})\\
&\qquad+a_5d(T(x_{2n+1},y_{2n+1},z_{2n+1}),gx_{2n+1})\\
&\qquad+a_6d(S(x_{2n},y_{2n},z_{2n}),gx_{2n+1})\\
&\qquad+a_7d(T(x_{2n+1},y_{2n+1},z_{2n+1}),gx_{2n})+\frac{h^n}{6}\\
&\quad\le a_1d(gx_{2n},gx_{2n+1})+a_2d(gy_{2n},gy_{2n+1})\\
&\qquad+a_3d(gz_{2n},gz_{2n+1})+a_4d(gx_{2n+1},gx_{2n})\\
&\qquad+a_5d(gx_{2n+2},gx_{2n+1})+a_6d(gx_{2n+1},gx_{2n+1})\\
&\qquad+a_7d(gx_{2n+2},gx_{2n})+\frac{h^n}{6},
\end{aligned}\tag{16}$$

which further gives

$$\begin{aligned}
&(1-a_5-a_7)\,d(gx_{2n+1},gx_{2n+2})\\
&\quad\le(a_1+a_4+a_7)\,d(gx_{2n},gx_{2n+1})\\
&\qquad+a_2d(gy_{2n},gy_{2n+1})+a_3d(gz_{2n},gz_{2n+1})+\frac{h^n}{6}.
\end{aligned}\tag{17}$$

Similarly it can be shown that

$$\begin{aligned}
&(1-a_5-a_7)\,d(gy_{2n+1},gy_{2n+2})\\
&\quad\le(a_1+a_4+a_7)\,d(gy_{2n},gy_{2n+1})\\
&\qquad+a_2d(gz_{2n},gz_{2n+1})+a_3d(gx_{2n},gx_{2n+1})+\frac{h^n}{6},\\
&(1-a_5-a_7)\,d(gz_{2n+1},gz_{2n+2})\\
&\quad\le(a_1+a_4+a_7)\,d(gz_{2n},gz_{2n+1})\\
&\qquad+a_2d(gx_{2n},gx_{2n+1})+a_3d(gy_{2n},gy_{2n+1})+\frac{h^n}{6}.
\end{aligned}\tag{18}$$

Again

$$\begin{aligned} &d(gx_{2n+2}, gx_{2n+1}) \\ &\quad \le H(T(x_{2n+1}, y_{2n+1}, z_{2n+1}), S(x_{2n}, y_{2n}, z_{2n})) + \frac{h^n}{6} \\ &\quad \le a_1 d(gx_{2n+1}, gx_{2n}) + a_2 d(gy_{2n+1}, gy_{2n}) \\ &\qquad + a_3 d(gz_{2n+1}, gz_{2n}) \\ &\qquad + a_4 d(T(x_{2n+1}, y_{2n+1}, z_{2n+1}), gx_{2n+1}) \\ &\qquad + a_5 d(S(x_{2n}, y_{2n}, z_{2n}), gx_{2n}) \\ &\qquad + a_6 d(T(x_{2n+1}, y_{2n+1}, z_{2n+1}), gx_{2n}) \\ &\qquad + a_7 d(S(x_{2n}, y_{2n}, z_{2n}), gx_{2n+1}) + \frac{h^n}{6} \\ &\quad \le a_1 d(gx_{2n+1}, gx_{2n}) + a_2 d(gy_{2n+1}, gy_{2n}) \\ &\qquad + a_3 d(gz_{2n+1}, gz_{2n}) + a_4 d(gx_{2n+2}, gx_{2n+1}) \\ &\qquad + a_5 d(gx_{2n+1}, gx_{2n}) + a_6 d(gx_{2n+2}, gx_{2n}) \\ &\qquad + a_7 d(gx_{2n+1}, gx_{2n+1}) + \frac{h^n}{6}, \end{aligned} \tag{19}$$

which implies

$$\begin{aligned} &(1 - a_4 - a_6) d(gx_{2n+2}, gx_{2n+1}) \\ &\quad \le (a_1 + a_5 + a_6) d(gx_{2n+1}, gx_{2n}) \\ &\qquad + a_2 d(gy_{2n+1}, gy_{2n}) + a_3 d(gz_{2n+1}, gz_{2n}) + \frac{h^n}{6}. \end{aligned} \tag{20}$$

Similarly, it can be shown that

$$\begin{aligned} &(1 - a_4 - a_6) d(gy_{2n+2}, gy_{2n+1}) \\ &\quad \le (a_1 + a_5 + a_6) d(gy_{2n+1}, gy_{2n}) \\ &\qquad + a_2 d(gz_{2n+1}, gz_{2n}) + a_3 d(gx_{2n+1}, gx_{2n}) + \frac{h^n}{6}, \\ &(1 - a_4 - a_6) d(gz_{2n+2}, gz_{2n+1}) \\ &\quad \le (a_1 + a_5 + a_6) d(gz_{2n+1}, gz_{2n}) \\ &\qquad + a_2 d(gx_{2n+1}, gx_{2n}) + a_3 d(gy_{2n+1}, gy_{2n}) + \frac{h^n}{6}. \end{aligned} \tag{21}$$

Let

$$\begin{aligned} \delta_{2n+1} = {} & d(gx_{2n+1}, gx_{2n+2}) + d(gy_{2n+1}, gy_{2n+2}) \\ & + d(gz_{2n+1}, gz_{2n+2}). \end{aligned} \tag{22}$$

From (17) and (18), we get

$$(1 - a_5 - a_7)\delta_{2n+1} \le (a_1 + a_2 + a_3 + a_4 + a_7)\delta_{2n} + \frac{h^n}{2}. \tag{23}$$

From (20) and (21) we get

$$(1 - a_4 - a_6)\delta_{2n+1} \le (a_1 + a_2 + a_3 + a_5 + a_6)\delta_{2n} + \frac{h^n}{2}. \tag{24}$$

Adding (23) and (24), we obtain

$$\begin{aligned} &(2 - a_4 - a_5 - a_6 - a_7)\delta_{2n+1} \\ &\quad \le (2a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 + a_7)\delta_{2n} + h^n. \end{aligned} \tag{25}$$

Since by inequality (5), we get

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \le h < 1. \tag{26}$$

Hence

$$\begin{aligned} 2&(a_1 + a_2 + a_3) + a_4 + a_5 + a_6 + a_7 \\ &= 2(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7) \\ &\quad - (a_4 + a_5 + a_6 + a_7) \\ &\le 2h - (a_4 + a_5 + a_6 + a_7) \\ &\le 2h - h(a_4 + a_5 + a_6 + a_7) \\ &\le h(2 - a_4 - a_5 - a_6 - a_7). \end{aligned} \tag{27}$$

Then from (27), we get

$$\begin{aligned} &(2 - a_4 - a_5 - a_6 - a_7)\delta_{2n+1} \\ &\quad \le h(2 - a_4 - a_5 - a_6 - a_7)\delta_{2n} + h^n. \end{aligned} \tag{28}$$

As $1/(2 - a_4 - a_5 - a_6 - a_7) < 1$,

$$\delta_{2n+1} \le h\delta_{2n} + h^n. \tag{29}$$

By the similar process as above, we can show that

$$\delta_{2n} \le h\delta_{2n-1} + h^n. \tag{30}$$

Thus we have

$$\begin{aligned} \delta_{2n+1} &\le h(h\delta_{2n-1} + h^n) + h^n \\ &= h^2\delta_{2n-1} + h^{n+1} + h^n \\ &\le h^2\left(h\delta_{2n-2} + h^{n-1}\right) + h^{n+1} + h^n \\ &= h^3\delta_{2n-2} + 2h^{n+1} + h^n. \end{aligned} \tag{31}$$

Continuing this process, we obtain

$$\begin{aligned} \delta_{2n+1} &\le h^{n+1}\delta_2 + nh^{n+1} + h^n \\ &\le h^{n+1}(h\delta_1 + h) + nh^{n+1} + h^n \\ &= h^{n+2}\delta_1 + h^{n+2} + nh^{n+1} + h^n \\ &\le h^{n+2}(h\delta_0 + 1) + h^{n+2} + nh^{n+1} + h^n \\ &= h^{n+3}\delta_0 + 2h^{n+2} + nh^{n+1} + h^n \\ &\le h^n\delta_0 + (n + 3)h^n. \end{aligned} \tag{32}$$

Similarly

$$\delta_{2n} \le h\left(h\delta_{2n-2} + h^{n-1}\right) + h^n = h^2\delta_{2n-2} + 2h^n. \tag{33}$$

Continuation of this process implies that

$$\begin{aligned}\delta_{2n} &\le h^{2n-1}\delta_1 + (2n-1)h^n \\ &\le h^{2n-1}(h\delta_0 + 1) + (2n-1)h^n \\ &= h^{2n}\delta_0 + h^{2n-1} + (2n-1)h^n \\ &\le h^n\delta_0 + 2nh^n.\end{aligned} \tag{34}$$

By (32) and (34), we have

$$\begin{aligned}\delta_n &\le h^{n+3}\delta_0 + 2h^{n+2} + (n+1)h^n \\ &\le h^n\delta_0 + 2nh^n \quad \forall n \ge 3.\end{aligned} \tag{35}$$

That is,

$$\delta_n \le h^n\delta_0 + 2nh^n \tag{36}$$

holds true for all $n \ge 3$, where

$$\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}). \tag{37}$$

Now for every $m, n \in N$ with $m > n \ge 3$, we have

$$\begin{aligned}&d(gx_n, gx_{m+n}) + d(gy_n, gy_{m+n}) + d(gz_n, gz_{m+n}) \\ &\quad\le d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) \\ &\qquad + \cdots + d(gx_{n+m-1}, gx_{n+m}) + d(gy_n, gy_{n+1}) \\ &\qquad + d(gy_{n+1}, gy_{n+2}) + \cdots + d(gy_{n+m-1}, gy_{n+m}) \\ &\qquad + d(gz_n, gz_{n+1}) + d(gz_{n+1}, gz_{n+2}) \\ &\qquad + \cdots + d(gz_{n+m-1}, gz_{n+m}) \\ &\quad\le (h^n\delta_0 + 2nh^n) + \left(h^{n+1}\delta_0 + 2(n+1)h^{n+1}\right) \\ &\qquad + \cdots + \left(h^{n+m-1}\delta_0 + 2(n+m-1)h^{n+m-1}\right) \\ &\quad\le \left(\sum_{i=n}^{m+n-1} h^i\delta_0 + 2\sum_{i=n}^{m+n-1} ih^i\right).\end{aligned} \tag{38}$$

Since $h < 1$, we conclude that $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$, there exists $x, y, z \in X$ such that $gx_n \to gx$, $gy_n \to gy$, and $gz_n \to gz$. Then from (4), we get

$$d(T(x,y,z), gx) \le d(T(x,y,z), gx_{2n+1}) + d(gx_{2n+1}, gx)$$

$$\begin{aligned}&d(T(x,y,z), gx) \\ &\quad\le H(T(x,y,z), S(x_{2n}, y_{2n}, z_{2n})) + d(gx_{2n+1}, gx) \\ &\quad\le a_1 d(gx, gx_{2n}) + a_2 d(gy, gy_{2n}) + a_3 d(gz, gz_{2n}) \\ &\qquad + a_4 d(T(x,y,z), gx) + a_5 d(S(x_{2n}, y_{2n}, z_{2n}), gx_{2n}) \\ &\qquad + a_6 d(T(x,y,z), gx_{2n}) + a_7 d(S(x_{2n}, y_{2n}, z_{2n}), gx) \\ &\qquad + d(gx_{2n+1}, gx) \\ &\quad\le a_1 d(gx, gx_{2n}) + a_2 d(gy, gy_{2n}) + a_3 d(gz, gz_{2n}) \\ &\qquad + a_4 d(T(x,y,z), gx) + a_5 d(gx_{2n+1}, gx_{2n}) \\ &\qquad + a_6 d(T(x,y,z), gx_{2n}) + a_7 d(gx_{2n+1}, gx) \\ &\qquad + d(gx_{2n+1}, gx).\end{aligned} \tag{39}$$

On taking limits as $n \to \infty$, we get

$$d(T(x,y,z), gx) \le (a_4 + a_6)\, d(T(x,y,z), gx), \tag{40}$$

which implies that

$$d(T(x,y,z), gx) = 0. \tag{41}$$

And hence $gx \in T(x,y,z)$. Similarly $gy \in T(y,z,x)$, $gz \in T(z,x,y)$. And $gx \in S(x,y,z), gy \in S(y,z,x)$, $gz \in S(z,x,y)$. Thus (x,y,z) is a tripled coincidence point of (S,g) and (T,g). Suppose that (i) holds; then, for some $(x,y,z) \in \Upsilon(g,S)$ and $\Upsilon(g,T)$, we have $gx \in S(x,y,z), gy \in S(y,z,x)$, $gz \in S(z,x,y)$, and $gx \in T(x,y,z)$, $gy \in T(y,z,x)$, $gz \in S(z,x,y)$. Since (S,g) and (T,g) are w-compatible, we have

$$\begin{aligned}g(S(x,y,z)) &\subseteq S(gx, gy, gz), \\ g(T(x,y,z)) &\subseteq T(gx, gy, gz),\end{aligned} \tag{42}$$

for $(x,y,z) \in \Upsilon(g,S)$ and $\Upsilon(g,T)$. Since $gx \in S(x,y,z), gy \in S(y,z,x)$ and $gz \in S(z,x,y)$. So $g^2x \in S(gx,gy,gz), g^2y \in S(gy,gz,gx)$, and $g^2z \in S(gz,gx,gy)$ $(\Rightarrow (gx,gy,gz) \in \Upsilon(g,S))$. Similarly $g^2x \in T(gx,gy,gz), g^2y \in T(gy,gz,gx)$, and $g^2z \in T(gz,gx,gy)(\Rightarrow (gx,gy,gz) \in \Upsilon(g,T))$. Continuing in this way, we get

$$\begin{aligned}g^n x &\in S\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), \\ g^n y &\in S\left(g^{n-1}y, g^{n-1}z, g^{n-1}x\right), \\ g^n z &\in S\left(g^{n-1}z, g^{n-1}x, g^{n-1}y\right),\end{aligned} \tag{43}$$

which implies

$$\begin{aligned}\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right) &\in \Upsilon(g,S), \\ \left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right) &\in \Upsilon(g,T)\end{aligned} \tag{44}$$

for all $n \geq 1$ and

$$g^n x \in S\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right),$$
$$g^n x \in T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right). \tag{45}$$

Since $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for $(x, y, z) \in \Upsilon(S, g)$ and $\Upsilon(T, g)$ and g is continuous at u, v and w, so we have $u = gu$, $v = gv$, and $w = gw$. Now using (4), we get

$$\begin{aligned}
d(gu, S(u,v,w)) &\leq d(gu, g^n x) + d(g^n x, S(u,v,w)), \\
d(gu, S(u,v,w)) &\\
&\leq d(gu, g^n x) + H\left(T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), S(u,v,w)\right), \\
d(gu, S(u,v,w)) &\\
&\leq d(gu, g^n x) + a_1 d(g^n x, gu) + a_2 d(g^n y, gv) \\
&\quad + a_3 d(g^n z, gw) + a_4 d\left(T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), g^n x\right) \\
&\quad + a_5 d(S(u,v,z), gu) \\
&\quad + a_6 d\left(T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), gu\right) \\
&\quad + a_7 d(S(u,v,w), g^n x) \\
&\leq d(gu, g^n x) + a_1 d(g^n x, gu) \\
&\quad + a_2 d(g^n y, gv) + a_3 d(g^n z, gw) \\
&\quad + a_4 d(g^n x, g^n x) + a_5 d(S(u,v,z), gu) \\
&\quad + a_6 d(g^n x, gu) + a_7 d(S(u,v,w), g^n x).
\end{aligned} \tag{46}$$

On taking limits as $n \to \infty$, we get

$$d(gu, S(u,v,w)) \leq (a_5 + a_7)\, d(gu, S(u,v,w)), \tag{47}$$

which implies that

$$d(gu, S(u,v,w)) = 0, \tag{48}$$

and hence

$$gu \in S(u,v,w). \tag{49}$$

Similarly,

$$gv \in S(v,w,u), \qquad gw \in S(w,u,v). \tag{50}$$

Consequently,

$$u = gu \in S(u,v,w), \qquad v = gv \in S(v,w,u),$$
$$w = gw \in S(w,u,v). \tag{51}$$

Similarly,

$$u = gu \in T(u,v,w), \qquad v = gv \in T(v,w,u)$$
$$w = gw \in T(w,u,v). \tag{52}$$

Hence (u, v, w) is a tripled fixed point of (S, g) and (T, g). Now suppose that (ii) holds. Since g is S, T-idempotent for some $(x, y, z) \in \Upsilon(g, S)$ and $(x, y, z) \in \Upsilon(g, T)$, we have $g^2 x \in S(gx, gy, gz)$, $g^2 y \in S(gy, gz, gx)$, $g^2 z \in S(gz, gx, gy)$, and $g^2 x \in T(gx, gy, gz)$, $g^2 y \in T(gy, gz, gx)$, $g^2 z \in T(gz, gx, gy)$. Since we have $g^2 x = gx$, $g^2 y = gy$ and $g^2 z = gz$. So

$$gx = g^2 x \in S(gx, gy, gz), \qquad gy = g^2 y \in S(gy, gz, gx),$$
$$gz = g^2 z \in S(gz, gx, gy),$$
$$gx = g^2 x \in T(gx, gy, gz), \qquad gy = g^2 y \in T(gy, gz, gx),$$
$$gz = g^2 z \in T(gz, gx, gy). \tag{53}$$

Hence (gx, gy, gz) is a tripled common fixed point of (S, g) and (T, g). Now suppose that (iii) holds. For some $(x, y, z) \in \Upsilon(g, S)$ and $\Upsilon(g, T)$, we get

$$gx \in S(x,y,z), \qquad gy \in S(y,z,x),$$
$$gz \in S(z,x,y), \qquad gx \in T(x,y,z), \tag{54}$$
$$gy \in T(y,z,x), \qquad gz \in T(z,x,y).$$

Since g is continuous at x and y, we get $gx = x$, $gy = y$, and $gz = z$. Thus

$$x = gx \in S(x,y,z), \qquad y = gy \in S(y,z,x),$$
$$z = gz \in S(z,x,y),$$
$$x = gx \in T(x,y,z), \qquad y = gy \in T(y,z,x), \tag{55}$$
$$z = gz \in T(z,x,y).$$

This implies that (x, y, z) is a tripled common fixed point of (S, g) and (T, g). □

If in Theorem 7, $S = T, g = I$ (the identity mapping), then we have the following result.

Corollary 8. *Let (X, d) be a metric space and $T : X \times X \times X \to CB(X)$ such that*

$$\begin{aligned}
H(T(x,y,z), T(u,v,w)) &\\
&\leq a_1 d(x,u) + a_2 d(y,v) + a_3 d(z,w) \\
&\quad + a_4 d(T(x,y,z), x) + a_5 d(T(u,v,w), u) \\
&\quad + a_6 d(T(x,y,z), u) + a_7 d(T(u,v,z), x),
\end{aligned} \tag{56}$$

for all $x, y, z, u, v, w \in X$, where $a_i = a_i(x, y, z, u, v, w)$, $i = 1, 2, \ldots, 7$, are nonnegative real satisfy (5)*. Then T has a fixed point.*

Corollary 9. *Let (X, d) be a metric space and $S, T : X \times X \times X \to CB(X)$ and $g : X \to X$ be mappings such that*

$$\begin{aligned}
H(S(x,y,z), T(u,v,w)) &\\
&\leq \frac{k}{2}\left[d(gx, gu) + d(gy, gv) + d(gz, gw)\right]
\end{aligned} \tag{57}$$

for all $x, y, z, u, v, w \in X$, where $k \in [0, 1)$. If $S(X \times X \times X) \cup T(X \times X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X. Then (S, g) and (T, g) have tripled coincidence point. Moreover (S, g) and (T, g) have tripled common fixed point if anyone of the conditions (i)–(iii) of Theorem 7 holds.

Theorem 10. *Let (X, d) be a metric space and $S, T : X \times X \times X \to CB(X)$ and $g : X \to X$ be mappings such that*

$$H(S(x, y, z), T(u, v, w)) \leq h \max \left\{ d(gx, gu), d(gy, gv), d(gz, gw), d(S(x, y, z), gx), d(T(u, v, w), gu), \frac{d(S(x, y, z), gx) + d(T(u, v, z), gu)}{2} \right\} \tag{58}$$

for all $x, y, z, u, v, w \in X$, where $h \in [0, 1)$. If $S(X \times X \times X) \cup T(X \times X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X. Then (S, g) and (T, g) have tripled coincidence point. Moreover (S, g) and (T, g) have tripled common fixed point if one of the conditions (i)–(iii) of Theorem 7 holds.

Proof. Let $x_0, y_0, z_0 \in X$ be arbitrary. Choose $x_1, y_1, z_1 \in X$ such that $gx_1 \in S(x_0, y_0, z_0)$, $gy_1 \in S(y_0, z_0, x_0)$ and $gz_1 \in S(z_0, x_0, y_0)$. Note that $T(x_1, y_1, z_1)$, $T(y_1, z_1, x_1)$ and $T(z_1, x_1, y_1)$ are well defined. Choose $x_2, y_2, z_2 \in X$ such that $gx_2 \in T(x_1, y_1, z_1)$, $gy_2 \in T(y_1, z_1, x_1)$, and $gz_2 \in T(z_1, x_1, y_1)$. If $h = 0$, then following similar arguments to those given in Theorem 7, we obtain that (x_1, y_1, z_1) and (x_2, y_2, z_2) are tripled coincidence point (T, g) and (S, g), respectively. Now assume that $h > 0$, set $k = 1/\sqrt{h}$. Then $k > 1$, so there exists

$$\begin{aligned} &t_1 \in T(x_1, y_1, z_1), \qquad t_2 \in T(y_1, z_1, x_1), \\ &t_3 \in T(z_1, x_1, y_1), \qquad t_4 \in S(x_2, y_2, z_2), \\ &t_5 \in S(y_2, z_2, x_2), \qquad t_6 \in S(z_2, x_2, y_2), \end{aligned} \tag{59}$$

such that

$$\begin{aligned} d(gx_1, t_1) &\leq kH(S(x_0, y_0, z_0), T(x_1, y_1, z_1)), \\ d(gy_1, t_2) &\leq kH(S(y_0, z_0, x_0), T(y_1, z_1, x_1)), \\ d(gz_1, t_3) &\leq kH(S(z_0, x_0, y_0), T(z_1, x_1, y_1)), \\ d(gx_2, t_4) &\leq kH(T(x_1, y_1, z_1), S(x_2, y_2, z_2)), \\ d(gy_2, t_5) &\leq kH(T(y_1, z_1, x_1), S(y_2, z_2, x_2)), \\ d(gz_2, t_6) &\leq kH(T(z_1, x_1, y_1), S(z_2, x_2, y_2)). \end{aligned} \tag{60}$$

Since $S(X \times X \times X) \cup T(X \times X \times X) \subseteq g(X)$, there exist x_2, y_2, z_2, x_3, y_3, and z_4 in X such that $t_1 = gx_2$, $t_2 = gy_2$, $t_3 = gz_2$, $t_4 = gx_3$, $t_5 = gy_3$, and $t_6 = gz_3$. Thus

$$\begin{aligned} d(gx_1, gx_2) &\leq kH(S(x_0, y_0, z_0), T(x_1, y_1, z_0)), \\ d(gy_1, gy_2) &\leq kH(S(y_0, z_0, x_0), T(y_1, z_1, x_1)), \\ d(gz_1, gz_2) &\leq kH(S(z_0, x_0, y_0), T(z_1, x_1, y_1)), \\ d(gx_2, gx_3) &\leq kH(T(x_1, y_1, z_1), S(x_2, y_2, z_2)), \\ d(gy_2, gy_3) &\leq kH(T(y_1, z_1, x_1), S(y_2, z_2, x_2)), \\ d(gz_2, gz_3) &\leq kH(T(z_1, x_1, y_1), S(z_2, x_2, y_2)). \end{aligned} \tag{61}$$

Continuing this process, we obtain sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in X as $gx_{2n+1} \in S(x_{2n}, y_{2n}, z_{2n})$, $gy_{2n+1} \in S(y_{2n}, z_{2n}, x_{2n})$, $gz_{2n+1} \in S(z_{2n}, x_{2n}, y_{2n})$, and $gx_{2n+2} \in T(x_{2n+1}, y_{2n+1}, z_{2n+1})$, $gy_{2n+2} \in T(y_{2n+1}, z_{2n+1}, x_{2n+1})$, $gz_{2n+2} \in T(z_{2n+1}, x_{2n+1}, y_{2n+1})$ such that

$$\begin{aligned} &d(gx_{2n+1}, gx_{2n+2}) \\ &\quad \leq kH(S(x_{2n}, y_{2n}, z_{2n}), T(x_{2n+1}, y_{2n+1}, z_{2n+1})), \\ &d(gy_{2n+1}, gy_{2n+2}) \\ &\quad \leq kH(S(y_{2n}, z_{2n}, x_{2n}), T(y_{2n+1}, z_{2n+1}, x_{2n+1})), \\ &d(gz_{2n+1}, gz_{2n+2}) \\ &\quad \leq kH(S(z_{2n}, x_{2n}, y_{2n}), T(z_{2n+1}, x_{2n+1}, y_{2n+1})), \\ &d(gx_{2n+2}, gx_{2n+3}) \\ &\quad \leq kH(T(x_{2n+1}, y_{2n+1}, z_{2n+1}), S(x_{2n+2}, y_{2n+2}, z_{2n+2})), \\ &d(gy_{2n+2}, gy_{2n+3}) \\ &\quad \leq kH(T(y_{2n+1}, z_{2n+1}, x_{2n+1}), S(y_{2n+2}, z_{2n+2}, x_{2n+2})), \\ &d(gz_{2n+2}, gz_{2n+3}) \\ &\quad \leq kH(T(z_{2n+1}, x_{2n+1}, y_{2n+1}), S(z_{2n+2}, x_{2n+2}, y_{2n+2})). \end{aligned} \tag{62}$$

From (58), we have

$$\begin{aligned}
& d\left(gx_{2n+1}, gx_{2n+2}\right) \\
&\quad \leq kH\left(S\left(x_{2n}, y_{2n}, z_{2n}\right), T\left(x_{2n+1}, y_{2n+1}, z_{2n+1}\right)\right) \\
&\quad \leq \sqrt{h}\max\Big\{d\left(gx_{2n}, gx_{2n+1}\right), d\left(gy_{2n}, gy_{2n+1}\right), \\
&\qquad d\left(gz_{2n}, gz_{2n+1}\right), d\left(S\left(x_{2n}, y_{2n}, z_{2n}\right), gx_{2n}\right), \\
&\qquad d\left(S\left(x_{2n}, y_{2n}, z_{2n}\right), gx_{2n}\right), \\
&\qquad \left(T\left(x_{2n+1}, y_{2n+1}, z_{2n+1}\right), gx_{2n+1}\right. \\
&\qquad \left. + d\left(S\left(x_{2n}, y_{2n}, z_{2n}\right), gx_{2n}\right)\right)\frac{1}{2}\Big\} \\
&\quad \leq \sqrt{h}\max\Big\{d\left(gx_{2n}, gx_{2n+1}\right), d\left(gy_{2n}, gy_{2n+1}\right), \\
&\qquad d\left(gz_{2n}, gz_{2n+1}\right), d\left(gx_{2n+1}, gx_{2n}\right), \\
&\qquad d\left(gx_{2n+1}, gx_{2n}\right), \left(d\left(gx_{2n+2}, gx_{2n+1}\right)\right. \\
&\qquad \left. + d\left(gx_{2n+1}, gx_{2n}\right)\right)\frac{1}{2}\Big\} \\
&\quad \leq \sqrt{h}\max\left\{d\left(gx_{2n-1}, gx_{2n}\right), d\left(gy_{2n-1}, gy_{2n}\right),\right. \\
&\qquad \left. d\left(gz_{2n-1}, gz_{2n}\right)\right\}, \\
& d\left(gx_{2n}, gx_{2n+1}\right) \\
&\quad \leq \sqrt{h}\max\Big\{d\left(gx_{2n-1}, gx_{2n}\right), d\left(gy_{2n-1}, gy_{2n}\right), \\
&\qquad d\left(gz_{2n-1}, gz_{2n}\right), d\left(gx_{2n}, gx_{2n-1}\right), \\
&\qquad d\left(gx_{2n+1}, gx_{2n}\right), \\
&\qquad \frac{d\left(gx_{2n}, gx_{2n-1}\right) + d\left(gx_{2n+1}, gx_{2n}\right)}{2}\Big\} \\
&\quad \leq \sqrt{h}\max\left\{d\left(gx_{2n-1}, gx_{2n}\right), d\left(gy_{2n-1}, gy_{2n}\right),\right. \\
&\qquad \left. d\left(gz_{2n-1}, gz_{2n}\right), d\left(gx_{2n+1}, gx_{2n}\right)\right\}.
\end{aligned} \tag{63}$$

Hence, if we suppose that $d(gx_{2n}, gx_{2n+1}) \leq \sqrt{h}d(gx_{2n+1}, gx_{2n})$, then

$$d\left(gx_{2n}, gx_{2n+1}\right) = 0. \tag{64}$$

Therefore,

$$\begin{aligned}
& d\left(gx_{2n}, gx_{2n+1}\right) \\
&\quad \leq \sqrt{h}\max\left\{d\left(gx_{2n-1}, gx_{2n}\right), d\left(gy_{2n-1}, gy_{2n}\right),\right. \\
&\qquad \left. d\left(gz_{2n-1}, gz_{2n}\right)\right\}.
\end{aligned} \tag{65}$$

Similarly, we obtain

$$\begin{aligned}
& d\left(gy_{2n}, gy_{2n+1}\right) \\
&\quad \leq \sqrt{h}\max\left\{d\left(gy_{2n-1}, gy_{2n}\right), d\left(gz_{2n-1}, gz_{2n}\right),\right. \\
&\qquad \left. d\left(gx_{2n-1}, gx_{2n}\right)\right\}, \\
& d\left(gz_{2n}, gz_{2n+1}\right) \\
&\quad \leq \sqrt{h}\max\left\{d\left(gz_{2n-1}, gz_{2n}\right), d\left(gx_{2n-1}, gx_{2n}\right),\right. \\
&\qquad \left. d\left(gy_{2n-1}, gy_{2n}\right)\right\}.
\end{aligned} \tag{66}$$

Using (65) and (66), we obtain for all $n \in \mathbb{N}$

$$\begin{aligned}
d\left(gx_n, gx_{n+1}\right) &\leq \left(\sqrt{h}\right)^n\delta, \\
d\left(gy_n, gy_{n+1}\right) &\leq \left(\sqrt{h}\right)^n\delta, \\
d\left(gz_n, gz_{n+1}\right) &\leq \left(\sqrt{h}\right)^n\delta,
\end{aligned} \tag{67}$$

where $\delta = \max\{d(gx_0, gx_1), d(gy_0, gy_1), d(gz_0, gz_1)\}$.Thus for $m, n \in N$ with $m > n$.

$$\begin{aligned}
& d\left(gx_n, gx_{m+n}\right) \\
&\quad \leq d\left(gx_n, gx_{n+1}\right) + d\left(gx_{n+1}, gx_{n+2}\right) \\
&\qquad + \cdots + d\left(gx_{n+m-1}, gx_{n+m}\right) \\
&\quad \leq \left(\sqrt{h}\right)^n\delta + \left(\sqrt{h}\right)^{n+1}\delta + \cdots + \left(\sqrt{h}\right)^{n+m-1}\delta \\
&\quad \leq \sum_{i=n}^{m+n-1}\left(\sqrt{h}\right)^i\delta.
\end{aligned} \tag{68}$$

Hence we conclude that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Similarly, we obtain that $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exists $x, y, z \in X$ such that $gx_n \to x$, $gy_n \to y$, and $gz_n \to z$. Thus from (58),

$$\begin{aligned}
& d\left(T\left(x, y, z\right), gx\right) \\
&\quad \leq d\left(T\left(x, y, z\right), gx_{2n+1}\right) + d\left(gx_{2n+1}, gx\right) \\
&\quad \leq H\left(T\left(x, y, z\right), S\left(x_{2n}, y_{2n}, z_{2n}\right)\right) + d\left(gx_{2n+1}, gx\right) \\
&\quad \leq h\max\Big\{d\left(gx, gx_{2n}\right), d\left(gy, gy_{2n}\right), d\left(gz, gz_{2n}\right), \\
&\qquad d\left(T\left(x, y, z\right), gx\right), d\left(S\left(x_{2n}, y_{2n}, gz_{2n}\right), gx_{2n}\right), \\
&\qquad \left(d\left(T\left(x, y, z\right), gx_{2n}\right)\right. \\
&\qquad \left. + d\left(S\left(x_{2n}, y_{2n}, gz_{2n}\right), gx\right)\right)\frac{1}{2}\Big\} \\
&\qquad + d\left(gx_{2n+1}, gx\right)
\end{aligned}$$

$$\leq h \max \left\{ d\left(gx, gx_{2n}\right), d\left(gy, gy_{2n}\right), d\left(gz, gz_{2n}\right), \right.$$
$$d\left(T\left(x,y,z\right), gx\right), d\left(gx_{2n+1}, gx_{2n}\right),$$
$$\left. \frac{d\left(T\left(x,y,z\right), gx_{2n}\right) + d\left(gx_{2n+1}, gx\right)}{2} \right\}$$
$$+ d\left(gx_{2n+1}, gx\right). \tag{69}$$

On taking limit as $n \to \infty$, we get

$$d\left(T\left(x,y,z\right), gx\right) \leq hd\left(T\left(x,y,z\right), gx\right), \tag{70}$$

which implies that

$$d\left(T\left(x,y,z\right), gx\right) = 0. \tag{71}$$

As $T(x, y, z)$ is closed, $gx \in T(x, y, z)$. Similarly, $gy \in T(y, z, x)$, $gz \in T(z, x, y)$. Therefore (x, y, z) is a tripled coincidence point of T and g. Similarly, (x, y, z) is a tripled coincidence point of S and g. Thus (x, y, z) is a tripled coincidence point of (S, g) and (T, g). Suppose that (i) holds; then, for some $(x, y, z) \in \Upsilon(g, S)$ and $\Upsilon(g, T)$, we have $gx \in S(x, y, z)$, $gy \in S(y, z, x)$, $gz \in S(z, x, y)$ and $gx \in T(x, y, z)$, $gy \in T(y, z, x)$, $gz \in S(z, x, y)$. Since (S, g) and (T, g) are w-compatible, we have

$$\begin{gathered} g\left(S\left(x,y,z\right)\right) \subseteq S\left(gx, gy, gz\right), \\ g\left(T\left(x,y,z\right)\right) \subseteq T\left(gx, gy, gz\right), \end{gathered} \tag{72}$$

for $(x, y, z) \in \Upsilon(g, S)$ and $\Upsilon(g, T)$. Since $gx \in S(x, y, z)$, $gy \in S(y, z, x)$, and $gz \in S(z, x, y)$. So $g^2x \in S(gx, gy, gz)$, $g^2y \in S(gy, gz, gx)$, and $g^2z \in S(gz, gx, gy)$ ($\Rightarrow$ $(gx, gy, gz) \in \Upsilon(g, S)$). Similarly $g^2x \in T(gx, gy, gz)$, $g^2y \in T(gy, gz, gx)$ and $g^2z \in T(gz, gx, gy)$, ($\Rightarrow$ $(gx, gy, gz) \in \Upsilon(g, T)$). Continuing in this way, we get

$$\begin{gathered} \left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right) \in \Upsilon\left(g, S\right), \\ \left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right) \in \Upsilon\left(g, T\right) \end{gathered} \tag{73}$$

for all $n \geq 1$ and

$$\begin{gathered} g^{n}x \in S\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), \\ g^{n}x \in T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right). \end{gathered} \tag{74}$$

And $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for $(x, y, z) \in \Upsilon(S, g)$ and $(x, y, z) \in \Upsilon(T, g)$. Since g is continuous at u, v, and w, we have $u = gu$, $v = gv$, and $w = gw$. Now using (58), we get

$$d\left(gu, S\left(u,v,w\right)\right)$$
$$\leq d\left(gu, g^{n}x\right) + d\left(g^{n}x, S\left(u,v,w\right)\right)$$
$$\leq d\left(gu, g^{n}x\right)$$
$$+ H\left(T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), S\left(u,v,w\right)\right)$$
$$\leq d\left(gu, g^{n}x\right)$$
$$+ h \max \left\{ d\left(g^{n}x, gu\right), d\left(g^{n}y, gv\right), d\left(g^{n}z, gw\right), \right.$$
$$d\left(T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), g^{n}x\right),$$
$$d\left(S\left(u,v,w\right), gu\right),$$
$$\left(d\left(T\left(g^{n-1}x, g^{n-1}y, g^{n-1}z\right), g^{n}x\right)\right.$$
$$\left. + \ d\left(S\left(u,v,w\right), gu\right)\right) \left.\frac{1}{2}\right\},$$
$$\leq d\left(gu, g^{n}x\right)$$
$$+ h \max \left\{ d\left(g^{n}x, gu\right), d\left(g^{n}y, gv\right), d\left(g^{n}z, gw\right), \right.$$
$$d\left(g^{n}x, g^{n}x\right), d\left(S\left(u,v,w\right), gu\right),$$
$$\left. \frac{d\left(g^{n}x, g^{n}x\right) + d\left(S\left(u,v,w\right), gu\right)}{2} \right\}. \tag{75}$$

Taking limit as $n \to \infty$, we get

$$d\left(gu, S\left(u,v,w\right)\right) \leq hd\left(S\left(u,v,w\right), gu\right). \tag{76}$$

Hence

$$d\left(gu, S\left(u,v,w\right)\right) = 0, \tag{77}$$

and therefore

$$gu \in S\left(u,v,w\right). \tag{78}$$

Similarly,

$$gv \in S\left(v,w,u\right), \qquad gw \in S\left(w,u,v\right). \tag{79}$$

Consequently,

$$\begin{gathered} u = gu \in S\left(u,v,w\right), \qquad v = gv \in S\left(v,w,u\right), \\ w = gw \in S\left(w,u,v\right). \end{gathered} \tag{80}$$

Similarly,

$$\begin{gathered} u = gu \in T\left(u,v,w\right), \qquad v = gv \in T\left(v,w,u\right), \\ w = gw \in T\left(w,u,v\right). \end{gathered} \tag{81}$$

Hence (u, v, w) is a tripled common fixed point of (S, g) and (T, g). Now suppose that (ii) holds. Since g is S, T-idempotent for some $(x, y, z) \in \Upsilon(g, S)$ and $(x, y, z) \in \Upsilon(g, T)$, we have $g^2x \in S(gx, gy, gz)$, $g^2y \in S(gy, gz, gx)$, $g^2z \in S(gz, gx, gy)$, and $g^2x \in T(gx, gy, gz)$, $g^2y \in T(gy, gz, gx)$, $g^2z \in T(gz, gx, gy)$. Since we have $g^2x = gx$, $g^2y = gy$ and $g^2z = gz$. So,

$$\begin{aligned} gx &= g^2x \in S(gx, gy, gz), \\ gy &= g^2y \in S(gy, gz, gx), \\ gz &= g^2z \in S(gz, gx, gy), \\ gx &= g^2x \in T(gx, gy, gz), \\ gy &= g^2y \in T(gy, gz, gx), \\ gz &= g^2z \in T(gz, gx, gy). \end{aligned} \tag{82}$$

Hence (gx, gy, gz) is a tripled common fixed point of (S, g) and (T, g). Now suppose that (iii) holds. For some $(x, y, z) \in \Upsilon(g, S)$ and $(x, y, z) \in \Upsilon(g, T)$, we get

$$\begin{aligned} & gx \in S(x, y, z), \qquad gy \in S(y, w, x), \\ & gz \in S(z, x, y), \qquad gx \in T(x, y, z), \\ & gy \in T(y, w, x), \qquad gz \in T(z, x, y). \end{aligned} \tag{83}$$

Since g is continuous at x and y, we get $gx = x$, $gy = y$, and $gz = z$. Thus

$$\begin{aligned} & x = gx \in S(x, y, z), \qquad y = gy \in S(y, w, x), \\ & z = gz \in S(z, x, y), \qquad x = gx \in T(x, y, z), \\ & y = gy \in T(y, w, x), \qquad z = gz \in T(z, x, y). \end{aligned} \tag{84}$$

This implies that (x, y, z) is a tripled common fixed point of (S, g) and (T, g). □

Conclusion. In this paper, we established tripled coincidence and common fixed point results for a pair of hybrid mappings in the context of complete metric spaces. These results are extensions of results in [4, 5, 11] to the case tripled coincidence and common fixed points. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly to writing this paper. All authors read and approved the final paper.

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