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Research Article

Interpolation of Gentle Spaces

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The notion of gentle spaces, introduced by Jaffard, describes what would be an "ideal" function space to work with wavelet coefficients. It is based mainly on the separability, the existence of bases, the homogeneity, and the γ-stability. We prove that real and complex interpolation spaces between two gentle spaces are also gentle. This shows the relevance and the stability of this notion. We deduce that Lorentz spaces $L^{p,q}$ and $H^{p,q}$ spaces are gentle. Further, an application to nonlinear approximation is presented.

1. Introduction

Interpolation is a powerful technique for proving continuity of linear operators. Let us recall some basic notions concerning interpolation between Banach spaces. Let X_0 and X_1 be two Banach spaces. We say that (X_0, X_1) is a compatible couple if X_0 and X_1 are continuously embedded in a common Hausdorff topological vector space $\mathscr{H}$ (we write $X_i \hookrightarrow H$ for $i = 0, 1$).

Let (X_0, X_1) be a compatible couple. Then $X_0 \cap X_1$ and $X_0 + X_1$ are Banach spaces under the norms

$$\|f\|_{X_0\cap X_1} = \max\left\{\|f\|_{X_0}, \|f\|_{X_1}\right\},$$
$$\|f\|_{X_0+X_1} = \inf\left\{\|f_0\|_{X_0} + \|f_1\|_{X_1} : f = f_0 + f_1\right\}, \quad (1)$$

where the infimum extends over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

If (X_0, X_1) is a compatible couple, then a Banach space X is said to be an intermediate space between X_0 and X_1 if X is continuously embedded between $X_0 \cap X_1$ and $X_0 + X_1$.

Let (Y_0, Y_1) be a second compatible couple. A linear operator T defined on $X_0 + X_1$ and taking values in $Y_0 + Y_1$ is said to be admissible with respect to couples (X_0, X_1) and (Y_0, Y_1) if, for each $i = 0, 1$, the restriction of T to X_i is a linear continuous operator from X_i into Y_i (if $X_i = Y_i$, then T is said to be admissible with respect to (X_0, X_1)). One looks for intermediate spaces X and Y of the couples (X_0, X_1) and (Y_0, Y_1), respectively, such that every admissible operator T maps X into Y. The pair (X, Y) is called interpolation pair relative to (X_0, X_1) and (Y_0, Y_1). If $X_i = Y_i$ for $i = 0, 1$ and $X = Y$, then X is called an interpolation space between X_0 and X_1.

Note that X_0, X_1, $X_0 \cap X_1$, and $X_0 + X_1$ are examples of interpolation spaces between X_0 and X_1. Other examples can be constructed by several methods.

In 1926, Riesz found the first interpolation method for $(L^p(d\mu), L^q(d\mu))$. A generalized version was given by Thorin in 1939/1948 and is known as the Convexity Theorem of Riesz and Thorin or the Riesz-Thorin interpolation theorem. There are many extensions of this theorem. In this connection, we mention the Marcinkiewicz interpolation theorem (in 1939) which extends the Riesz-Thorin interpolation theorem to couples of weak L^p-spaces and which was proved by Zygmund in 1956. In 1958, Stein and Weiss generalized the method for couples $(L^p(d\mu), L^q(d\nu))$ with different measures μ and ν. At the end of 1958, Lions gave the first proof of the interpolation theorem for quadratic interpolation between Hilbert spaces. Since then several authors have introduced and developed different interpolation methods for couples of general Banach spaces. We mention here essentially two methods: the real interpolation method introduced by Lions and Peetre and the complex method developed by Lions, Calderón, and Krejn. In general, these methods lead to different interpolation spaces. Let us quote the example of usual complex spaces $L^p = L^p(\mathbb{R}^d)$ of p-integrable functions and homogeneous Bessel-potential spaces $\dot{W}^{s,p} = \dot{W}^{s,p}(\mathbb{R}^d)$. If we denote by $(X_0, X_1)_{[\theta]}$ and $(X_0, X_1)_{\theta,q}$, with $0 < \theta < 1$, the interpolation spaces obtained, respectively, by the complex and the real method, then we have (see for instance [1])

(i) $(L^{p_0}, L^{p_1})_{[\theta]} = L^p$, $1 \le p_0, p_1 \le \infty$ and $1/p = ((1-\theta)/p_0) + (\theta/p_1)$.

(ii) $(L^{p_0}, L^{p_1})_{\theta,q} = L^{p,q}$, $1 \le p_0, p_1 \le \infty$, $1/p = ((1-\theta)/p_0) + (\theta/p_1)$ and $p_0 < q \le \infty$.

(iii) $(\dot{W}^{s,p_0}, \dot{W}^{s,p_1})_{\theta,p} = \dot{W}^{s,p}$, $s \in \mathbb{R}, 1 \le p_0, p_1 \le \infty$ and $(1/p) = ((1-\theta)/p_0) + (\theta/p_1)$.

(iv) $(\dot{W}^{s_0,p}, \dot{W}^{s_1,p})_{\theta,q} = \dot{B}_p^{s,q}$, $s_0 \ne s_1$, $s = (1-\theta)s_0 + \theta s_1$ and $1 \le p, q \le \infty$.

(v) $(\dot{W}^{s_0,p_0}, \dot{W}^{s_1,p_1})_{[\theta]} = \dot{W}^{s,p}$, $s_0 \ne s_1$, $s = (1-\theta)s_0 + \theta s_1$ and $(1/p) = ((1-\theta)/p_0) + (\theta/p_1)$.

Here, $L^{p,q} = L^{p,q}(\mathbb{R}^d)$ and $\dot{B}_p^{s,q} = \dot{B}_p^{s,q}(\mathbb{R}^d)$ are, respectively, Lorentz and homogenous Besov spaces (see [1–3]). The spaces $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, $\dot{W}^{s,p}(\mathbb{R}^d)$ where $s \in \mathbb{R}$ and $1 < p < \infty$, and $\dot{B}_p^{s,q}(\mathbb{R}^d)$ with $s \in \mathbb{R}$ and $0 < p, q \le \infty$ are gentle spaces (see [4]). Nonetheless, inhomogeneous Besov spaces $B_p^{s,q}(\mathbb{R}^d)$ are not gentle because the homogeneity property (the second requirement in Definition 1) is not verified. Recall that

$$B_p^{s,q}\left(\mathbb{R}^d\right) = L^p\left(\mathbb{R}^d\right) + \dot{B}_p^{s,q}\left(\mathbb{R}^d\right) \quad \text{if } s < 0, 1 \le p, q \le \infty,$$
$$B_p^{s,q}\left(\mathbb{R}^d\right) = L^p\left(\mathbb{R}^d\right) \cap \dot{B}_p^{s,q}\left(\mathbb{R}^d\right) \quad \text{if } s > 0, 1 \le p, q \le \infty. \tag{2}$$

This means that the interpolation space between two gentle spaces is not always gentle (recall that, above, we said that X_0, X_1, $X_0 \cap X_1$, and $X_0 + X_1$ are examples of interpolation spaces between X_0 and X_1). However, in the third section of this paper, we will prove that gentleness is stable by real and complex interpolation methods. In the next section, we give all the necessary recalls concerning these methods.

The notion of gentleness was introduced by Jaffard [5]. It describes what would be an "ideal" function space to work with wavelet coefficients, in "any" wavelet basis. This is the case for Sobolev spaces in PDEs (see [6, 7] for instance), and for Besov spaces in statistics, see [8]. Moreover, many signals and images are stored, denoised, or transmitted by their wavelet coefficients (see [9]). One often needs to obtain local or global information on signals or images by conditions bearing on the moduli of their wavelet coefficients. These conditions should be independent of the chosen wavelet basis.

Gentleness is based mainly on separability, existence of bases, homogeneity, and γ-stability; let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all complex valued rapidly decreasing C^∞ functions on $\mathbb{R}^d$. Set

$$\mathcal{S}_\infty\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}\left(\mathbb{R}^d\right) : \int_{\mathbb{R}^d} x^\beta f(x)\, dx = 0 \,\forall \beta \in \mathbb{N}^d \right\}, \tag{3}$$

(where $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$ for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and all $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$). By $\mathcal{S}'_\infty(\mathbb{R}^d)$ we denote its topological dual (also called the space of tempered distributions modulo polynomials).

Definition 1. Let $H \in \mathbb{R}$. A function space E is gentle of order H if we have the following.

(i) E is a Banach or a quasi-Banach space of distributions.

(ii) E is homogeneous of order H; that is, there exists a constant $C > 0$ such that for all $f \in E$, all $a \in \mathbb{R}^d$ and all $r > 0$

$$\|\tau_a f\|_E \le C\|f\|_E, \qquad \|h_r f\|_E \le C r^H \|f\|_E, \tag{4}$$

where τ_a and h_r are the shift and the dilation operators defined by

$$\tau_a f(x) = f(x-a), \qquad h_r f(x) = f(rx). \tag{5}$$

(iii) $\mathcal{S}_\infty(\mathbb{R}^d) \hookrightarrow E \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^d)$.

(iv) If E is separable, then $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in E, and if E is the dual of a separable space F, then $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in F.

(v) There exists $\gamma > 0$ such that E is γ-stable.

The first requirement is explained in the following definition introduced by Bourdaud in [10]. Denote by $\mathcal{D}'(\mathbb{R}^d)$ the dual space of the space of all complex valued compactly supported C^∞ functions on $\mathbb{R}^d$.

Definition 2. A Banach (resp., quasi-Banach) space of distributions is a vector subspace E of $\mathcal{D}'(\mathbb{R}^d)$ endowed with a complete norm (resp., quasi-norm) such that the embedding $E \hookrightarrow \mathcal{D}'(\mathbb{R}^d)$ is continuous.

Recall that a quasi-Banach space is a complete topological vector space endowed with a quasi-norm. A quasi-norm (see [1] page 59) satisfies the requirements of a norm except for the triangular inequality which is replaced by the weaker condition

$$\exists C > 0; \quad \forall x, y \in E,$$
$$\|x+y\|_E \le C\left(\|x\|_E + \|y\|_E\right). \tag{6}$$

We say that $\|\cdot\|_E$ is a p-norm where $0 < p \le 1$ if in addition

$$\forall x, y \in E, \quad \|x+y\|_E^p \le \|x\|_E^p + \|y\|_E^p. \tag{7}$$

Note that a quasi-norm is always equivalent to a p-norm (see [11]). The real Hardy spaces $H^p(\mathbb{R}^d)$ and Besov spaces $\dot{B}_p^{s,p}(\mathbb{R}^d)$, with $s \in \mathbb{R}$ and $0 < p < 1$, are quasi-Banach spaces.

We should point out that the definitions of intermediate and interpolation spaces carry over without change for a compatible couple of quasi-Banach spaces (see [12–14]).

In the second requirement in Definition 1, the shift invariance implies that the definition of pointwise regularity introduced by Jaffard in [5] (which describes how the norm of f (properly renormalized by substracting a polynomial) behaves in small neighbourhoods of a given point x_0) is the same at every point, and the dilation invariance is implicit in pointwise regularity through scaling invariance. This requirement and the third one imply that Meyer-Lemarié wavelets [15] belong to E since they are obtained by translations and dyadic dilations of a basic function (mother wavelet) in $\mathcal{S}_\infty(\mathbb{R}^d)$.

The fourth requirement shows that wavelet bases are either unconditional bases or unconditional $*$-weak bases of gentle spaces. Examples of nonseparable spaces for which wavelets are $*$-weak bases include homogeneous Hölder spaces $\dot{C}^s(\mathbb{R}^d)$ and, more generally, homogeneous Besov spaces $\dot{B}_p^{s,q}(\mathbb{R}^d)$ with $p = \infty$ or $q = \infty$.

The last requirement implies that the characterization of gentle space by wavelet coefficients does not depend on the particular r-smooth wavelet basis (see [15]) which is chosen for $\gamma < r$; let $\mathcal{M}_\gamma$ be the space of infinite matrices $M(\lambda, \lambda')$ indexed by dyadic cubes $\lambda = k2^{-j} + [0, 2^{-j}[^d$ and $\lambda' = k'2^{-j'} + [0, 2^{-j'}[^d$ (where $j, j' \in \mathbb{Z}$ and $k, k' \in \mathbb{Z}^d$) and satisfying

$$\exists C > 0 : \forall \left(\lambda, \lambda'\right) \quad \left|M\left(\lambda, \lambda'\right)\right| \leq C\omega_\gamma\left(\lambda, \lambda'\right), \tag{8}$$

where

$$\omega_\gamma\left(\lambda, \lambda'\right) = \frac{2^{-((d/2)+\gamma)|j-j'|}}{\left(1 + \left(j - j'\right)^2\right)\left(1 + 2^{\inf\{j,j'\}}\left|k2^{-j} - k'2^{-j'}\right|\right)^{d+\gamma}} \tag{9}$$

(here $|\cdot|$ denotes the Euclidean norm). Meyer proved that $\mathcal{M}_\gamma$ is algebra. Besides, he defined $\mathcal{O}p(\mathcal{M}_\gamma)$ as the algebra of bounded operators on $L^2(\mathbb{R}^d)$ whose matrices on a r-smooth wavelet basis (for a $r > \gamma$) belong to $\mathcal{M}_\gamma$ and showed that this definition does not depend on the chosen wavelet basis. In particular, we can use compactly supported wavelet bases.

Definition 3. Let $\gamma > 0$. A Banach or a quasi-Banach space of distributions E is γ-stable if the operators of $\mathcal{O}p(\mathcal{M}_\gamma)$ are continuous on E.

In [16], we extended the notion of gentle spaces to include anisotropic homogeneous Besov spaces.

In the fourth section of this paper, we will apply our results to exhibit new examples of gentle spaces, namely, Lorentz spaces $L^{p,q}$ (see [2]) and $H^{p,q}$ spaces (see [17]). We will also prove that if the Jackson and Bernstein inequalities are valid, then "nonlinear approximation space" as defined in [18] associated to a gentle space is gentle. Note that there are different types of nonlinear approximations. The n-term approximation is one of the dominant types. We can mention, for example, approximation by splines with n free knots or by rational functions of degree n; see DeVore and Popov [19]. Approximation by a linear combination with n-term of φ-function was developed by DeVore et al. in [20]. A generalization of n-term approximation (called restricted approximation) by a linear combination of compactly supported biorthogonal wavelets was presented by Cohen et al. in [21]. In this paper, we consider approximation by a linear combination of Lemarié-Meyer wavelets as was done by Kyriazis in [18]. This form of approximation occurs in several applications including image processing, statistical estimation, and numerical solutions of differential equations.

2. Real and Complex Interpolation Methods

Originally, real and complex interpolation methods were developed for Banach spaces. The extension of real interpolation for quasi-Banach spaces causes no serious problem (see [12, 13, 22]). However, for the complex method the situation is quite different (see [23–25]). Let us recall briefly some basic definitions and notations related to these two methods. For more details see [1, 12, 13, 22, 26–28].

Definition 4. Let (X_0, X_1) be a compatible couple of Banach or quasi-Banach spaces.

(1) The K-functional is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$\begin{aligned} K(f,t) &= K(f,t,X_0,X_1) \\ &= \inf\left\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\right\}, \end{aligned} \tag{10}$$

where the infimum extends over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

(2) The J-functional is defined for each $f \in X_0 \cap X_1$ and $t > 0$ by

$$J(f,t) = J(f,t,X_0,X_1) = \max\left\{\|f\|_{X_0}, t\|f\|_{X_1}\right\}. \tag{11}$$

The K-functional and J-functional, introduced By Peetre, are nonnegative concave and increasing functions.

Definition 5. Let (X_0, X_1) be a compatible couple of Banach or quasi-Banach spaces.

(1) Let $0 < \theta < 1$ and $0 < q < \infty$ or let $0 \leq \theta \leq 1$ and $q = \infty$. The space $(X_0, X_1)_{\theta,q,K}$ consists of all $f \in X_0 + X_1$ such that

$$\|f\|_{\theta,q,K} := \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{-\theta}K(f,t)\right)^q \frac{dt}{t}\right)^{1/q}, & 0 < \theta < 1,\ 0 < q < \infty, \\ \displaystyle\sup_{t>0} t^{-\theta}K(f,t), & 0 \leq \theta \leq 1,\ q = \infty \end{cases} \tag{12}$$

is finite.

(2) Let $0 < \theta < 1$ and $0 < q \le \infty$. The space $(X_0, X_1)_{\theta,q,J}$ consists of all $f \in X_0 + X_1$ that are represented by Bochner-integral

$$f = \int_0^\infty u(s)\frac{ds}{s}, \tag{13}$$

where u is measurable with values in $X_0 \cap X_1$ and such that

$$\|f\|_{\theta,q,J} := \inf \begin{cases} \left(\int_0^\infty \left(s^{-\theta} J(u(s), s)\right)^q \frac{ds}{s}\right)^{1/q}, & q < \infty, \\ \sup_{s>0} s^{-\theta} J(u(s), s), & q = \infty \end{cases} \tag{14}$$

is finite, where the infimum is taken over all representations (13) of f.

Remark 6. There is a discrete representation of the space $(X_0, X_1)_{\theta,q,J}$ (see [1]); in fact $f \in (X_0, X_1)_{\theta,q,J}$ if and only if there exists a sequence $(u_\nu)_{\nu\in\mathbb{Z}}$ in $X_0 \cap X_1$ such that

$$f = \sum_{\nu\in\mathbb{Z}} u_\nu, \tag{15}$$

$$\left(\sum_{\nu\in\mathbb{Z}} \left(2^{-\nu\theta} J(u_\nu, 2^\nu)\right)^q\right)^{1/q} < \infty, \quad \text{if } q < \infty \tag{16}$$

or

$$\sup_{\nu\in\mathbb{Z}} 2^{-\nu\theta} J(u_\nu, 2^\nu) < \infty, \quad \text{if } q = \infty. \tag{17}$$

Moreover,

$\|f\|_{\theta,q,J}$ is equivalent to

$$\inf \begin{cases} \left(\sum_{\nu\in\mathbb{Z}} \left(2^{-\nu\theta} J(u_\nu, 2^\nu)\right)^q\right)^{1/q}, & q < \infty, \\ \sup_{\nu\in\mathbb{Z}} 2^{-\nu\theta} J(u_\nu, 2^\nu), & q = \infty, \end{cases} \tag{18}$$

where the infimum extends over all sequences $(u_\nu)_{\nu\in\mathbb{Z}}$ satisfying (15).

The following result is given in [1] (see also [29]).

Theorem 7. *Let (X_0, X_1) be a compatible couple of Banach (resp., quasi-Banach) spaces. Then, spaces $(X_0, X_1)_{\theta,q,K}$, with $0 < \theta < 1$ and $0 < q < \infty$ or $0 \le \theta \le 1$ and $q = \infty$, and $(X_0, X_1)_{\theta,q,J}$, with $0 < \theta < 1$ and $0 < q \le \infty$, equipped, respectively, by the norms (resp., quasi-norms) (12) and (14) are Banach (resp., quasi-Banach) spaces and are interpolation spaces between X_0 and X_1.*

Furthermore, if $0 < \theta < 1$ and $0 < q \le \infty$, then $(X_0, X_1)_{\theta,q,K} = (X_0, X_1)_{\theta,q,J}$ with equivalence of norms (resp., quasi-norms).

Real interpolation method means either the K- or J-method. We will write $(X_0, X_1)_{\theta,q}$ instead of $(X_0, X_1)_{\theta,q,K}$ or $(X_0, X_1)_{\theta,q,J}$, if $0 < \theta < 1$. If $\theta = 0$ or 1 and $q = \infty$, then $(X_0, X_1)_{\theta,q}$ denotes $(X_0, X_1)_{\theta,q,K}$. By $\|\cdot\|_{\theta,q}$ we denote the norm or quasi-norm on $(X_0, X_1)_{\theta,q}$ depending whether this space is Banach or quasi-Banach.

In the complex case, we will first restrict ourselves to Banach spaces. There are two interpolation spaces whose norms are equivalent under some conditions. Set

$$S = \{z \in \mathbb{C}, 0 \le \Re(z) \le 1\}, \qquad S_0 = \{z \in \mathbb{C}, 0 < \Re(z) < 1\}. \tag{19}$$

Let (X_0, X_1) be a compatible couple of Banach spaces. We denote by $\mathscr{F}$ the space of all functions $f : S \to X_0 + X_1$ that are bounded, continuous on S, and analytic on S_0, such that functions $t \mapsto f(k + it)$, $k = 0, 1$, from $\mathbb{R}$ into X_k, are continuous and tend to zero as $|t| \to \infty$.

By $\mathscr{G}$ we denote the space of all functions $g : S \to X_0 + X_1$ that are continuous on S and analytic on S_0, satisfying

$$\|g(z)\|_{X_0+X_1} \le C(1 + |z|) \tag{20}$$

such that $g(k + it_1) - g(k + it_2)$ has values in X_k, $k = 0, 1$, for any $-\infty < t_1 < t_2 < \infty$.

Spaces $\mathscr{F}$ and $\mathscr{G}$ provided, respectively, with

$$\|f\|_{\mathscr{F}} := \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{X_1}\right\}, \tag{21}$$

$$\|g\|_{\mathscr{G}} := \max\left\{\sup_{t_1,t_2\in\mathbb{R}} \left\|\frac{g(it_1) - g(it_2)}{t_1 - t_2}\right\|_{X_0}, \sup_{t_1,t_2\in\mathbb{R}} \left\|\frac{g(1 + it_1) - g(1 + it_2)}{t_1 - t_2}\right\|_{X_1}\right\} \tag{22}$$

are Banach spaces.

Definition 8. Let (X_0, X_1) be a compatible couple of Banach spaces. For all $0 \le \theta \le 1$ (resp. $0 < \theta < 1$) we define $(X_0, X_1)_{[\theta]}$ (resp., $(X_0, X_1)^{[\theta]}$) as the space of all $x \in X_0 + X_1$ such that $x = f(\theta)$ (resp., $x = g'(\theta)$) for some $f \in \mathscr{F}$ (resp., $g \in \mathscr{G}$).

Theorem 9. *Let (X_0, X_1) be a compatible couple of Banach spaces. Then, spaces $(X_0, X_1)_{[\theta]}$, where $0 \le \theta \le 1$ and $(X_0, X_1)^{[\theta]}$, where $0 < \theta < 1$, equipped respectively with*

$$\|x\|_{[\theta]} := \inf\{\|f\|_{\mathscr{F}} : f(\theta) = x, f \in \mathscr{F}\}, \tag{23}$$

$$\|x\|^{[\theta]} := \inf\{\|g\|_{\mathscr{G}} : g'(\theta) = x, g \in \mathscr{G}\} \tag{24}$$

are Banach spaces and are interpolation spaces between X_0 and X_1.

Remark 10. We have $(X_0, X_1)_{[\theta]} \subset (X_0, X_1)^{[\theta]}$. In general, $(X_0, X_1)_{[\theta]}$ and $(X_0, X_1)^{[\theta]}$ are not equal. However, if either X_0 or X_1 is reflexive and if $0 < \theta < 1$, then $(X_0, X_1)_{[\theta]} = (X_0, X_1)^{[\theta]}$ and $\|x\|^{[\theta]} = \|x\|_{[\theta]}$, for all $x \in X_0 + X_1$.

The extension of this method to quasi-Banach spaces is not routine; one cannot use duality as was done for Banach spaces. The duality theorem is not true in general in the quasi-Banach case, and the maximum principle fails for functions taking values in a quasi-Banach space (see [30]).

There are several possible ways to define complex interpolation spaces. For example, in [23] complex interpolation was defined in the framework of Fourier analysis, while in [24] complex interpolation was defined as in the Banach setting but by adding in $\|f\|_{\mathcal{F}}$ (which was given in (21)) a third term $\sup_{z\in S_0}\|f(z)\|_{X_0+X_1}$. In [25], the authors described a new approach to interpolate by the complex method some quasi-Banach spaces; let $A(S)$ be the space of all scalar valued functions f continuous and bounded on S and analytic on S_0. Let (X_0, X_1) be a compatible couple of quasi-Banach spaces. Denote by $\mathscr{A}(X_0, X_1)$ the collection of all functions f that can be written as a finite sum $f(z) = \sum_k f_k(z)a_k$ where $f_k \in A(S)$ and $a_k \in X_0 \cap X_1$. We put

$$\|f\|_{\mathscr{A}(X_0, X_1)} := \max\left\{\sup_{t\in\mathbb{R}}\|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}}\|f(1+it)\|_{X_1}\right\}, \tag{25}$$

and for all $x \in X_0 \cap X_1$ and $0 < \theta < 1$ let

$$\|x\|_\theta := \inf\left\{\|f\|_{\mathscr{A}(X_0, X_1)} : f(\theta) = x, f \in \mathscr{A}(X_0, X_1)\right\}. \tag{26}$$

This functional is a semi-quasi-norm. Let

$$N = \{x \in X_0 \cap X_1 : \|x\|_\theta = 0\}; \tag{27}$$

then, $((X_0\cap X_1)/N, \|\cdot\|_\theta)$ is a quasi-normed space. We define $[X_0, X_1]_\theta$ as the completion of $((X_0\cap X_1)/N, \|\cdot\|_\theta)$ (see [31]).

3. Real and Complex Interpolation between Gentle Spaces

Assume that E_0 and E_1 are two gentle spaces. Let E be an interpolation space between them. Then, E is a Banach or a quasi-Banach space of distributions and $\mathcal{S}_\infty(\mathbb{R}^d) \hookrightarrow E \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^d)$.

On the other hand, there exist two constants $\gamma_0 > 0$ and $\gamma_1 > 0$ such that E_i is γ_i-stable, $i = 0, 1$. Set $\gamma = \min\{\gamma_0, \gamma_1\}$. If $T \in \mathcal{O}p(\mathcal{M}_\gamma)$, then $T \in \mathcal{O}p(\mathcal{M}_{\gamma_i})$. So T is continuous on E_i and therefore continuous on E. Thus, E is γ-stable.

We will now prove the homogeneity property.

Proposition 11. *Let E_0 and E_1 be two gentle spaces of order H_0 and H_1, respectively. Then, spaces*

(1) $(E_0, E_1)_{\theta,q}$, *with* $0 < \theta < 1$ *and* $1 \le q < \infty$ *or* $0 \le \theta \le 1$ *and* $q = \infty$,

(2) $(E_0, E_1)_{[\theta]}$, *with* $0 \le \theta \le 1$ *(in the Banach setting),*

(3) $(E_0, E_1)^{[\theta]}$, *with* $0 < \theta < 1$ *(in the Banach setting),*

(4) $[E_0, E_1]_\theta$, *with* $0 < \theta < 1$ *(in the quasi-Banach setting)*

are homogeneous spaces of order $(1-\theta)H_0 + \theta H_1$.

Proof. (1) Let $0 < \theta < 1$ and $0 < q < \infty$ or let $0 \le \theta \le 1$ and $q = \infty$. Let $f \in (E_0, E_1)_{\theta,q}$, $f_0 \in E_0$, and $f_1 \in E_1$ such that $f = f_0 + f_1$. Let $t > 0$. From the homogeneity of E_0 and E_1 we get

$$\begin{aligned} K(\tau_a(f), t) &\le \|\tau_a(f_0)\|_{E_0} + t\|\tau_a(f_1)\|_{E_1} \\ &\le C\left(\|f_0\|_{E_0} + t\|f_1\|_{E_1}\right), \\ K(h_r(f), t) &\le \|h_r(f_0)\|_{E_0} + t\|h_r(f_1)\|_{E_1} \\ &\le Cr^{H_0}\left(\|f_0\|_{E_0} + tr^{H_1-H_0}\|f_1\|_{E_1}\right). \end{aligned} \tag{28}$$

By taking the infimum over all such decompositions $f = f_0 + f_1$ of f, we obtain

$$\begin{gathered} K(\tau_a(f), t) \le CK(f, t) \qquad \text{so } \|\tau_a(f)\|_{\theta,q} \le C\|f\|_{\theta,q}, \\ K(h_r(f), t) \le Cr^{H_0}K\left(f, tr^{H_1-H_0}\right). \end{gathered} \tag{29}$$

(i) If $0 < \theta < 1$ and $0 < q < \infty$, then

$$\begin{aligned} \|h_r(f)\|_{\theta,q} &= \left(\int_0^\infty \left(t^{-\theta}K(h_r(f), t)\right)^q \frac{dt}{t}\right)^{1/q} \\ &\le Cr^{H_0}\left(\int_0^\infty \left(t^{-\theta}K\left(f, tr^{H_1-H_0}\right)\right)^q \frac{dt}{t}\right)^{1/q}. \end{aligned} \tag{30}$$

It follows, by a simple change of variable, that

$$\begin{aligned} \|h_r(f)\|_{\theta,q} &\le Cr^{H_0}r^{-\theta(H_0-H_1)}\left(\int_0^\infty \left(t^{-\theta}K(f, t)\right)^q \frac{dt}{t}\right)^{1/q} \\ &= Cr^{(1-\theta)H_0+\theta H_1}\|f\|_{\theta,q}. \end{aligned} \tag{31}$$

(i) If $0 \le \theta \le 1$ and $q = \infty$, we obtain

$$\|h_r(f)\|_{\theta,q} = \sup_{t>0} t^{-\theta}K(h_r(f), t) \le Cr^{(1-\theta)H_0+\theta H_1}\|f\|_{\theta,q}. \tag{32}$$

Therefore, the space $(E_0, E_1)_{\theta,q}$ is homogeneous of order $(1-\theta)H_0 + \theta H_1$.

(2) Now, let $f \in (E_0, E_1)_{[\theta]}$; there exists $g \in \mathcal{F}$ such that $f = g(\theta)$. Clearly

$$\tau_a(g(z))(x) = g(z)(x-a), \qquad h_r(g(z))(x) = g(z)(rx). \tag{33}$$

Thus, $z \mapsto \tau_a(g(z))$ and $z \mapsto h_r(g(z))$ are analytic functions.

From (23), (21), and the homogeneity of E_0 and E_1, we can easily see that

$$\|\tau_a(f)\|_{[\theta]} \le C\|f\|_{[\theta]}. \tag{34}$$

Put

$$F(z) = r^{(z-1)H_0}r^{-zH_1}h_r(g(z)). \tag{35}$$

Therefore,

$$h_r(f) = r^{(1-\theta)H_0+\theta H_1} F(\theta). \tag{36}$$

Hence,

$$\begin{aligned}\|h_r(f)\|_{[\theta]} &\le r^{(1-\theta)H_0+\theta H_1}\|F\|_{\mathcal{F}} \\ &= r^{(1-\theta)H_0+\theta H_1}\max\left\{\sup_{t\in\mathbb{R}}\|F(it)\|_{E_0},\right. \\ &\qquad \left.\sup_{t\in\mathbb{R}}\|F(1+it)\|_{E_1}\right\}.\end{aligned} \tag{37}$$

It follows from (35) and the homogeneity of E_0 and E_1 that

$$\begin{aligned}\|F(it)\|_{E_0} &= r^{-H_0}\left\|h_r(g(it))\right\|_{E_0} \le C\left\|g(it)\right\|_{E_0}, \\ \|F(1+it)\|_{E_1} &= r^{-H_1}\left\|h_r(g(1+it))\right\|_{E_1} \\ &\le C\left\|g(1+it)\right\|_{E_1}.\end{aligned} \tag{38}$$

This implies that

$$\|F\|_{\mathcal{F}} \le C\|g\|_{\mathcal{F}}, \qquad \left\|h_r(f)\right\|_{[\theta]} \le Cr^{(1-\theta)H_0+\theta H_1}\|g\|_{\mathcal{F}}. \tag{39}$$

Taking the infimum over all $g \in \mathcal{F}$ such that $g(\theta) = f$, we obtain

$$\left\|h_r(f)\right\|_{[\theta]} \le Cr^{(1-\theta)H_0+\theta H_1}\|f\|_{[\theta]}. \tag{40}$$

(3) If $f \in (E_0, E_1)^{[\theta]}$, then there exists $g \in \mathcal{G}$ such that $f = g'(\theta)$. So as previously $z \to \tau_a(g(z))$ and $z \to h_r(g(z))$ are analytic functions. Clearly, from (24), (22), and the homogeneity of E_0 and E_1, we have

$$\left\|\tau_a(f)\right\|^{[\theta]} \le C\|f\|^{[\theta]}. \tag{41}$$

Set

$$G(z) = \int_0^z r^{(\eta-1)H_0}r^{-\eta H_1}h_r\left(g'(\eta)\right)d\eta. \tag{42}$$

Hence,

$$G'(\theta) = r^{(\theta-1)H_0}r^{-\theta H_1}h_r(f). \tag{43}$$

Thus,

$$\left\|h_r(f)\right\|^{[\theta]} \le r^{(1-\theta)H_0+\theta H_1}\|G\|_{\mathcal{G}}. \tag{44}$$

Let $-\infty < t_1 < t_2 < \infty$. From (42) and the fact that E_0 is homogeneous of order H_0, we get

$$\begin{aligned}\left\|G(it_1) - G(it_2)\right\|_{E_0} &\le r^{-H_0}\int_{t_1}^{t_2}\left\|h_r\left(g'(it)\right)\right\|_{E_0}dt \\ &\le C\int_{t_1}^{t_2}\left\|g'(it)\right\|_{E_0}dt \\ &\le C\left\|g(it_2) - g(it_1)\right\|_{E_0}.\end{aligned} \tag{45}$$

Similarly

$$\left\|G(1+it_1) - G(1+it_2)\right\|_{E_1} \le C\left\|g(1+it_2) - g(1+it_1)\right\|_{E_1}. \tag{46}$$

Therefore,

$$\|G\|_{\mathcal{G}} \le C\|g\|_{\mathcal{G}}, \qquad \text{so } \left\|h_r(f)\right\|^{[\theta]} \le Cr^{(1-\theta)H_0+\theta H_1}\|g\|_{\mathcal{G}}. \tag{47}$$

By taking the infimum over all $g \in \mathcal{G}$ such that $g'(\theta) = f$, we obtain

$$\left\|h_r(f)\right\|^{[\theta]} \le Cr^{(1-\theta)H_0+\theta H_1}\|f\|^{[\theta]}. \tag{48}$$

(4) Now, let $f \in [E_0, E_1]_\theta$. There exists a sequence $(f_n)_n \in (E_0 \cap E_1)/N$ such that $\|f - f_n\|_\theta \to 0$. Let $g \in \mathcal{A}(E_0, E_1)$ such that $g(\theta) = f - f_n$ and g is written as a finite sum $g(z) = \sum_k g_k(z)a_k$ where $g_k \in A(S)$ and $a_k \in E_0 \cap E_1$. We define

$$\begin{aligned}\tau_a(g(z)) &= \sum_k \tau_a(g_k(z))a_k, \\ h_r(g(z)) &= \sum_k h_r(g_k(z))a_k,\end{aligned} \tag{49}$$

where

$$\begin{aligned}\tau_a(g_k(z))(x) &= g_k(z)(x-a), \\ h_r(g_k(z))(x) &= g_k(z)(rx),\end{aligned} \tag{50}$$

and $z \mapsto \tau_a(g_k(z))$ and $z \mapsto h_r(g_k(z))$ are analytic functions. By arguing similarly as in $(E_0, E_1)_{[\theta]}$, by taking $f - f_n$ instead of f, we get the desired result. □

Let us now prove the density property.

Proposition 12. *If $q < \infty$, then $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in $(E_0, E_1)_{\theta,q}$.*

Proof. Since $q < \infty$, then $E_0 \cap E_1$ is dense in $(E_0, E_1)_{\theta,q}$ (see e.g., [1]). Therefore it suffices to prove that every function $f \in E_0 \cap E_1$ can be approached by a sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{S}_\infty(\mathbb{R}^d)$ in $(E_0, E_1)_{\theta,q}$. The assumption $q < \infty$ implies that $0 < \theta < 1$, so by the equivalence between real interpolation spaces we will prove this approach in $(E_0, E_1)_{\theta,q,J}$.

On the other hand, since E_0 and E_1 are gentle spaces, then wavelets are either unconditional bases or unconditional $*$-weak bases, depending on the separability of these spaces. Using the γ-stability, we can take Lemarié-Meyer wavelet basis; let ψ^0 and ψ^1 be, respectively, the Lemarié-Meyer [32] father and mother wavelets; that is, ψ^0 and ψ^1 are in the Schwartz class such that all moments of ψ^1 vanish, $\int_{\mathbb{R}}\psi^0(x)dx = 1$, and the collection of the union of $(\psi^0(\cdot - k))_{k\in\mathbb{Z}}$ and $(2^{j/2}\psi^1(2^j\cdot - k))_{j\in\mathbb{N},k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Let V be the set of nonzero vertices of the unit cube in $\mathbb{R}^d$, for each vertex $v = (v_1, \ldots, v_d) \in V$ we set

$$\psi^v(x) := \psi^{v_1}(x_1)\cdots\psi^{v_d}(x_d), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \tag{51}$$

and we define

$$\psi_\lambda^\nu(x) := 2^{dj/2}\psi^\nu\left(2^j x - k\right), \tag{52}$$

indexed by dyadic cube $\lambda = 2^{-j}[k, k+1]$, where $k \in \mathbb{Z}^d$ and $j \in \mathbb{Z}$. Denote by Λ (resp., Λ_j) the set of all dyadic cubes (resp., dyadic cubes at scale j). Then, the collection

$$\{\psi_\lambda^\nu : \lambda \in \Lambda, \nu \in V\} \tag{53}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$. Thus any function $f \in L^2(\mathbb{R}^d)$ can be written as

$$f = \sum_{\lambda\in\Lambda}\sum_{\nu\in V} c_\lambda^\nu \psi_\lambda^\nu, \quad \text{where } c_\lambda^\nu = \langle f, \psi_\lambda^\nu\rangle. \tag{54}$$

Let $f \in E_0 \cap E_1$. Put $f_j = \sum_{\lambda\in\Lambda_j}\sum_{\nu\in V} c_\lambda^\nu\psi_\lambda^\nu$; then, $(f_j)_{j\in\mathbb{Z}}$ belongs to $\mathcal{S}_\infty(\mathbb{R}^d)$, so to $E_0 \cap E_1$ and $f = \sum_{j\in\mathbb{Z}} f_j$. Since $E_0 \cap E_1 \subset (E_0, E_1)_{\theta,q,J}$, we therefore obtain from (15) and (16) that

$$\left(\sum_{j\in\mathbb{Z}}\left(2^{-j\theta}J\left(f_j, 2^j\right)\right)^q\right)^{1/q} < \infty. \tag{55}$$

Set now

$$f_n = \sum_{j=-n}^{n} f_j. \tag{56}$$

It follows from (18) that

$$\|f - f_n\|_{\theta,q,J} \le \left(\sum_{|j|>n}\left(2^{-j\theta}J\left(f_j, 2^j\right)\right)^q\right)^{1/q}, \tag{57}$$

which tends to 0 as n tends to infinity. □

Proposition 13. *Suppose that $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in E_0 and in E_1.*

(1) *Assume that E_0 and E_1 are Banach spaces. Then*

(a) *if $0 \le \theta \le 1$, then $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in $(E_0, E_1)_{[\theta]}$.*

(b) *if $0 < \theta < 1$, then $\mathcal{S}_\infty(\mathbb{R}^d)$ dense in $(E_0, E_1)^{[\theta]}$.*

(2) *Assume that E_0 and E_1 are quasi-Banach spaces. Then $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in $[E_0, E_1]_\theta$, where $0 < \theta < 1$.*

Proof. Let $f \in (E_0, E_1)_{[\theta]}$. There exists $g \in \mathcal{F}$ such that $f = g(\theta)$.

Since $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in both E_0 and E_1, then, for all $\varepsilon > 0$, there exist two functions g_0 and g_1 in $\mathcal{S}_\infty(\mathbb{R}^d)$ such that for all $t \in \mathbb{R}$,

$$\|g(it) - g_0\|_{E_0} \le \varepsilon, \qquad \|g(1+it) - g_1\|_{E_1} \le \varepsilon. \tag{58}$$

Let $H \in \mathcal{F}$ such that H takes values in $\mathcal{S}_\infty(\mathbb{R}^d)$, $H(it) = g_0$, and $H(1+it) = g_1$ for all $t \in \mathbb{R}$. Put $h = H(\theta)$. Thus, from (23), (21), and (58), we obtain

$$\|f - h\|_{[\theta]} \le \|g - H\|_{\mathcal{F}} \le \varepsilon. \tag{59}$$

Since $h \in \mathcal{S}_\infty(\mathbb{R}^d)$, then this implies the density of $\mathcal{S}_\infty(\mathbb{R}^d)$ in $(E_0, E_1)_{[\theta]}$.

Now let $f \in (E_0, E_1)^{[\theta]}$. Arguing similarly as previously but by replacing $\mathcal{F}$ by $\mathcal{G}$, g by g', and (58) by

$$\begin{gathered}\left\|\frac{g(it_1) - g(it_2)}{t_1 - t_2} - g_0\right\|_{E_0} \le \varepsilon, \\ \left\|\frac{g(1+it_1) - g(1+it_2)}{t_1 - t_2} - g_1\right\|_{E_1} \le \varepsilon.\end{gathered} \tag{60}$$

Taking $H \in \mathcal{G}$ such that H takes values in $\mathcal{S}_\infty(\mathbb{R}^d)$, $H(it) = tg_0$ and $H(1+it) = tg_1$ for all $t \in \mathbb{R}$. To finish the proof, it suffices to put $h = H'(\theta)$.

For the proof of the density of $\mathcal{S}_\infty(\mathbb{R}^d)$ in $[E_0, E_1]_\theta$, we argue similarly as in $(E_0, E_1)_{[\theta]}$ by taking $f - f_n$ instead of f, where $(f_n)_n$ is a sequence in $(E_0 \cap E_1)/N$ such that $\|f - f_n\|_\theta \to 0$. □

Consequence. Suppose that $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in both X_0 and X_1, where $X_i' = E_i$, with $i = 0, 1$. Let $0 < \theta < 1$. It follows from the previous proposition that $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in both $(X_0, X_1)_{[\theta]}$ and $(X_0, X_1)^{[\theta]}$. On the other hand, since

$$\mathcal{S}_\infty\left(\mathbb{R}^d\right) \hookrightarrow X_0 \cap X_1 \hookrightarrow X_i, \quad i = 0, 1, \tag{61}$$

then $X_0 \cap X_1$ is dense in X_i, $i = 0, 1$. From the duality theorem (see [1]), it follows that

$$\begin{gathered}(X_0, X_1)'_{[\theta]} = \left(X_0', X_1'\right)^{[\theta]} = (E_0, E_1)^{[\theta]}, \\ \left((X_0, X_1)^{[\theta]}\right)' = (X_0, X_1)'_{[\theta]}.\end{gathered} \tag{62}$$

Furthermore, if either E_0 or E_1 is reflexive (then either X_0 or X_1 is reflexive), then

$$\begin{gathered}(X_0, X_1)'_{[\theta]} = (E_0, E_1)_{[\theta]}, \\ \left((X_0, X_1)^{[\theta]}\right)' = \left(X_0', X_1'\right)^{[\theta]} = (E_0, E_1)^{[\theta]}.\end{gathered} \tag{63}$$

4. Applications

As a first application we show that Lorentz spaces $L^{p,q} = L^{p,q}(\mathbb{R}^d)$ are gentle. This follows from the fact that $L^{p,q} = (L^{p_0}, L^{p_1})_{\theta,q}$ (see [1]), where $0 < \theta < 1$, $1 < p_0, p_1 < \infty$, $(1/p) = ((1-\theta)/p_0) + (\theta/p_1)$ and $p_0 < q \le \infty$ and $L^{p_0} = L^{p_0}(\mathbb{R}^d)$ and $L^{p_1} = L^{p_1}(\mathbb{R}^d)$ (where $1 < p_0, p_1 < \infty$) are gentle spaces (see [4]).

In the second application, we consider the real interpolation between Hardy spaces $H^{p_0} = H^{p_0}(\mathbb{R}^d)$ and

$\mathrm{BMO} = \mathrm{BMO}(\mathbb{R}^d)$ spaces. In [4], we investigated the gentleness property of some functional spaces and we proved the stability of this property by duality. We also proved that homogeneous Lizorkin-Triebel spaces $\dot{F}_p^{s,q}(\mathbb{R}^d)$ (see [33]), where $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$ are gentle spaces. In particular, $H^p(\mathbb{R}^d) = \dot{F}_p^{0,2}(\mathbb{R}^d)$, where $0 < p \leq 1$ are gentle. As a consequence, since BMO is the topological dual of H^1, then BMO is gentle. In [34], it was shown that, for $0 < \theta < 1$, $0 < q \leq \infty$, $0 < p_0 < 1$, and $p = p_0/1 - \theta$, $(H^{p_0}, \mathrm{BMO})_{\theta,q} = H^{p,q}$, where $H^{p,q} = H^{p,q}(\mathbb{R}^d)$ is the space of all tempered distributions f on $\mathbb{R}^d$ such that

$$\sup_{t>0} t^{-d} \left|\phi_t * f\right| \in L^{p,q}, \tag{64}$$

where $\phi_t(x) = \phi(x/t)$ and ϕ is a smooth function with $\int_{\mathbb{R}^d} \phi(x)\,dx \neq 0$ (see [17]). Consequently $H^{p,q}$ are gentle spaces. Note that $H^{p,q} = H^p$ for $p = q$.

Our last application refers to some nonlinear approximation spaces introduced in [18]. Take the Lemarié-Meyer orthonormal basis (53). Let Σ_n be the set of all functions

$$S = \sum_{\lambda\in\Lambda(n)} A_\lambda (S), \tag{65}$$

where $A_\lambda(S) = \sum_{\nu\in V} c_\lambda^\nu \psi_\lambda^\nu$ and $\Lambda(n)$ is a subset of Λ with cardinality $\sharp\Lambda(n) \leq n$. For a given distribution f and any quasi-normed subspace $X \subset \mathcal{S}'_\infty(\mathbb{R}^d)$, we define

$$\sigma_n (f, X) := \inf_{S\in\Sigma_n} \|f - S\|_X. \tag{66}$$

For $0 < q \leq \infty$ and $\alpha > 0$, we define the approximation class $\mathcal{A}_q^\alpha(X)$ as the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^d)$ such that

$$|f|_{\mathcal{A}_q^\alpha(X)} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha}\sigma_{2^j} (f, X)\right)^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{j\alpha}\sigma_{2^j} (f, X), & q = \infty, \end{cases} \tag{67}$$

is finite.

In [18], Kyriazis proved that if X and Y are quasi-normed spaces that verify, for some $r > 0$, Jackson inequality (for all $f \in Y, \sigma_n(f, X) \leq Cn^{-r/d}\|f\|_Y$) and Bernstein inequality (for all $S \in \Sigma_n, \|S\|_Y \leq Cn^{r/d}\|S\|_X$), then for each $0 < \alpha < r$ and $0 < q \leq \infty$

$$\mathcal{A}_q^{\alpha/d} (X) = (X, Y)_{\alpha/r,q}. \tag{68}$$

This result was also proved in [21], where Cohen et al. have used compactly supported biorthogonal wavelets, and in [35], by DeVore and Lorentz, where Y is embedded in X.

In the literature complete characterizations are known for the cases $X = H^p$, $0 < p < \infty$; $X = \dot{B}_p^{0,p}$, $0 < p < \infty$; and $X = \dot{F}_p^{\beta,t}$, $\beta \in \mathbb{R}$, $0 < p < \infty$, and $0 < t \leq \infty$. DeVore et al. have shown, in [20], that for any $0 < \alpha < s$ and any $0 < q \leq \infty$

$$\mathcal{A}_q^{\alpha/d} (H^p) = \left(H^p, \dot{B}_\tau^{s,\tau}\right)_{\alpha/s,q}, \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}. \tag{69}$$

This result was proved by simpler techniques in [21], where the authors proved also that for any $0 < \alpha < s$ and any $0 < q \leq \infty$

$$\mathcal{A}_q^{\alpha/d} \left(\dot{B}_p^{0,p}\right) = \left(\dot{B}_p^{0,p}, \dot{B}_\tau^{s,\tau}\right)_{\alpha/s,q} \tag{70}$$

and for any $\alpha > 0$ and any $0 < q \leq \infty$

$$\mathcal{A}_q^{\alpha/d} (H^p) = \mathcal{A}_q^{\alpha/d} \left(\dot{B}_p^{0,p}\right). \tag{71}$$

Approximation in $\dot{F}_p^{\beta,t}$ was investigated in [18]. More precisely, it has been shown that, for every $0 < p < \infty$, $0 < q$, $t \leq \infty$, $\alpha > 0$, and $\beta \in \mathbb{R}$,

$$\mathcal{A}_q^{\alpha/d} \left(\dot{F}_p^{\beta,t}\right) = \left(\dot{F}_p^{\beta,t}, \dot{B}_\tau^{\gamma,\tau}\right)_{\alpha/(\gamma-\beta),q}, \tag{72}$$

where $\alpha < \gamma - \beta$ and $1/\tau = \gamma - \beta/d + 1/p$.

Since H^p, $\dot{B}_p^{0,p}$, and $\dot{F}_p^{\beta,t}$ are gentle spaces, then their approximation spaces (which are real interpolation spaces) are also gentle. Moreover, the pairs $(H^p, \dot{B}_\tau^{s,\tau})$, $(\dot{B}_p^{0,p}, \dot{B}_\tau^{s,\tau})$, and $(\dot{F}_p^{\beta,t}, \dot{B}_\tau^{\gamma,\tau})$, under the above conditions on all parameters p, q, s, t, τ, and γ, verify the Jackson and Bernstein inequalities. More generally, we deduce that if X and Y are gentle spaces and if the Jackson and Bernstein inequalities are valid, then the approximation space is also gentle.

In [18, 21], the authors restricted themselves to particular wavelets. They remarked that all of their theorems hold in more generality. Using our result, this remark is now confirmed using the gentleness stability by real interpolation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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