

Research Article

Subnormal Weighted Shifts on Directed Trees and Composition Operators in L^2 -Spaces with Nondensely Defined Powers

Piotr Budzyński,¹ Piotr Dymek,^{1,2} Zenon Jan Jabłoński,² and Jan Stochel²

¹ *Katedra Zastosowań Matematyki, Uniwersytet Rolniczy w Krakowie, Ulica Balicka 253c, 30-198 Kraków, Poland*

² *Instytut Matematyki, Uniwersytet Jagielloński, Ulica Łojasiewicza 6, 30-348 Kraków, Poland*

Correspondence should be addressed to Piotr Budzyński; piotr.budzynski@ur.krakow.pl

Received 15 October 2013; Accepted 3 December 2013; Published 19 February 2014

Academic Editor: Henryk Hudzik

Copyright © 2014 Piotr Budzyński et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

It is shown that for every positive integer n there exists a subnormal weighted shift on a directed tree (with or without root) whose n th power is densely defined while its $(n + 1)$ th power is not. As a consequence, for every positive integer n there exists a nonsymmetric subnormal composition operator C in an L^2 -space over a σ -finite measure space such that C^n is densely defined and C^{n+1} is not.

1. Introduction

The question of when powers of a closed densely defined linear operator are densely defined has attracted considerable attention. In 1940 Naimark gave a surprising example of a closed symmetric operator whose square has trivial domain (see [1]; see also [2] for a different construction). More than four decades later, Schmüdgen discovered another pathological behaviour of domains of powers of symmetric operators (cf. [3]). It is well known that symmetric operators are subnormal (cf. [4, Theorem 1 in Appendix I.2]). Hence, closed subnormal operators may have nondensely defined powers. In turn, quasinormal operators, which are subnormal as well (see [5, 6]), have all powers densely defined (cf. [6]). In the present paper we discuss the above question in the context of subnormal weighted shifts on directed trees and subnormal composition operators in L^2 -spaces (over σ -finite measure spaces).

As recently shown (cf. [7, Proposition 3.1]), formally normal (in particular symmetric) weighted shifts on directed trees are automatically bounded and normal (in general, formally normal operators are not subnormal, cf. [8]). The same applies to symmetric composition operators in L^2 -spaces (cf. [9, Proposition B.1]). Formally normal composition operators in L^2 -spaces, which may be unbounded (see [9, Appendix C]), are still normal (cf. [10, Theorem 9.4]). As a

consequence, all powers of such operators are densely defined (see, e.g., [11, Corollary 5.28]).

The above discussion suggests the question of whether for every positive integer n there exists a subnormal weighted shift on a directed tree whose n th power is densely defined while its $(n + 1)$ th power is not. A similar question can be asked for composition operators in L^2 -spaces. To answer both of them, we proceed as follows. First, by applying a recently established criterion for subnormality of weighted composition operators in L^2 -spaces which makes no appeal to density of C^∞ -vectors (see Theorem 1), we show that a densely defined weighted shift on a directed tree which admits a consistent system of probability measures (i.e., a system $\{\mu_v\}_{v \in V}$ of Borel probability measures on \mathbb{R}_+ which satisfies (6)) is subnormal and, what is more, its n th power is densely defined if and only if all moments of these measures up to degree n are finite (cf. Theorem 3). The particular case of directed trees with one branching vertex is examined in Theorem 5 and Corollary 6. Using these two results, we answer both questions in the affirmative (see Example 1 and Remark 8). It is worth pointing out that though directed trees with one branching vertex have simple structure, they provide many examples which are important in operator theory (see e.g., [12, 13]).

Now we introduce some notation and terminology. In what follows, \mathbb{Z} , \mathbb{Z}_+ , \mathbb{N} , \mathbb{R}_+ , and \mathbb{C} stand for the sets of

integers, nonnegative integers, positive integers, nonnegative real numbers and complex numbers, respectively. Set $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. We write $\mathfrak{B}(\mathbb{R}_+)$ for the σ -algebra of all Borel subsets of \mathbb{R}_+ . Given $t \in \mathbb{R}_+$, we denote by δ_t the Borel probability measure on \mathbb{R}_+ concentrated on $\{t\}$.

The domain of an operator A in a complex Hilbert space \mathcal{H} is denoted by $\mathcal{D}(A)$ (all operators considered in this paper are linear). Set $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$. Recall that a closed densely defined operator A in \mathcal{H} is said to be *normal* if $AA^* = A^*A$ (see [11, 14, 15] for more on this class of operators). We say that a densely defined operator A in \mathcal{H} is *subnormal* if there exist a complex Hilbert space \mathcal{K} and a normal operator N in \mathcal{K} such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Ah = Nh$ for all $h \in \mathcal{D}(S)$. We refer the reader to [6, 16–19] for the foundations of the theory of bounded and unbounded subnormal operators, respectively.

2. Weighted Composition Operators

Assume that (X, \mathcal{A}, ν) is a σ -finite measure space, $w : X \rightarrow \mathbb{C}$ is an \mathcal{A} -measurable function, and $\phi : X \rightarrow X$ is an \mathcal{A} -measurable mapping. Define the σ -finite measure $\nu_w : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$ by $\nu_w(\Delta) = \int_\Delta |w|^2 d\nu$ for $\Delta \in \mathcal{A}$. Let $\nu_w \circ \phi^{-1} : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$ be the measure given by $\nu_w \circ \phi^{-1}(\Delta) = \nu_w(\phi^{-1}(\Delta))$ for $\Delta \in \mathcal{A}$. Assume that $\nu_w \circ \phi^{-1}$ is absolutely continuous with respect to ν . By the Radon-Nikodym theorem (cf. [20, Theorem 2.2.1]), there exists a unique (up to a.e. $[\nu]$ equivalence) \mathcal{A} -measurable function $h = h_{\phi,w} : X \rightarrow \overline{\mathbb{R}}_+$ such that

$$\nu_w \circ \phi^{-1}(\Delta) = \int_\Delta h d\nu, \quad \Delta \in \mathcal{A}. \tag{1}$$

Then the operator $C = C_{\phi,w}$ in $L^2(\nu)$, given by

$$\begin{aligned} \mathcal{D}(C) &= \{f \in L^2(\nu) : w \cdot (f \circ \phi) \in L^2(\nu)\}, \\ Cf &= w \cdot (f \circ \phi), \quad f \in \mathcal{D}(C), \end{aligned} \tag{2}$$

is well defined (cf. [21, Proposition 7]). Call C a *weighted composition operator*. By [21, Proposition 10], C is densely defined if and only if $h < \infty$ a.e. $[\nu]$; moreover, if this is the case, then $\nu_w|_{\phi^{-1}(\mathcal{A})}$ is σ -finite and, by the Radon-Nikodym theorem, for every \mathcal{A} -measurable function $f : X \rightarrow \overline{\mathbb{R}}_+$ there exists a unique (up to a.e. $[\nu_w]$ equivalence) $\phi^{-1}(\mathcal{A})$ -measurable function $E(f) = E_{\phi,w}(f) : X \rightarrow \overline{\mathbb{R}}_+$ such that

$$\int_{\phi^{-1}(\Delta)} f d\nu_w = \int_{\phi^{-1}(\Delta)} E(f) d\nu_w, \quad \Delta \in \mathcal{A}. \tag{3}$$

We call $E(f)$ the *conditional expectation* of f with respect to $\phi^{-1}(\mathcal{A})$ (see [21] for more information). A mapping $P : X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$ is called an *\mathcal{A} -measurable family of probability measures* if the set-function $P(x, \cdot)$ is a probability measure for every $x \in X$ and the function $P(\cdot, \sigma)$ is \mathcal{A} -measurable for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$.

The following criterion (read: a sufficient condition) for subnormality of unbounded weighted composition operators is extracted from [21, Theorem 29].

Theorem 1. *If C is densely defined, $h > 0$ a.e. $[\nu_w]$, and there exists an \mathcal{A} -measurable family of probability measures $P : X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$ such that*

$$E(P(\cdot, \sigma))(x) = \frac{\int_\sigma tP(\phi(x), dt)}{h(\phi(x))}, \tag{CC}$$

for ν_w -a.e. $x \in X$, $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,

then C is subnormal.

Regarding Theorem 1, recall that if C is subnormal, then $h > 0$ a.e. $[\nu_w]$ (cf. [21, Corollary 13]).

3. Weighted Shifts on Directed Trees

Let $\mathcal{T} = (V, E)$ be a directed tree (V and E stand for the sets of vertices and edges of \mathcal{T} , resp.). Set $\text{Chi}(u) = \{v \in V : (u, v) \in E\}$ for $u \in V$. Denote by par the partial function from V to V which assigns to a vertex $u \in V$ its parent $\text{par}(u)$ (i.e., a unique $v \in V$ such that $(v, u) \in E$). A vertex $u \in V$ is called a *root* of \mathcal{T} if u has no parent. A root is unique (provided it exists); we denote it by root . Set $V^\circ = V \setminus \{\text{root}\}$ if \mathcal{T} has a root and $V^\circ = V$ otherwise. We say that $u \in V$ is a *branching vertex* of V and write $u \in V_\prec$, if $\text{Chi}(u)$ consists of at least two vertices. We refer the reader to [12] for all facts about directed trees needed in this paper.

By a *weighted shift on \mathcal{T}* with weights $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ we mean the operator S_λ in $\ell^2(V)$ defined by

$$\begin{aligned} \mathcal{D}(S_\lambda) &= \{f \in \ell^2(V) : \Lambda_{\mathcal{T}} f \in \ell^2(V)\}, \\ S_\lambda f &= \Lambda_{\mathcal{T}} f, \quad f \in \mathcal{D}(S_\lambda), \end{aligned} \tag{4}$$

where $\Lambda_{\mathcal{T}}$ is the mapping defined on functions $f : V \rightarrow \mathbb{C}$ via

$$(\Lambda_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\text{par}(v)), & \text{if } v \in V^\circ, \\ 0, & \text{if } v = \text{root}. \end{cases} \tag{5}$$

As usual, $\ell^2(V)$ is the Hilbert space of square summable complex functions on V with standard inner product. For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the one-point set $\{u\}$. Then $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$.

The following useful lemma is an extension of part (iv) of [13, Theorem 3.2.2].

Lemma 2. *Let S_λ be a weighted shift on a directed tree $\mathcal{T} = (V, E)$ with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$ and let $n \in \mathbb{Z}_+$. Then S_λ^n is densely defined if and only if $e_u \in \mathcal{D}(S_\lambda^n)$ for every $u \in V_\prec$.*

Proof. In view of [13, Theorem 3.2.2(iv)], S_λ^n is densely defined if and only if $e_u \in \mathcal{D}(S_\lambda^n)$ for every $u \in V$. Note that if $u \in V$ and $\text{Chi}(u) = \{v\}$, then $e_u \in \mathcal{D}(S_\lambda)$ and $S_\lambda e_u = \lambda_v e_v$, which implies that $e_u \in \mathcal{D}(S_\lambda^{n+1})$ whenever $e_v \in \mathcal{D}(S_\lambda^n)$. In turn, if $\text{Chi}(u) = \emptyset$, then clearly $e_u \in \mathcal{D}^\infty(S_\lambda)$. Using the above and an induction argument (related to paths in \mathcal{T}), we deduce that S_λ^n is densely defined if and only if $e_u \in \mathcal{D}(S_\lambda^n)$ for every $u \in V_\prec$. \square

It is worth mentioning that if $V_{\prec} = \emptyset$, then, by Lemma 2 and [13, Theorem 3.2.2(iv)] (or by the proof of Lemma 2), $\mathcal{D}^{\infty}(S_{\lambda})$ is dense in $\ell^2(V)$. In particular, this covers the case of classical weighted shifts and their adjoints.

Now we give a criterion for subnormality of weighted shifts on directed trees. As opposed to [22, Theorem 5.1.1], we do not assume the density of C^{∞} -vectors in the underlying ℓ^2 -space. Moreover, we do not assume that the underlying directed tree is rootless and leafless, which is required in [9, Theorem 47], and that weights are nonzero. The only restriction we impose is that the directed tree is countably infinite. This is always satisfied if the weighted shift in question is densely defined and has nonzero weights (cf. [12, Proposition 3.1.10]). Here, and later, we adopt the conventions that $0 \cdot \infty = \infty \cdot 0 = 0$, $1/0 = \infty$ and $\sum_{v \in \phi} \xi_v = 0$; we also write \int_0^{∞} in place of $\int_{\mathbb{R}_+}$.

Theorem 3. *Let S_{λ} be a weighted shift on a countably infinite directed tree $\mathcal{T} = (V, E)$ with weights $\lambda = \{\lambda_v\}_{v \in V^{\circ}}$. Suppose that there exist a system $\{\mu_v\}_{v \in V}$ of Borel probability measures on \mathbb{R}_+ and a system $\{\varepsilon_v\}_{v \in V}$ of nonnegative real numbers such that*

$$\mu_u(\sigma) = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_{\sigma} \frac{1}{t} \mu_v(dt) + \varepsilon_u \delta_0(\sigma), \quad (6)$$

$$\sigma \in \mathfrak{B}(\mathbb{R}_+), \quad u \in V.$$

Then the following two assertions hold:

- (i) if S_{λ} is densely defined, then S_{λ} is subnormal,
- (ii) if $n \in \mathbb{N}$, then S_{λ}^n is densely defined if and only if $\int_0^{\infty} s^n d\mu_u(s) < \infty$ for all $u \in V_{\prec}$.

Proof. (i) Assume that S_{λ} is densely defined. Set $X = V$ and $\mathcal{A} = 2^V$. Let $\nu : \mathcal{A} \rightarrow \overline{\mathbb{R}_+}$ be the counting measure on X (ν is σ -finite because V is countable). Define the weight function $w : X \rightarrow \mathbb{C}$ and the mapping $\phi : X \rightarrow X$ by

$$w(x) = \begin{cases} \lambda_x, & \text{if } x \in V^{\circ}, \\ 0, & \text{if } x = \text{root}, \end{cases} \quad (7)$$

$$\phi(x) = \begin{cases} \text{par}(x), & \text{if } x \in V^{\circ}, \\ \text{root}, & \text{if } x = \text{root}. \end{cases}$$

Clearly, the measure $\nu_w \circ \phi^{-1}$ is absolutely continuous with respect to ν and

$$h(x) = \nu_w(\phi^{-1}(\{x\})) = \nu_w(\text{Chi}(x)) = \sum_{y \in \text{Chi}(x)} |\lambda_y|^2, \quad (8)$$

$$x \in X.$$

Thus, by [12, Proposition 3.1.3], $h(x) < \infty$ for every $x \in X$. We claim that $h > 0$ a.e. $[\nu_w]$. This is the same as to show that if $x \in V^{\circ}$ and $\nu_w(\text{Chi}(x)) = 0$, then $\lambda_x = 0$. Thus, if $x \in V^{\circ}$ and $\nu_w(\text{Chi}(x)) = 0$, then applying (6) to $u = x$, we deduce that $\mu_x = \delta_0$; in turn, applying (6) to $u = \text{par}(x)$ with $\sigma = \{0\}$, we get $\lambda_x = 0$, which proves our claim.

Note that $X = \bigsqcup_{x \in X} \phi^{-1}(\{x\})$ (the disjoint union). Hence, the conditional expectation $E(f)$ of a function $f : X \rightarrow \overline{\mathbb{R}_+}$ with respect to $\phi^{-1}(\mathcal{A})$ is given by

$$E(f)(z) = \frac{\int_{\text{Chi}(z)} f d\nu_w}{h(z)}, \quad z \in \phi^{-1}(\{x\}), \quad x \in X_+, \quad (9)$$

where $X_+ := \{x \in X : \nu_w(\text{Chi}(x)) > 0\}$ (see also (8)); on the remaining part of X we can put $E(f) = 0$.

Substituting $\sigma = \{0\}$ into (6), we see that $\mu_y(\{0\}) = 0$ for every $y \in V^{\circ}$ such that $\lambda_y \neq 0$. Thus, using the standard measure-theoretic argument and (6), we deduce that

$$\int_{\sigma} t d\mu_x(t) = \sum_{y \in \text{Chi}(x)} |\lambda_y|^2 \mu_y(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \quad x \in X. \quad (10)$$

Set $P(x, \sigma) = \mu_x(\sigma)$ for $x \in X$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. It follows from (9) and (10) that $P : X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$ is a (\mathcal{A} -measurable) family of probability measures which fulfils the following equality:

$$E(P(\cdot, \sigma))(z) = \frac{\int_{\sigma} t P(\phi(z), dt)}{h(\phi(z))}, \quad z \in \phi^{-1}(\{x\}), \quad x \in X_+. \quad (11)$$

This implies that P satisfies (CC). Hence, by Theorem 1, the weighted composition operator C (see (2)) is subnormal. Since $S_{\lambda} = C$, assertion (i) is proved.

(ii) It is easily seen that if μ is a finite positive Borel measure on \mathbb{R}_+ and $\int_0^{\infty} s^n d\mu(s) < \infty$ for some $n \in \mathbb{N}$, then $\int_0^{\infty} s^k d\mu(s) < \infty$ for every $k \in \mathbb{N}$ such that $k \leq n$. This fact combined with Lemma 2 and [22, Lemmata 2.3.1(i) and 4.2.2(i)] implies assertion (ii). \square

Remark 4. Assume that S_{λ} is a densely defined weighted shift on a countably infinite directed tree $\mathcal{T} = (V, E)$ with weights $\lambda = \{\lambda_v\}_{v \in V^{\circ}}$. A careful inspection of the proof of Theorem 3 reveals that if $\{\mu_x\}_{x \in X}$ (with $X = V$) is a system of Borel probability measures on \mathbb{R}_+ which satisfies (6), then $h > 0$ a.e. $[\nu_w]$, the family P defined by $P(x, \cdot) = \mu_x$ for $x \in X$ satisfies (CC), and $\mu_x = \delta_0$ for every $x \in X \setminus X_+$. We claim that if $h > 0$ a.e. $[\nu_w]$ and $P : X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$ is any family of probability measures which satisfies (CC), then the system $\{\tilde{\mu}_x\}_{x \in X}$ of probability measures defined by

$$\tilde{\mu}_x = \begin{cases} P(x, \cdot), & \text{if } x \in X_+, \\ \delta_0, & \text{otherwise,} \end{cases} \quad (12)$$

satisfies (6) with $\{\tilde{\mu}_x\}_{x \in X}$ in place of $\{\mu_x\}_{x \in X}$. Indeed, (CC) implies (11). Hence, by (9), equality in (10) holds for every $x \in X_+$ with $\mu_x = P(z, \cdot)$ for $z \in X$. This implies via the standard measure-theoretic argument that equality in (6) holds for every $u \in X_+$. Since $h > 0$ a.e. $[\nu_w]$, we deduce that equality in (6) holds for every $u \in X_+$ with $\{\tilde{\mu}_x\}_{x \in X}$ in place of $\{\mu_x\}_{x \in X}$. Clearly, this is also the case for $u \in X \setminus X_+$. Thus, our claim is proved.

4. Trees with One Branching Vertex

Theorem 3 will be applied in the case of weighted shifts on leafless directed trees with one branching vertex. First, we recall the models of such trees (see Figure 1). For $\eta, \kappa \in \mathbb{Z}_+ \cup \{\infty\}$ with $\eta \geq 2$, we define the directed tree $\mathcal{T}_{\eta, \kappa} = (V_{\eta, \kappa}, E_{\eta, \kappa})$ as follows (the symbol “ \sqcup ” denotes disjoint union of sets):

$$\begin{aligned} V_{\eta, \kappa} &= \{-k : k \in J_\kappa\} \sqcup \{0\} \sqcup \{(i, j) : i \in J_\eta, j \in \mathbb{N}\}, \\ E_{\eta, \kappa} &= E_\kappa \sqcup \{(0, (i, 1)) : i \in J_\eta\} \\ &\sqcup \{((i, j), (i, j+1)) : i \in J_\eta, j \in \mathbb{N}\}, \end{aligned} \quad (13)$$

$$E_\kappa = \{(-k, -k+1) : k \in J_\kappa\},$$

where $J_n = \{k \in \mathbb{N} : k \leq n\}$ for $n \in \mathbb{Z}_+ \cup \{\infty\}$. Clearly, $\mathcal{T}_{\eta, \kappa}$ is leafless and 0 is its only branching vertex. From now on, we write $\lambda_{i,j}$ instead of the more formal expression $\lambda_{(i,j)}$ whenever $(i, j) \in V_{\eta, \kappa}$.

Theorem 5. *Let $\eta, \kappa \in \mathbb{Z}_+ \cup \{\infty\}$ be such that $\eta \geq 2$ and let S_λ be a weighted shift on a directed tree $\mathcal{T}_{\eta, \kappa}$ with nonzero weights $\lambda = \{\lambda_v\}_{v \in V_{\eta, \kappa}^\circ}$. Suppose that there exists a sequence $\{\mu_i\}_{i=1}^\eta$ of Borel probability measures on \mathbb{R}_+ such that*

$$\int_0^\infty s^n d\mu_i(s) = \left| \prod_{j=2}^{n+1} \lambda_{i,j} \right|^2, \quad n \in \mathbb{N}, \quad i \in J_\eta, \quad (14)$$

and that one of the following three disjunctive conditions is satisfied:

(i) $\kappa = 0$ and

$$\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s} d\mu_i(s) \leq 1, \quad (15)$$

(ii) $0 < \kappa < \infty$ and

$$\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s} d\mu_i(s) = 1, \quad (16)$$

$$\left| \prod_{j=0}^{l-1} \lambda_{-j} \right|^2 \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{l+1}} d\mu_i(s) = 1, \quad l \in J_{\kappa-1}, \quad (17)$$

$$\left| \prod_{j=0}^{\kappa-1} \lambda_{-j} \right|^2 \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{\kappa+1}} d\mu_i(s) \leq 1, \quad (18)$$

(iii) $\kappa = \infty$ and equalities (16) and (17) are valid.

Then the following two assertions hold:

- (a) if S_λ is densely defined, then S_λ is subnormal,
 (b) if $n \in \mathbb{N}$, then S_λ^n is densely defined if and only if

$$\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty s^{n-1} d\mu_i(s) < \infty. \quad (19)$$

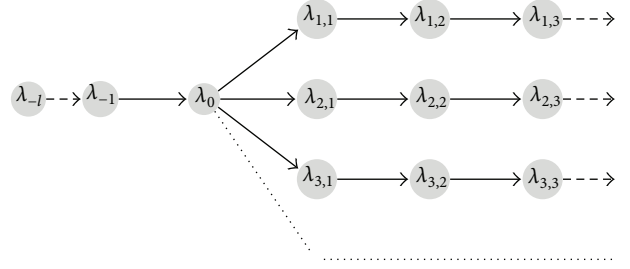


FIGURE 1

Proof. As in the proof of [23, Theorem 4.1], we define the system $\{\mu_v\}_{v \in V_{\eta, \kappa}}$ of Borel probability measures on \mathbb{R}_+ and verify that $\{\mu_v\}_{v \in V_{\eta, \kappa}}$ satisfies (6). Hence, assertion (a) is a direct consequence of Theorem 3(i).

(b) Fix $n \in \mathbb{N}$. It follows from Theorem 3(ii) that S_λ^n is densely defined if and only if $\int_0^\infty s^n d\mu_0(s) < \infty$. Using the explicit definition of μ_0 and applying the standard measure-theoretic argument, we see that

$$\int_0^\infty s^n d\mu_0(s) = \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty s^{n-1} d\mu_i(s). \quad (20)$$

This completes the proof of assertion (b) (the case of $n = 1$ can also be settled without using the definition of μ_0 simply by applying Lemma 2 and [12, Proposition 3.1.3(iii)]). \square

Note that Theorem 5 remains true if its condition (ii) is replaced by the condition (iii) of [23, Theorem 4.1] (see also [23, Lemma 4.2] and its proof).

Corollary 6. *Under the assumptions of Theorem 5, if $n \in \mathbb{N}$, then the following two assertions are equivalent:*

- (i) S_λ^n is densely defined and S_λ^{n+1} is not,
 (ii) the condition (19) holds and $\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty s^n d\mu_i(s) = \infty$.

5. The Example

It follows from [22, Lemma 2.3.1(i)] that if S_λ is a weighted shift on $\mathcal{T}_{\eta, \kappa}$ and $\eta < \infty$, then $\mathcal{D}^\infty(S_\lambda)$ is dense in $\ell^2(V_{\eta, \kappa})$ (this means that Corollary 6 is interesting only if $\eta = \infty$). If $\eta = \infty$, the situation is completely different. Using Theorem 5 and Corollary 6, we show that for every $n \in \mathbb{N}$ and for every $\kappa \in \mathbb{Z}_+ \cup \{\infty\}$, there exists a subnormal weighted shift S_λ on $\mathcal{T}_{\infty, \kappa}$ such that S_λ^n is densely defined and S_λ^{n+1} is not. For this purpose, we adapt [12, Procedure 6.3.1] to the present context. In the original procedure, one starts with a sequence $\{\mu_i\}_{i=1}^\infty$ of Borel probability measures on \mathbb{R}_+ (whose n th moments are finite for every $n \in \mathbb{Z}$ such that $n \geq -(\kappa + 1)$) and then constructs a system of nonzero weights $\lambda = \{\lambda_v\}_{v \in V_{\infty, \kappa}^\circ}$ that satisfies the assumptions of Theorem 5 (in fact, using Lemma 7 below, we can also maintain the condition (19)). However, in general, it is not possible to maintain the condition (ii) of Corollary 6 even if $\{\mu_i\}_{i=1}^\infty$ are measures with two-point supports (this question is not discussed here).

Example 1. Assume that $\eta = \infty$. Consider the measures $\mu_i = \delta_{q_i}$ with $q_i \in (0, \infty)$ for $i \in \mathbb{N}$. By [12, Notation 6.1.9 and Procedure 6.3.1], $S_\lambda \in \mathbf{B}(\ell^2(V_{\infty, \kappa}))$ if and only if $\sup\{q_i : i \in \mathbb{N}\} < \infty$. Hence, there is no loss of generality in assuming that $\sup\{q_i : i \in \mathbb{N}\} = \infty$. To cover all possible choices of $\kappa \in \mathbb{Z}_+ \sqcup \{\infty\}$, we look for a system of nonzero weights $\{\lambda_\nu\}_{\nu \in V_{\infty, \infty}}$ which satisfies (14), (16), (17) with $\kappa = \infty$, (19) and the equality $\sum_{i=1}^\infty |\lambda_{i,1}|^2 \int_0^\infty s^n d\mu_i(s) = \infty$. Setting $\lambda_{i,1} = \sqrt{\alpha_i q_i}$ for $i \in \mathbb{N}$, we reduce our problem to find a sequence $\{\alpha_i\}_{i=1}^\infty \subseteq (0, \infty)$ such that

$$\sum_{i=1}^\infty \alpha_i q_i^l < \infty, \quad l \in \mathbb{Z}, l \leq n, \tag{21}$$

$$\sum_{i=1}^\infty \alpha_i q_i^{n+1} = \infty.$$

Indeed, if $\{\alpha_i\}_{i=1}^\infty$ is such a sequence, then multiplying its terms by an appropriate positive constant, we may assume that $\{\alpha_i\}_{i=1}^\infty$ satisfies (21) and (16). Next we define the weights $\{\lambda_{-j} : j \in \mathbb{Z}_+\}$ recursively so as to satisfy (17) with $\kappa = \infty$, and finally we set $\lambda_{i,j} = \sqrt{q_i}$ for all $i, j \in \mathbb{N}$ such that $j \geq 2$. The so constructed weights $\{\lambda_\nu\}_{\nu \in V_{\infty, \infty}}$ meet our requirements.

The following lemma turns out to be helpful when solving the reduced problem.

Lemma 7. *If $[a_{i,j}]_{i,j=1}^\infty$ is an infinite matrix with entries $a_{i,j} \in \mathbb{R}_+$, then there exists a sequence $\{\alpha_i\}_{i=1}^\infty \subseteq (0, \infty)$ such that*

$$\sum_{i=1}^\infty \alpha_i a_{i,j} < \infty, \quad j \in \mathbb{N}. \tag{22}$$

Proof. First observe that, for every $i \in \mathbb{N}$, there exists $\alpha_i \in (0, \infty)$ such that $\alpha_i \sum_{k=1}^i a_{i,k} \leq 2^{-i}$. Hence, $\sum_{i=j}^\infty \alpha_i a_{i,j} \leq 1$ for every $j \in \mathbb{N}$. \square

Since $\sup\{q_i : i \in \mathbb{N}\} = \infty$, there exists a subsequence $\{q_{i_k}\}_{k=1}^\infty$ of the sequence $\{q_i\}_{i=1}^\infty$ such that $q_{i_k} \geq k$ for every $k \in \mathbb{N}$. Set $\Omega = \{i_k : k \in \mathbb{N}\}$. By Lemma 7, there exists $\{\alpha_i\}_{i \in \mathbb{N} \setminus \Omega} \subseteq (0, \infty)$ such that

$$\sum_{i \in \mathbb{N} \setminus \Omega} \alpha_i q_i^l < \infty, \quad l \in \mathbb{Z}, l \leq n. \tag{23}$$

Define the system $\{\alpha_i\}_{i \in \Omega} \subseteq (0, \infty)$ by

$$\alpha_{i_k} = \frac{1}{k^2 q_{i_k}^n}, \quad k \in \mathbb{N}. \tag{24}$$

Since $q_{i_k} \geq k$ for all $k \in \mathbb{N}$, we get

$$\begin{aligned} \sum_{i \in \Omega} \alpha_i q_i^l &= \sum_{k=1}^\infty \alpha_{i_k} q_{i_k}^l = \sum_{k=1}^\infty \frac{1}{k^2 q_{i_k}^{n-l}} \\ &\leq \sum_{k=1}^\infty \frac{1}{k^2} < \infty, \quad l \in \mathbb{Z}, l \leq n, \end{aligned} \tag{25}$$

$$\sum_{i \in \Omega} \alpha_i q_i^{n+1} = \sum_{k=1}^\infty \alpha_{i_k} q_{i_k}^{n+1} = \sum_{k=1}^\infty \frac{q_{i_k}}{k^2} \geq \sum_{k=1}^\infty \frac{1}{k} = \infty.$$

Combining (23) and (25), we get (21), which solves the reduced problem and consequently gives the required example.

Remark 8. It is worth mentioning that if $\kappa = \infty$, then any weighted shift S_λ on $\mathcal{T}_{\infty, \infty}$ with nonzero weights is unitarily equivalent to an injective composition operator in an L^2 -space over a σ -finite measure space (cf. [13, Lemma 4.3.1]). This fact combined with Example 1 shows that, for every $n \in \mathbb{N}$, there exists a subnormal composition operator C in an L^2 -space over a σ -finite measure space such that C^n is densely defined and C^{n+1} is not.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research of the first author was supported by the NCN (National Science Center) Grant DEC-2011/01/D/ST1/05805. The research of the third and the fourth authors was supported by the MNiSzW (Ministry of Science and Higher Education) Grant NN201 546438 (2010–2013).

References

- [1] M. Naimark, “On the square of a closed symmetric operator,” *Doklady Akademii Nauk SSSR*, vol. 26, pp. 866–870, 1940.
- [2] P. R. Chernoff, “A semibounded closed symmetric operator whose square has trivial domain,” *Proceedings of the American Mathematical Society*, vol. 89, no. 2, pp. 289–290, 1983.
- [3] K. Schmüdgen, “On domains of powers of closed symmetric operators,” *Journal of Operator Theory*, vol. 9, no. 1, pp. 53–75, 1983.
- [4] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, vol. 2, Dover, New York, NY, USA, 1993.
- [5] A. Brown, “On a class of operators,” *Proceedings of the American Mathematical Society*, vol. 4, pp. 723–728, 1953.
- [6] J. Stochel and F. H. Szafraniec, “On normal extensions of unbounded operators. II,” *Acta Universitatis Szegediensis*, vol. 53, no. 1-2, pp. 153–177, 1989.
- [7] Z. J. Jabłoński, I. B. Jung, and J. Stochel, “Normal extensions escape from the class of weighted shifts on directed trees,” *Complex Analysis and Operator Theory*, vol. 7, no. 2, pp. 409–419, 2013.
- [8] E. A. Coddington, “Formally normal operators having no normal extensions,” *Canadian Journal of Mathematics*, vol. 17, pp. 1030–1040, 1965.
- [9] P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, “Unbounded subnormal composition operators in L^2 -spaces,” <http://arxiv.org/abs/1303.6486>.
- [10] P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, “On unbounded composition operators in L^2 -spaces,” *Annali di Matematica Pura ed Applicata*, 2012.
- [11] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, vol. 265 of *Graduate Texts in Mathematics*, Springer, Dordrecht, The Netherlands, 2012.

- [12] Z. J. Jabłoński, I. B. Jung, and J. Stochel, “Weighted shifts on directed trees,” *Memoirs of the American Mathematical Society*, vol. 216, no. 1017, p. viii+107, 2012.
- [13] Z. J. Jabłoński, I. B. Jung, and J. Stochel, “A non-hyponormal operator generating Stieltjes moment sequences,” *Journal of Functional Analysis*, vol. 262, no. 9, pp. 3946–3980, 2012.
- [14] M. Sh. Birman and M. Z. Solomjak, *Spectral Theory of Selfadjoint Operators in Hilbert Space*, Reidel, Dordrecht, The Netherlands, 1987.
- [15] J. Weidmann, *Linear Operators in Hilbert Spaces*, vol. 68, Springer, New York, NY, USA, 1980.
- [16] J. B. Conway, *The Theory of Subnormal Operators*, vol. 36, American Mathematical Society, Providence, RI, USA, 1991.
- [17] J. Stochel and F. H. Szafraniec, “On normal extensions of unbounded operators. I,” *Journal of Operator Theory*, vol. 14, no. 1, pp. 31–55, 1985.
- [18] J. Stochel and F. H. Szafraniec, “On normal extensions of unbounded operators. III. Spectral properties,” *Kyoto University. Research Institute for Mathematical Sciences. Publications*, vol. 25, no. 1, pp. 105–139, 1989.
- [19] J. Stochel and F. H. Szafraniec, “The complex moment problem and subnormality: a polar decomposition approach,” *Journal of Functional Analysis*, vol. 159, no. 2, pp. 432–491, 1998.
- [20] R. B. Ash, *Probability and Measure Theory*, Harcourt/Academic Press, Burlington, Mass, USA, 2nd edition, 2000.
- [21] P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, “Unbounded weighted composition operators in L^2 -spaces,” <http://arxiv.org/abs/1310.3542>.
- [22] P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, “Unbounded subnormal weighted shifts on directed trees,” *Journal of Mathematical Analysis and Applications*, vol. 394, no. 2, pp. 819–834, 2012.
- [23] P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, “Unbounded subnormal weighted shifts on directed trees. II,” *Journal of Mathematical Analysis and Applications*, vol. 398, no. 2, pp. 600–608, 2013.