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Research Article

Periodic Boundary Value Problems for First-Order Impulsive Functional Integrodifferential Equations with Integral-Jump Conditions

Chatthai Thaiprayoon,[1,2] Decha Samana,[1,2] and Jessada Tariboon[2,3]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand*
[2] *Centre of Excellence in Mathematics, CHE, Sri Ayutthaya Road, Bangkok 10400, Thailand*
[3] *Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*

Correspondence should be addressed to Jessada Tariboon; jessadat@kmutnb.ac.th

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By developing a new comparison result and using the monotone iterative technique, we are able to obtain existence of minimal and maximal solutions of periodic boundary value problems for first-order impulsive functional integrodifferential equations with integral-jump conditions. An example is also given to illustrate our results.

1. Introduction

The theory of impulsive differential equations is now being recognized as not only being richer than the corresponding theory of differential equations without impulses, but also representing a more natural framework for mathematical modelling of many real world phenomena and applications; see [1–5] and the references therein. Monotone iterative technique coupled with the method of upper and lower solutions has provided an effective mechanism to prove constructive existence results for initial and boundary value problems for nonlinear differential equations; see [6]. However, many papers have studied applications of the monotone iterative technique to impulsive problems; see, for example, [7–15]. In those articles, the authors assumed that $\Delta x(t_k) = I_k(x(t_k^-))$, that is, a short-term rapid change of the state (jump condition) at impulse point t_k, depends on the left side of the limit of $x(t_k)$.

In [16–18] the authors discussed some classes of first-order impulsive problems with the impulsive integral conditions:

$$\Delta x(t_k) = I_k\left(\int_{t_k-\tau_k}^{t_k} x(s)\,ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} x(s)\,ds\right), \quad (1)$$

where $0 < \sigma_{k-1} \le (t_k - t_{k-1})/2$ and $0 \le \tau_k \le (t_k - t_{k-1})/2$, $k = 1, 2, \ldots, m$. Furthermore, Thaiprayoon et al. [19] have used such technique to investigate the existence criteria of extremal solutions of multipoint impulsive problems to include multipoint jump conditions

$$\Delta x(t_k) = I_k\left(\sum_{l=1}^{c_k}\rho_l^k x(\eta_l^k)\right), \quad c_k \in \{1, 2, \ldots\}, \quad (2)$$

for $t_{k-1} < \eta_1^k < \eta_2^k < \cdots < \eta_{c_k}^k \le t_k$, $k = 1, 2, \ldots, m$. Recently, Thiramanus and Tariboon [20] have given some results on impulsive differential inequalities with integral-jump conditions of the form:

$$x'(t) \le p(t)m(t) + q(t), \quad t \ne t_k,$$
$$x(t_k^+) \le d_k x(t_k) + c_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + b_k, \quad k = 1, 2, \ldots, \quad (3)$$

where $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$. We note that if $d_k = 1$, $\sigma_k < \tau_k$, $k = 1, 2, \ldots$, then the above inequalities mean that the bound of jump condition at t_k is a functional of past states on the interval $(t_k - \tau_k, t_k - \sigma_k]$ before the impulse point t_k.

In spirit of the results from [20], this paper considers the periodic boundary value problem for first-order impulsive functional integrodifferential equation (PBVP) with integral-jump conditions:

$$\begin{aligned} x'(t) &= f(t, x(t), x(\theta(t)), (Kx)(t), (Sx)(t)), \\ &\qquad t \in J = [0, T], \quad t \neq t_k, \\ \Delta x(t_k) &= I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds\right), \quad k = 1, 2, \ldots, m, \\ x(0) &= x(T), \end{aligned} \tag{4}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f \in C(J \times R^4, R)$, $\theta \in C(J, J)$, $I_k \in C(R, R)$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$,

$$\begin{aligned} (Kx)(t) &= \int_0^t k(t, s)\, x(s)\, ds, \\ (Sx)(t) &= \int_0^T h(t, s)\, x(s)\, ds, \end{aligned} \tag{5}$$

$k(t, s) \in C(D, R^+)$, $h(t, s) \in C(J \times J, R^+)$, $D = \{(t, s) \in R^2, 0 \le s \le t \le T\}$, and $R^+ = [0, +\infty)$.

We first introduce a new concept of lower and upper solutions, then establish a new comparison principle, and discuss the existence and uniqueness of the solutions for first-order impulsive functional integrodifferential equations with integral-jump conditions. By using the method of upper and lower solutions and monotone iterative technique, we obtain the existence of extreme solution of PBVP (4). Finally, we give an example to illustrate the obtained results.

2. Preliminaries

Let $J_0 = J \setminus \{t_1, t_2, \ldots, t_m\}$, $k_0 = \max\{k(t, s); (t, s) \in D\}$, $h_0 = \max\{h(t, s); (t, s) \in J \times J\}$, and $PC(J, R) = \{x : J \to R;\ x(t)$ be continuous everywhere except for some t_k, at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$, and let $PC^1(J, R) = \{x \in PC(J, R);\ x'(t)$ be continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k)$, $k = 1, 2, \ldots, m\}$. Clearly, $PC(J, R)$ is a Banach space with the norm $\|x\|_{PC} = \sup\{x(t) : t \in J\}$. Let $E = PC(J, R) \cap PC^1(J, R)$. A function $x \in E$ is called a solution of PBVP (4) if it satisfies (4).

Definition 1. We say that the functions α, $\beta \in E$ are lower and upper solutions of PBVP (4) if there exist $M > 0$, $W \ge 0$, $N \ge 0$, $L \ge 0$, $L_k \ge 0$, and $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$ such that

$$\begin{aligned} \alpha'(t) &\le f(t, \alpha(t), \alpha(\theta(t)), (K\alpha)(t), (S\alpha)(t)) - a_\alpha(t), \\ &\qquad t \in J_0, \\ \Delta\alpha(t_k) &\le I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha(s)\,ds\right) - b_{\alpha k}, \quad k = 1, 2, \ldots, m, \end{aligned} \tag{6}$$

where

$$a_\alpha(t) = \begin{cases} 0, & \text{if } \alpha(0) \le \alpha(T), \\ \left((M(T-t) + W \times (T - \theta(t)) + 1) \times (T)^{-1} + \dfrac{N\int_0^t k(t,s)(T-s)\,ds}{T} + \dfrac{L\int_0^T h(t,s)(T-s)\,ds}{T} \right) \\ \times [\alpha(0) - \alpha(T)], & \text{if } \alpha(0) > \alpha(T), \end{cases}$$

$$b_{\alpha k} = \begin{cases} 0, & \text{if } \alpha(0) \le \alpha(T), \\ \dfrac{L_k(\tau_k - \sigma_k)}{2T} \times (2(T - t_k) + \tau_k + \sigma_k) \\ \times [\alpha(0) - \alpha(T)], & \text{if } \alpha(0) > \alpha(T), \end{cases}$$

$$\begin{aligned} \beta'(t) &\ge f(t, \beta(t), \beta(\theta(t)), (K\beta)(t), (S\beta)(t)) + \bar{a}_\beta(t), \\ &\qquad t \in J_0, \\ \Delta\beta(t_k) &\ge I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta(s)\,ds\right) + \bar{b}_{\beta k}, \quad k = 1, 2, \ldots, m, \end{aligned} \tag{7}$$

where

$$\bar{a}_\beta(t) = \begin{cases} 0, & \text{if } \beta(0) \ge \beta(T), \\ \left((M(T-t) + W \times (T - \theta(t)) + 1) \times (T)^{-1} + \dfrac{N\int_0^t k(t,s)(T-s)\,ds}{T} + \dfrac{L\int_0^T h(t,s)(T-s)\,ds}{T} \right) \\ \times [\beta(T) - \beta(0)], & \text{if } \beta(0) < \beta(T), \end{cases}$$

$$\bar{b}_{\beta k} = \begin{cases} 0, & \text{if } \beta(0) \ge \beta(T), \\ \dfrac{L_k(\tau_k - \sigma_k)}{2T} \times (2(T - t_k) + \tau_k + \sigma_k) \\ \times [\beta(T) - \beta(0)], & \text{if } \beta(0) < \beta(T). \end{cases} \tag{8}$$

Denote $l = \max\{k : t \ge t_k, k = 1, 2, \ldots\}$. We prove the comparison principle by using the following lemma (see [20]).

Lemma 2. *Let $r \in \{t_0, t_1, \dots, t_m\}$, $c_k \geq 0$, and $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, γ_k, $k = 1, \dots, m$, be constants and let $q \in PC(J, R)$, $x \in PC^1(J, R)$. If*

$$x'(t) \geq q(t), \quad t \in (r, T), \ t \neq t_k,$$
$$\Delta x(t_k) \geq c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\, ds + \gamma_k, \quad t_k \in (r, T), \tag{9}$$

then, for $t \in (r, T]$,

$$\begin{aligned} x(t) \geq{} & x(r^+)\left(\prod_{r<t_k<t}[1 + c_k(\tau_k - \sigma_k)]\right) \\ & + \sum_{r<t_k<t}\Bigg[\prod_{t_k<t_j<t}\left(1 + c_j(\tau_j - \sigma_j)\right) \\ & \times \Bigg([1 + c_k(\tau_k - \sigma_k)] \int_{t_{k-1}}^{t_k-\tau_k} q(s)\, ds \\ & + \int_{t_k-\tau_k}^{t_k-\sigma_k}[1 + c_k(t_k - \sigma_k - s)]\, q(s)\, ds \\ & + \int_{t_k-\sigma_k}^{t_k} q(s)\, ds + \gamma_k\Bigg)\Bigg] + \int_{t_l}^{t} q(s)\, ds. \end{aligned} \tag{10}$$

Now we are in the position to establish a new comparison principle, which plays an important role in monotone iterative technique.

Lemma 3. *Assume that $x \in E$ satisfies*

$$x'(t) \geq Mx(t) + Wx(\theta(t)) + N\int_0^t k(t,s)\, x(s)\, ds + L\int_0^T h(t,s)\, x(s)\, ds + \overline{a}_x(t), \quad t \in J_0, \tag{11}$$

$$\Delta x(t_k) \geq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\, ds + \overline{b}_{xk}, \quad k = 1, 2, \dots, m,$$

where constants $M > 0$, $W \geq 0$, $N \geq 0$, $L \geq 0$, $L_k \geq 0$, and $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, $k = 1, 2, \dots, m$. If

$$\left(\widehat{L} + \sum_{k=1}^{m} B_k + \int_0^T r(s)\, ds\right)\prod_{k=1}^{m} A_k \leq 1, \tag{12}$$

where $\overline{a}_x(t)$, $t \in J_0$, and $\overline{b}_{xk}$ are given by Definition 1 with $\beta = x$ and

$$\begin{aligned} \widehat{L} &= \max\{L_k(\tau_k - \sigma_k); k = 1, 2, \dots, m\}, \\ r(t) &= M + W + N\int_0^t k(t,s)\, ds + L\int_0^T h(t,s)\, ds, \\ A_k &= 1 + L_k(\tau_k - \sigma_k), \\ B_k &= A_k \int_{t_{k-1}}^{t_k-\tau_k} r(s)\, ds \\ &\quad + \int_{t_k-\tau_k}^{t_k-\sigma_k}[1 + L_k(t_k - \sigma_k - s)]\, r(s)\, ds \\ &\quad + \int_{t_k-\sigma_k}^{t_k} r(s)\, ds, \end{aligned} \tag{13}$$

then $x(t) \leq 0$, $t \in J$.

Proof.

Case 1. One has $x(0) \geq x(T)$. Suppose that there exists $t^* \in J$ such that $x(t^*) > 0$ and distinguish two cases.

Case (a). $x(t) \geq 0$ for all $t \in J$, $x \not\equiv 0$; then

$$\begin{aligned} x'(t) &\geq Mx(t) + Wx(\theta(t)) + N\int_0^t k(t,s)\, x(s)\, ds \\ &\quad + L\int_0^T h(t,s)\, x(s)\, ds \geq 0, \quad t \in J_0, \\ x(t_k^+) &\geq x(t_k) + L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\, ds \geq x(t_k), \\ &\qquad k = 1, 2, \dots, m, \end{aligned} \tag{14}$$

so that x is nondecreasing in J, and then $x(T) \geq x(0)$. Since $x(0) \geq x(T)$, then x is a constant function $x(t) = C > 0$, which implies that

$$\begin{aligned} x'(t) = 0 &\geq MC + WC + NC\int_0^t k(t,s)\, ds \\ &\quad + LC\int_0^T h(t,s)\, ds \geq MC > 0, \end{aligned} \tag{15}$$

getting a contradiction.

Case (b). $x(t) < 0$ for some $t \in J$. Let $\inf_{t\in J} x(t) = -\lambda < 0$; then there exists $\overline{t} \in (t_i, t_{i+1}]$, for some i such that $x(\overline{t}) = -\lambda$ or $x(t_i^+) = -\lambda$. Without loss of generality, we only consider $x(\overline{t}) = -\lambda$, and for the case $x(t_i^+) = -\lambda$ the proof is similar.

From (11), it is easy to see that

$$\begin{aligned} x'(t) \geq -\lambda\Bigg(& M + W + N\int_0^t k(t,s)\, ds \\ & + L\int_0^T h(t,s)\, ds\Bigg), \quad t \in J_0. \end{aligned} \tag{16}$$

We consider the inequalities

$$\begin{aligned} x'(t) &\geq -\lambda r(t), \quad t \in J_0, \\ \Delta x(t_k) &\geq L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\, ds, \quad k = 1, 2, \dots, m. \end{aligned} \tag{17}$$

From Lemma 2, we have

$$x(t) \geq x(0) \prod_{0<t_k<t} A_k - \lambda \left[\sum_{0<t_k<t} \left(\prod_{t_k<t_j<t} A_j B_k \right) + \int_{t_l}^{t} r(s)\,ds \right], \quad t \in J. \tag{18}$$

Let $t = \bar{t}$ in (18); then

$$x(\bar{t}) \geq x(0) \prod_{0<t_k<\bar{t}} A_k - \lambda \left[\sum_{0<t_k<\bar{t}} \left(\prod_{t_k<t_j<\bar{t}} A_j B_k \right) + \int_{t_i}^{\bar{t}} r(s)\,ds \right], \tag{19}$$

so that

$$x(0) \prod_{0<t_k<\bar{t}} A_k \leq -\lambda + \lambda \left[\sum_{0<t_k<\bar{t}} \left(\prod_{t_k<t_j<\bar{t}} A_j B_k \right) + \int_{t_i}^{\bar{t}} r(s)\,ds \right]. \tag{20}$$

If $x(0) > 0$, then (20) with $A_k \geq 1$, $B_k \geq 0$ for all k implies

$$1 < \sum_{0<t_k<\bar{t}} \left(\prod_{t_k<t_j<\bar{t}} A_j B_k \right) + \int_{t_i}^{\bar{t}} r(s)\,ds \leq \sum_{k=1}^{m} \left(\prod_{k<j<m} A_j B_k \right) + \int_{0}^{T} r(s)\,ds. \tag{21}$$

This contradicts the condition (12).

Suppose that $x(0) \leq 0$. If $t^* < \bar{t}$ for $t^* \in (t_\nu, t_{\nu+1}]$, then Lemma 2 provides that

$$x(t) \geq x(t_{\nu+1}^{+}) \prod_{t_{\nu+1}<t_k<t} A_k - \lambda \left[\sum_{t_{\nu+1}<t_k<t} \left(\prod_{t_k<t_j<t} A_j B_k \right) + \int_{t_l}^{t} r(s)\,ds \right], \quad t \geq t_{\nu+1}^{+}. \tag{22}$$

Since

$$x(t_{\nu+1}^{+}) \geq x(t_{\nu+1}) + L_{\nu+1} \int_{t_{\nu+1}-\tau_{\nu+1}}^{t_{\nu+1}-\sigma_{\nu+1}} x(s)\,ds \geq x(t_{\nu+1}) - \lambda L_{\nu+1} (\tau_{\nu+1} - \sigma_{\nu+1}), \tag{23}$$

and integrating (11) from t^* into $t_{\nu+1}$, we obtain

$$x(t_{\nu+1}) \geq x(t^*) - \lambda \int_{t^*}^{t_{\nu+1}} r(s)\,ds. \tag{24}$$

Hence,

$$\begin{aligned} x(t) &\geq x(t^*) \left(\prod_{t_{\nu+1}<t_k<t} A_k \right) - \lambda L_{\nu+1} (\tau_{\nu+1} - \sigma_{\nu+1}) \times \left(\prod_{t_{\nu+1}<t_k<t} A_k \right) - \lambda \left(\prod_{t_{\nu+1}<t_k<t} A_k \right) \int_{t^*}^{t_{\nu+1}} r(s)\,ds \\ &\quad - \lambda \left[\sum_{t_{\nu+1}<t_k<t} \left(\prod_{t_k<t_j<t} A_j B_k \right) + \int_{t_l}^{t} r(s)\,ds \right] \\ &\geq x(t^*) \left(\prod_{t_{\nu+1}<t_k<t} A_k \right) - \lambda L_{\nu+1} (\tau_{\nu+1} - \sigma_{\nu+1}) \left(\prod_{t_{\nu+1}<t_k<t} A_k \right) \\ &\quad - \lambda \left[\sum_{t_\nu<t_k<t} \left(\prod_{t_k<t_j<t} A_j B_k \right) + \int_{t_l}^{t} r(s)\,ds \right], \quad t \geq t^*. \end{aligned} \tag{25}$$

We note that if $t \in (t_\nu, t_{\nu+1}]$, then $t_l = t^*$.

Let $t = \bar{t}$ in (25); then

$$\begin{aligned} x(\bar{t}) &\geq x(t^*) \left(\prod_{t_{\nu+1}<t_k<\bar{t}} A_k \right) - \lambda L_{\nu+1} (\tau_{\nu+1} - \sigma_{\nu+1}) \left(\prod_{t_{\nu+1}<t_k<\bar{t}} A_k \right) \\ &\quad - \lambda \left[\sum_{t_\nu<t_k<\bar{t}} \left(\prod_{t_k<t_j<\bar{t}} A_j B_k \right) + \int_{t_l}^{\bar{t}} r(s)\,ds \right]. \end{aligned} \tag{26}$$

From (26), we have

$$\begin{aligned} 0 &< x(t^*) \left(\prod_{t_{\nu+1}<t_k<\bar{t}} A_k \right) \\ &\leq -\lambda + \lambda L_{\nu+1} (\tau_{\nu+1} - \sigma_{\nu+1}) \left(\prod_{t_{\nu+1}<t_k<\bar{t}} A_k \right) \\ &\quad + \lambda \left[\sum_{t_\nu<t_k<\bar{t}} \left(\prod_{t_k<t_j<\bar{t}} A_j B_k \right) + \int_{t_l}^{\bar{t}} r(s)\,ds \right], \end{aligned} \tag{27}$$

which gives

$$1 < \hat{L} \prod_{k=1}^{m} A_k + \sum_{0<t_k<T} \left(\prod_{t_k<t_j<T} A_j B_k \right) + \int_{0}^{T} r(s)\,ds, \tag{28}$$

contradicting condition (12). For the case $\bar{t} < t^*$, the proof is similar.

Case 2. One has $x(0) < x(T)$. Set $v(t) = x(t) + ((T-t)/T)(x(T) - x(0))$. It follows that $v(0) = x(T) = v(T)$, and for $t \in J_0$,

$$\begin{aligned} v'(t) &\geq Mx(t) + Wx(\theta(t)) + N\int_0^t k(t,s)\,x(s)\,ds \\ &\quad + L\int_0^T h(t,s)\,x(s)\,ds + \bar{a}_x(t) - \frac{1}{T}(x(T) - x(0)) \\ &= Mv(t) + Wv(\theta(t)) + N\int_0^t k(t,s)\,v(s)\,ds \\ &\quad + L\int_0^T h(t,s)\,v(s)\,ds + \bar{a}_x(t) \\ &\quad - \frac{M(T-t) + W(T-\theta(t)) + 1}{T}[x(T) - x(0)] \\ &\quad - \frac{N[x(T)-x(0)]}{T}\int_0^t k(t,s)(T-s)\,ds \\ &\quad - \frac{L[x(T)-x(0)]}{T}\int_0^t h(t,s)(T-s)\,ds \\ &\geq Mv(t) + Wv(\theta(t)) + N\int_0^t k(t,s)\,v(s)\,ds \\ &\quad + L\int_0^T h(t,s)\,v(s)\,ds, \end{aligned} \tag{29}$$

and for $t = t_k$, $k = 1, 2, \ldots, m$,

$$\begin{aligned} \Delta v(t_k) = \Delta x(t_k) &\geq L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds \\ &\quad + \frac{L_k(\tau_k - \sigma_k)[2(T-t_k) + \tau_k + \sigma_k]}{2T} \\ &\quad \times (x(T) - x(0)) = L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)\,ds. \end{aligned} \tag{30}$$

In view of Case 1, we see that $v(t) \leq 0$ on J, and therefore $x(t) \leq 0$ on J. This completes the proof. □

Corollary 4. *Assume that $x \in E$ satisfies*

$$\begin{aligned} x'(t) &\geq Mx(t) + Wx(\theta(t)) + N\int_0^t k(t,s)\,x(s)\,ds \\ &\quad + L\int_0^T h(t,s)\,x(s)\,ds, \quad t \in J_0, \\ \Delta x(t_k) &\geq L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds, \quad k = 1, 2, \ldots, m, \\ &\qquad x(0) \geq x(T), \end{aligned} \tag{31}$$

where constants $M > 0$, $W \geq 0$, $N \geq 0$, $L \geq 0$, $L_k \geq 0$, and $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, $k = 1, 2, \ldots, m$. If

$$\left(\widehat{L} + \sum_{k=1}^m B_k^0 + r_0 T\right)\prod_{k=1}^m A_k \leq 1, \tag{32}$$

where

$$\begin{aligned} r_0 &= M + W + k_0NT + h_0LT, \\ B_k^0 &= r_0A_k\int_{t_{k-1}}^{t_k-\tau_k} ds \\ &\quad + r_0\int_{t_k-\tau_k}^{t_k-\sigma_k}[1 + L_k(t_k - \sigma_k - s)]\,ds + r_0\int_{t_k-\sigma_k}^{t_k} ds \\ &= r_0\Big[(t_k - t_{k-1}) + \frac{L_k}{2}(\tau_k - \sigma_k) \\ &\qquad \times (2(t_k - t_{k-1}) - (\tau_k + \sigma_k))\Big], \end{aligned} \tag{33}$$

and $\widehat{L}$, A_k are given by Lemma 3, then $x(t) \leq 0$, for $t \in J$.

Corollary 5. *Assume that $x \in E$ satisfies*

$$\begin{aligned} &x'(t) \\ &\geq Mx(t) + Wx(\theta(t)) \\ &\quad + N\int_0^t k(t,s)\,x(s)\,ds + L\int_0^T h(t,s)\,x(s)\,ds \\ &\quad + \bigg(\frac{M(T-t) + W(T-\theta(t)) + 1}{T} \\ &\quad + \frac{k_0N(2Tt - t^2)}{2T} \\ &\quad + \frac{h_0LT}{2}\bigg)[x(T) - x(0)], \quad t \in J_0, \\ \Delta x(t_k) &\geq L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds \\ &\quad + \frac{L_k(\tau_k - \sigma_k)[2(T-t_k) + \tau_k + \sigma_k]}{2T} \\ &\quad \times (x(T) - x(0)), \quad k = 1, 2, \ldots, m, \\ &\qquad x(0) < x(T), \end{aligned} \tag{34}$$

where constants $M > 0$, $W \geq 0$, $N \geq 0$, $L \geq 0$, $L_k \geq 0$, and $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, $k = 1, 2, \ldots, m$, and condition (32) *holds. Then $x(t) \leq 0$, for $t \in J$.*

Let us consider the following linear problem of PBVP (4):

$$\begin{aligned} x'(t) - Mx(t) &= Wx(\theta(t)) + N\int_0^t k(t,s)\,x(s)\,ds \\ &\quad + L\int_0^T h(t,s)\,x(s)\,ds + g(t), \quad t \in J_0, \\ \Delta x(t_k) &= L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right) \\ &\quad - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds, \quad k = 1,2,\ldots,m, \\ x(0) &= x(T), \end{aligned} \tag{35}$$

where $M > 0$, $W \geq 0$, $N \geq 0$, $L \geq 0$, $L_k \geq 0$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, $\theta \in (J,J)$, $I_k \in C(J,R)$, $k = 1,2,\ldots,m$, $g \in PC(J,R)$, and $\eta \in PC^1(J,R)$.

Lemma 6. *A function $x \in E$ is a solution of* (35) *if and only if $x \in PC(J,R)$ is a solution of the following impulsive integral equation:*

$$\begin{aligned} x(t) &= -\int_0^T G(t,s)\,R(s)\,ds - \sum_{k=1}^m G(t,t_k) \\ &\quad \times \left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right)\right. \\ &\quad \left. - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right], \quad t \in J, \end{aligned} \tag{36}$$

where $R(t) = Wx(\theta(t)) + N\int_0^t k(t,s)x(s)ds + L\int_0^T h(t,s)x(s)ds + g(t)$ *and*

$$G(t,s) = \frac{1}{e^{MT}-1}\begin{cases} e^{M(t-s)}, & 0 \leq s < t \leq T, \\ e^{M(T+t-s)}, & 0 \leq t \leq s \leq T. \end{cases} \tag{37}$$

Proof. If $x(t)$ is a solution of (35), by directly integrating, we obtain

$$\begin{aligned} x(t) &= -\int_0^T G(t,s)\,R(s)\,ds - \sum_{k=1}^m G(t,t_k) \\ &\quad \times \left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right)\right. \\ &\quad \left. - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right], \quad t \in J. \end{aligned} \tag{38}$$

If $x(t)$ is a solution of the above-mentioned integral equation, then

$$\begin{aligned} x'(t) &= M\left\{-\int_0^t \frac{e^{M(t-s)}}{e^{MT}-1}R(s)\,ds - \int_t^T \frac{e^{M(T+t-s)}}{e^{MT}-1}R(s)\,ds\right. \\ &\quad - \sum_{0<t_k<t}\frac{e^{M(t-t_k)}}{e^{MT}-1}\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right) - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right] \\ &\quad \left. - \sum_{t\leq t_k<t}\frac{e^{M(T+t-t_k)}}{e^{MT}-1}\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right) - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right]\right\} \\ &\quad + R(t) \\ &= M\left\{-\int_0^T G(t,s)\,R(s)\,ds - \sum_{k=1}^m G(t,t_k)\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right) - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right]\right\} + R(t) \\ &= Mx(t) + R(t), \\ \Delta x(t_k) &= x(t_k^+) - x(t_k^-) \\ &= \frac{e^{MT}}{e^{MT}-1}\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right) - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right] \\ &\quad - \frac{1}{e^{MT}-1}\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right) - L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} \eta(s)\,ds\right], \\ x(0) &= -\int_0^T \frac{e^{M(T-s)}}{e^{MT}-1}R(s)\,ds + \sum_{k=1}^m \frac{e^{M(T-t_k)}}{e^{MT}-1} \end{aligned}$$

$$\times\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}x(s)\,ds+I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k}\eta(s)\,ds\right)\right.$$
$$\left.-L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\eta(s)\,ds\right]=x(T). \tag{39}$$

The proof is complete. □

Lemma 7. *Let $M>0$, $W\geq 0$, $N\geq 0$, $L\geq 0$, $L_k\geq 0$, $0\leq\sigma_k\leq\tau_k\leq t_k-t_{k-1}$, $\theta\in(J,J)$, $I_k\in C(J,R)$, $k=1,2,\ldots,m$, $g\in PC(J,R)$, and $\eta\in PC^1(J,R)$ and assume that*

$$\sup_{t\in J}\left\{\int_0^T G(t,s)\left(W+N\int_0^s k(s,r)\,dr\right.\right.$$
$$\left.\left.+L\int_0^T h(s,r)\,dr\right)ds\right\} \tag{40}$$
$$+\frac{1}{1-e^{-MT}}\sum_{k=1}^m L_k(\tau_k-\sigma_k)<1.$$

Then problem (35) *has a unique solution in $PC(J,R)$.*

Proof. We define the mapping $F:PC(J,R)\to PC(J,R)$ by

$$(Fx)(t)=-\int_0^T G(t,s)\left(Wx(\theta(s))+N\int_0^s k(s,r)x(r)\,dr\right.$$
$$\left.+L\int_0^T h(s,r)x(r)\,dr+g(s)\right)ds$$
$$-\sum_{k=1}^m G(t,t_k)\left(L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}x(s)\,ds\right.$$
$$+I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k}\eta(s)\,ds\right)$$
$$\left.-L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\eta(s)\,ds\right),\quad t\in J, \tag{41}$$

for any $x\in PC(J,R)$ and G is given by Lemma 6. Then

$$\|Fx-Fy\|_{PC}$$
$$=\sup_{t\in J}\left|-\int_0^T G(t,s)\left(W[x(\theta(s))-y(\theta(s))]\right.\right.$$
$$+N\int_0^s k(s,r)[x(r)-y(r)]\,dr$$
$$\left.+L\int_0^T h(s,r)[x(r)-y(r)]\,dr\right)ds$$
$$\left.-\sum_{k=1}^m G(t,t_k)\left(L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}[x(s)-y(s)]\,ds\right)\right|$$
$$\leq\sup_{t\in J}\left\{\int_0^T G(t,s)\left(W|x(\theta(s))-y(\theta(s))|\right.\right.$$
$$+N\int_0^s k(s,r)|x(r)-y(r)|\,dr$$
$$\left.+L\int_0^T h(s,r)|x(r)-y(r)|\,dr\right)ds$$
$$\left.+\sum_{k=1}^m G(t,t_k)\left(L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}|x(r)-y(r)|\,ds\right)\right\}$$
$$\leq\|x-y\|_{PC}$$
$$\times\left[\sup_{t\in J}\left\{\int_0^T G(t,s)\right.\right.$$
$$\left.\times\left(W+N\int_0^s k(s,r)\,dr+L\int_0^T h(s,r)\,dr\right)ds\right\}$$
$$\left.+\frac{1}{1-e^{-MT}}\sum_{k=1}^m L_k(\tau_k-\sigma_k)\right]. \tag{42}$$

The above result and condition (40) imply that F is a contractive mapping, which completes the proof. □

Corollary 8. *Let $M>0$, $W\geq 0$, $N\geq 0$, $L\geq 0$, $L_k\geq 0$, $0\leq\sigma_k\leq\tau_k\leq t_k-t_{k-1}$, $\theta\in(J,J)$, $I_k\in C(J,R)$, $k=1,2,\ldots,m$, $g\in PC(J,R)$, and $\eta\in PC^1(J,R)$ and assume that*

$$\frac{W+(k_0N+h_0L)T}{M}+\frac{1}{1-e^{-MT}}\sum_{k=1}^m L_k(\tau_k-\sigma_k)<1. \tag{43}$$

Then problem (35) *has a unique solution in $PC^1(J,R)$.*

3. Main Results

In this section, we establish existence criteria for solutions of the PBVP (4) by the method of lower and upper solutions and monotone iterative technique. For $\alpha,\beta\in E$, we write $\beta\leq\alpha$ if $\beta(t)\leq\alpha(t)$ for all $t\in J$. In such a case, we denote $[\beta,\alpha]=\{x\in E:\beta(t)\leq x(t)\leq\alpha(t),t\in J\}$.

Theorem 9. *Assume the existence of lower and upper solutions for PBVP* (4) *and also suppose that the following conditions hold.*

(H_1) *The function $f\in C(J\times R^4,R)$ satisfies*

$$f(t,z_1,z_2,z_3,z_4)-f(t,y_1,y_2,y_3,y_4)$$
$$\leq M(z_1-y_1)+W(z_2-y_2)$$
$$+N(z_3-y_3)+L(z_4-y_4),$$
$$f(t,y_1,y_2,y_3,y_4)-f(t,x_1,x_2,x_3,x_4)$$

$$\leq M(y_1 - x_1) + W(y_2 - x_2) + N(y_3 - x_3) + L(y_4 - x_4), \tag{44}$$

where $\beta(t) \leq x_1 \leq y_1 \leq z_1 \leq \alpha(t)$, $\beta(\theta(t)) \leq x_2 \leq y_2 \leq z_2 \leq \alpha(\theta(t))$, $(K\beta)(t) \leq x_3 \leq y_3 \leq z_3 \leq (K\alpha)(t)$, *and* $(S\beta)(t) \leq x_4 \leq y_4 \leq z_4 \leq (S\alpha)(t)$, $t \in J$, *where* $M > 0$, $W \geq 0$, $N \geq 0$, $L \geq 0$, *and* $\theta \in C(J, J)$.

(H_2) *The functions* $I_k \in C(R, R)$ *satisfy*

$$\begin{aligned} &I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha(s)\,ds\right) - I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} y(s)\,ds\right) \\ &\quad\leq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha(s) - y(s)\,ds, \\ &I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} y(s)\,ds\right) - I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta(s)\,ds\right) \\ &\quad\leq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} y(s) - \beta(s)\,ds, \end{aligned} \tag{45}$$

where $\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta(s)ds \leq \int_{t_k-\tau_k}^{t_k-\sigma_k} y(s)ds \leq \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha(s)ds$, $k = 1, 2, \ldots, m$, *where* $L_k \geq 0$ *and* $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, $k = 1, 2, \ldots, m$.

If inequalities (12) *and* (40) *hold, then there exists a solution* x *of PBVP* (4) *such that* $\beta(t) \leq x(t) \leq \alpha(t)$, *for* $t \in J$.

Proof. We consider the following modified problem relative to PBVP (4):

$$\begin{aligned} x'(t) - Mx(t) &= Wx(\theta(t)) + N(Kx)(t) + L(Sx)(t) + g_q(t), \quad t \in J_0, \\ \Delta x(t_k) &= L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} q(s, x(s))\,ds\right) \\ &\quad - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} q(s, x(s))\,ds, \quad k = 1, 2, \ldots, m, \\ x(0) &= x(T), \end{aligned} \tag{46}$$

where $g_q(t) = f(t, q(t), q(\theta(t)), (Kq)(t), (Sq)(t)) - Mq(t) - Wq(\theta(t)) - N(Kq)(t) - L(Sq)(t)$ and

$$\begin{aligned} q(t, x(t)) &= \max\{\beta(t), \min\{x(t), \alpha(t)\}\} \\ &= \begin{cases} \beta(t), & \text{if } x < \beta(t), \\ x(t), & \text{if } \beta(t) \leq x \leq \alpha(t), \\ \alpha(t), & \text{if } x > \alpha(t). \end{cases} \end{aligned} \tag{47}$$

If $x \in E$ is such that $\beta \leq x \leq \alpha$ on J, then x is a solution of PBVP (4) if and only if x is a solution of (46). We will show that (46) is solvable and that every solution of (46) satisfies $\beta \leq x \leq \alpha$ on J. Suppose that $x \in E$ is a solution of (46). We will show that $\beta \leq x$. Let $p = \beta - x$. Then, we have $p(0) - p(T) = \beta(0) - \beta(T)$ since $x(0) = x(T)$ and

$$\begin{aligned} p'(t) &= \beta'(t) - x'(t) \\ &\geq f(t, \beta(t), \beta(\theta(t)), (K\beta)(t), (S\beta)(t)) \\ &\quad + \bar{a}_\beta - \Big[Mx(t) + Wx(\theta(t)) + N(Kx)(t) \\ &\quad + L(Sx)(t) \\ &\quad + f(t, q(t), q(\theta(t)), (Kq)(t), (Sq)(t)) \\ &\quad - Mq(t) - Wq(\theta(t)) \\ &\quad - N(Kq)(t) - L(Sq)(t)\Big] \\ &\geq Mp(t) + Wp(\theta(t)) + N(Kp)(t) \\ &\quad + L(Sp)(t) + \bar{a}_p(t), \quad t \in J_0. \\ \Delta p(t_k) &= \Delta\beta(t_k) - \Delta x(t_k) \\ &\geq I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta(s)\,ds\right) + \bar{b}_{\beta k} \\ &\quad - \left[L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} q(s, x(s))\,ds\right)\right. \\ &\quad \left. - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} q(s, x(s))\,ds\right] \\ &\geq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s)\,ds + \bar{b}_{pk}, \quad k = 1, 2, \ldots m, \end{aligned} \tag{48}$$

By Lemma 3, we get $p(t) \leq 0$ on J; that is, $\beta \leq x$. Similar arguments show that $x \leq \alpha$.

It remains to prove that (46) possesses at least one solution. By Lemma 6, PBVP (46) is equivalent to the following impulsive integral equation:

$$\begin{aligned} x(t) &= -\int_0^T G(t, s) \\ &\quad \times \Big[Wx(\theta(s)) + N(Kx)(s) + L(Sx)(s) \\ &\quad + g_q(s)\Big]\,ds \\ &\quad - \sum_{k=1}^{m} G(t, t_k)\left[L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)\,ds + e_k\right], \quad t \in J, \end{aligned} \tag{49}$$

where $e_k = I_k(\int_{t_k-\tau_k}^{t_k-\sigma_k} q(s, x(s))ds) - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} q(s, x(s))ds$. For any $x \in E$, define a continuous compact operator F by

$$\begin{aligned} (Fx)(t) &= -\int_0^T G(t, s) \\ &\quad \times \Big[Wx(\theta(s)) + N(Kx)(s) + L(Sx)(s) \end{aligned}$$

$$+g_q(s)\Big]ds$$
$$-\sum_{k=1}^{m}G(t,t_k)\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}x(s)\,ds+e_k\right],\quad t\in J. \tag{50}$$

Let $\delta>0$, such that $|\alpha(t)|<\delta$, $|\beta(t)|<\delta$, $t\in J$, and take the compact sets $B=\{(t,y_1,y_2,y_3,y_4)\in R^5: t\in J, \beta(t)\le y_1\le \alpha(t), \beta(\theta(t))\le y_2\le\alpha(\theta(t)), (K\beta)(t)\le y_3\le (K\alpha)(t)$, and $(S\beta)(t)\le y_4\le (S\alpha)(t)\}$. Since f is continuous, then we can choose a constant $\rho>0$, such that $|f(t,y_1,y_2,y_3,y_4)|\le\rho$, $(t,y_1,y_2,y_3,y_4)\in B$. For $\lambda\in(0,1)$, we see that any solution of

$$x=\lambda Fx \tag{51}$$

satisfies

$$\begin{aligned}
\|x\|_{PC}&=\lambda\|Fx\|_{PC}\\
&\le\lambda\sup_{t\in J}\int_0^T G(t,s)\\
&\quad\times\Big[W|x(\theta(s))|+N|(Kx)(s)|\\
&\quad+L|(Sx)(s)|+\left|g_q(s)\right|\Big]ds\\
&\quad+\lambda\sum_{k=1}^{m}G(t,t_k)\left[L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}|x(s)|\,ds+|e_k|\right]\\
&\le\sup_{t\in J}\left\{\int_0^T G(t,s)\right.\\
&\quad\times\left[W+N\int_0^s k(s,r)\,dr\right.\\
&\quad\left.\left.+L\int_0^T h(s,r)\,dr\right]ds\right\}\|x\|_{PC}\\
&\quad+\frac{\rho+M\delta}{M}\\
&\quad+\sup_{t\in J}\sum_{k=1}^{m}G(t,t_k)\\
&\quad\times\left[L_k(\tau_k-\sigma_k)\|x\|_{PC}+|e_k|\right]\\
&\quad+\sup_{t\in J}\left\{\int_0^T G(t,s)\right.\\
&\quad\times\left[W+N\int_0^s k(s,r)\,dr\right.\\
&\quad\left.\left.+L\int_0^T h(s,r)\,dr\right]ds\right\}\delta.
\end{aligned} \tag{52}$$

From the continuity of I_k, $k=1,2,\ldots,m$, and $\beta(t)\le q(t,x(t))\le\alpha(t)$ on J, we can choose some $\omega>0$ such that $|e_k|<\omega$, $k=1,2,\ldots,m$. Then we have

$$\begin{aligned}
\|x\|_{PC}\le\frac{1}{1-\mu}&\left[\frac{\rho+M\delta}{M}\right.\\
&+\sup_{t\in J}\left\{\int_0^T G(t,s)\right.\\
&\times\left[W+N\int_0^s k(s,r)\,dr\right.\\
&\left.\left.+L\int_0^T h(s,r)\,dr\right]ds\right\}\delta\\
&\left.+\frac{\omega m}{1-e^{-MT}}\right],
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
\mu=\sup_{t\in J}&\left\{\int_0^T G(t,s)\right.\\
&\times\left[W+N\int_0^s k(s,r)\,dr\right.\\
&\left.\left.+L\int_0^T h(s,r)\,dr\right]ds\right\}\\
&+\frac{1}{1-e^{-MT}}\sum_{k=1}^{m}L_k(\tau_k-\sigma_k)<1.
\end{aligned} \tag{54}$$

Hence, by Schaefer's theorem [21], we get that F has at least a fixed point $x\in E$, which is a solution of (46). Such a solution lies between β and α and, consequently, is a solution of (4). Thus, the proof is complete. □

Theorem 10. *Assume that there exist lower and upper solutions for PBVP* (4) *and assume the following.*

(H_3) *The function $f\in C(J\times R^4,R)$ satisfies*

$$\begin{aligned}
&f(t,z_1,z_2,z_3,z_4)-f(t,\overline{z}_1,\overline{z}_2,\overline{z}_3,\overline{z}_4)\\
&\quad\le M(z_1-\overline{z}_1)+W(z_2-\overline{z}_2)\\
&\quad+N(z_3-\overline{z}_3)+L(z_4-\overline{z}_4),
\end{aligned} \tag{55}$$

where $\beta(t)\le\overline{z}_1\le z_1\le\alpha(t)$, $\beta(\theta(t))\le\overline{z}_2\le z_2\le\alpha(\theta(t))$, $(K\beta)(t)\le\overline{z}_3\le z_3\le(K\alpha)(t)$, and $(S\beta)(t)\le\overline{z}_4\le z_4\le(S\alpha)(t)$, $t\in J$, where $M>0$, $W\ge0$, $N\ge0$, $L\ge0$, and $\theta\in C(J,J)$.

(H_4) *The functions $I_k\in C(R,R)$ satisfy*

$$\begin{aligned}
&I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k}x(s)\,ds\right)-I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k}y(s)\,ds\right)\\
&\quad\le L_k\int_{t_k-\tau_k}^{t_k-\sigma_k}x(s)-y(s)\,ds,
\end{aligned} \tag{56}$$

where $\int_{t_k-\tau_k}^{t_k-\sigma_k}\beta(s)ds\le\int_{t_k-\tau_k}^{t_k-\sigma_k}y(s)ds\le\int_{t_k-\tau_k}^{t_k-\sigma_k}x(s)ds\le\int_{t_k-\tau_k}^{t_k-\sigma_k}\alpha(s)ds$, $k=1,2,\ldots,m$, where $L_k\ge0$ and $0\le\sigma_k\le\tau_k\le t_k-t_{k-1}$, $k=1,2,\ldots,m$.

Suppose that inequalities (12) *and* (40) *hold. Then there exist monotone sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *with* $\alpha_0 = \alpha$, $\beta_0 = \beta$, *which converge uniformly on* J *to the extremal solutions of the periodic boundary value problem* (4) *in* $[\beta, \alpha]$.

Proof. For any $\eta \in [\beta, \alpha]$, consider PBVP (35) with

$$g(t) = f(t, \eta(t), \eta(\theta(t)), (K\eta)(t), (S\eta)(t)) - M\eta(t) - W\eta(\theta(t)) - N(K\eta)(t) - L(S\eta)(t). \tag{57}$$

By Lemmas 6 and 7, PBVP (35) possesses a unique solution $x \in E$. We define an operator A by $u = A\eta$; then the operator A has the following properties:

(i) $\beta \le A\beta$, $A\alpha \le \alpha$;

(ii) $A\eta_1 \le A\eta_2$ for any $\eta_1, \eta_2 \in [\beta, \alpha]$ with $\eta_1 \le \eta_2$.

First we prove (i). Let $p = \beta_0 - \beta_1$, where $\beta_1 = A\beta_0$. Then, we have $p(0) - p(T) = \beta_0(0) - \beta_0(T)$ since $\beta_1(0) = \beta_1(T)$ and

$$\begin{aligned} p'(t) &= \beta_0'(t) - \beta_1'(t) \\ &\ge Mp(t) + Wp(\theta(t)) + N(Kp)(t) \\ &\quad + L(Sp)(t) + \bar{a}_p(t), \quad t \in J_0, \\ \Delta p(t_k) &= \Delta\beta_0(t_k) - \Delta\beta_1(t_k) \\ &\ge L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s)\,ds + \bar{b}_{pk}, \quad k = 1, 2, \ldots m. \end{aligned} \tag{58}$$

By Lemma 3, we get $p(t) \le 0$ on J; that is, $\beta \le A\beta$. Analogously, we have $A\alpha \le \alpha$.

Now, we claim (ii). Set $u_1 = A\eta_1$, $u_2 = A\eta_2$, where η_1, $\eta_2 \in [\beta, \alpha]$ with $\eta_1 \le \eta_2$. Let $p = u_1 - u_2$; by (H_3)-(H_4), we have

$$\begin{aligned} p'(t) &= u_1'(t) - u_2'(t) \\ &\ge Mp(t) + Wp(\theta(t)) + N(Kp)(t) \\ &\quad + L(Sp)(t), \quad t \in J_0, \\ \Delta p(t_k) &= \Delta u_1(t_k) - \Delta u_2(t_k) \\ &= L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s)\,ds, \quad k = 1, 2, \ldots m, \\ p(0) &= p(T). \end{aligned} \tag{59}$$

By Lemma 3, we have $p(t) \le 0$ on J and so $u_1 \le u_2$. Thus we may define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ by $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$, $\alpha_0 = \alpha$, and $\beta_0 = \beta$. From (i), (ii), we obtain

$$\begin{aligned} &\beta_0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_n \le \cdots \\ &\le \alpha_n \le \cdots \le \alpha_2 \le \alpha_1 \le \alpha_0 \quad \text{on } J, \end{aligned} \tag{60}$$

and each α_n, $\beta_n \in E$ $(\forall n \in N)$ satisfies

$$\begin{aligned} \alpha_n(t) = &-\int_0^T G(t,s)\left[W\alpha_n(\theta(s)) + N(K\alpha_n)(s) + L(S\alpha_n)(s) + g_{n-1}(s)\right] ds \\ &- \sum_{k=1}^m G(t, t_k)\left[L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_n(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_{n-1}(s)\,ds\right) - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_{n-1}(s)\,ds\right], \quad t \in J, \\ \beta_n(t) = &-\int_0^T G(t,s)\left[W\beta_n(\theta(s)) + N(K\beta_n)(s) + L(S\beta_n)(s) + \bar{g}_{n-1}(s)\right] ds \\ &- \sum_{k=1}^m G(t, t_k)\left[L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_n(s)\,ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_{n-1}(s)\,ds\right) - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_{n-1}(s)\,ds\right], \quad t \in J, \end{aligned} \tag{61}$$

where

$$\begin{aligned} g_{n-1}(t) &= f(t, \alpha_{n-1}(t), \alpha_{n-1}(\theta(t)), (K\alpha_{n-1})(t), (S\alpha_{n-1})(t)) \\ &\quad - M\alpha_{n-1}(t) - W\alpha_{n-1}(\theta(t)) \\ &\quad - N(K\alpha_{n-1})(t) - L(S\alpha_{n-1})(t), \\ \bar{g}_{n-1}(t) &= f(t, \beta_{n-1}(t), \beta_{n-1}(\theta(t)), (K\beta_{n-1})(t), (S\beta_{n-1})(t)) \\ &\quad - M\beta_{n-1}(t) - W\beta_{n-1}(\theta(t)) \\ &\quad - N(K\beta_{n-1})(t) - L(S\beta_{n-1})(t). \end{aligned} \tag{62}$$

Hence, there exist ξ, ψ such that $\lim_{n\to\infty}\alpha_n(t) = \psi(t)$ and $\lim_{n\to\infty}\beta_n(t) = \xi(t)$ uniformly on J. Clearly, ξ, ψ satisfy PBVP (4). We will prove that ξ, ψ are extreme solutions of PBVP (4). Let $x(t)$ be any solution of PBVP (4), which satisfies $\beta(t) \le x(t) \le \alpha(t)$, $t \in J$. Also suppose there exists a positive integer n such that for $t \in J$, $\beta_n(t) \le x(t) \le \alpha_n(t)$. Setting $p(t) = \beta_{n+1}(t) - x(t)$, then for $t \in J$,

$$\begin{aligned} p'(t) &= \beta_{n+1}'(t) - x'(t) \\ &\ge Mp(t) + Wp(\theta(t)) + N(Kp)(t) + L(Sp)(t), \\ &\qquad t \in J_0, \end{aligned}$$

$$\Delta p(t_k) = \Delta\beta_{n+1}(t_k) - \Delta x(t_k) \geq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s)\,ds, \quad k = 1, 2, \ldots m, \quad p(0) = p(T). \tag{63}$$

According to Lemma 3, we get that $p(t) \leq 0$ on J. Similarly, we obtain $x(t) \leq \alpha_{n+1}(t)$ on J. Since $\beta_0 \leq x(t) \leq \alpha_0(t)$ on J, by induction, we have $\beta_n(t) \leq x(t) \leq \alpha_n(t)$ on J for all n. Therefore, $\xi(t) \leq x(t) \leq \psi(t)$ on J by taking limit as $n \to \infty$. The proof is complete. □

4. An Example

In this section, in order to illustrate our results, we consider an example.

Example 11. Consider the following periodic boundary value problem:

$$u'(t) = \frac{1}{6}t^2(1+u(t)) + \frac{1}{16}tu\left(\frac{1}{2}t\right) + \frac{1}{36}\left[\int_0^t ts^3u(s)ds\right]^3 + \frac{1}{24}\left[\int_0^1 t^3su(s)ds\right]^2, \quad t \in J = [0,1],\ t \neq \frac{1}{2},$$
$$\Delta u\left(\frac{1}{2}\right) = \frac{1}{2}\int_{1/6}^{1/4} u(s)\,ds, \quad k = 1, \quad u(0) = u(1), \tag{64}$$

where $k(t,s) = ts^3$, $h(t,s) = t^3s$, $m = 1$, $t_1 = 1/2$, $\sigma_1 = 1/4$, $\tau_1 = 1/3$, and $T = 1$.

Obviously, $\alpha_0 = 0$, $\beta_0 = -5$ are lower and upper solutions for (64), respectively, and $\beta_0 \leq \alpha_0$.

Let

$$f(t, z_1, z_2, z_3, z_4) = \frac{1}{6}t^2(1+z_1) + \frac{1}{16}tz_2 + \frac{1}{36}z_3^3 + \frac{1}{24}z_4^2. \tag{65}$$

We have

$$f(t, z_1, z_2, z_3, z_4) - f(t, \overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4) \leq \frac{1}{6}(z_1 - \overline{z}_1) + \frac{1}{16}(z_2 - \overline{z}_2) + \frac{1}{3}(z_3 - \overline{z}_3) + \frac{1}{4}(z_4 - \overline{z}_4), \tag{66}$$

where $\beta(t) \leq \overline{z}_1 \leq z_1 \leq \alpha(t)$, $\beta((1/2)t) \leq \overline{z}_2 \leq z_2 \leq \alpha((1/2)t)$, $(K\beta)(t) \leq \overline{z}_3 \leq z_3 \leq (K\alpha)(t)$, and $(S\beta)(t) \leq \overline{z}_4 \leq z_4 \leq (S\alpha)(t)$, $t \in J$. It is easy to see that

$$I_1\left(\int_{1/6}^{1/4} x(s)\,ds\right) - I_1\left(\int_{1/6}^{1/4} y(s)\,ds\right) = \frac{1}{2}\int_{1/6}^{1/4} x(s) - y(s)\,ds, \tag{67}$$

where $\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta(s)ds \leq \int_{t_k-\tau_k}^{t_k-\sigma_k} y(s)ds \leq \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds \leq \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha(s)ds$, $k = 1$.

Taking $L_1 = 1/2$, $M = 1/6$, $W = 1/16$, $N = 1/3$, and $L = 1/4$, it follows that

$$\begin{aligned}
\widehat{L} &= L_k(\tau_k - \sigma_k) = \frac{1}{24},\\
r(t) &= M + W + N\int_0^t k(t,s)\,ds + L\int_0^T h(t,s)\,ds\\
&= \frac{1}{6} + \frac{1}{16} + \frac{1}{3}\int_0^t ts^3ds + \frac{1}{4}\int_0^1 t^3s\,ds,\\
A_k &= 1 + L_k(\tau_k - \sigma_k) = 1 + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{25}{24},\\
B_k &= A_k\int_{t_{k-1}}^{t_k-\tau_k} r(s)\,ds + \int_{t_k-\tau_k}^{t_k-\sigma_k} [1 + L_k(t_k - \sigma_k - s)]\,r(s)\,ds + \int_{t_k-\sigma_k}^{t_k} r(s)\,ds,\\
&= A_k\int_0^{(1/2)-(1/3)} r(s)\,ds + \int_{(1/2)-(1/3)}^{(1/2)-(1/4)} \left[1 + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} - s\right)\right] r(s)\,ds + \int_{(1/2)-(1/4)}^{1/2} r(s)\,ds.
\end{aligned} \tag{68}$$

Thus,

$$\left(\widehat{L} + \sum_{k=1}^m B_k + \int_0^T r(s)\,ds\right)\prod_{k=1}^m A_k \approx 0.4528309326 \leq 1,$$

$$\begin{aligned}
&\sup_{t\in J}\left\{\int_0^T G(t,s)\left(W + N\int_0^s k(s,r)\,dr + L\int_0^T h(s,r)\,dr\right)ds\right\} + \frac{1}{1-e^{-MT}}\sum_{k=1}^m L_k(\tau_k - \sigma_k)\\
&= \sup_{t\in J}\left\{\int_0^1 G(t,s)\left(\frac{1}{16} + \frac{1}{3}\int_0^s sr^3dr + \frac{1}{4}\int_0^1 s^3rdr\right)ds\right\} + \frac{1}{1-e^{-1/6}}\frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) \approx 0.9256866285 < 1.
\end{aligned} \tag{69}$$

Therefore, (64) satisfies all conditions of Theorem 10. So PBVP (64) has minimal and maximal solutions in the segment $[\beta_0, \alpha_0]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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