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Research Article

Viscosity Projection Algorithms for Pseudocontractive Mappings in Hilbert Spaces

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An explicit projection algorithm with viscosity technique is constructed for finding the fixed points of the pseudocontractive mapping in Hilbert spaces. Strong convergence theorem is demonstrated. Consequently, as an application, we can approximate to the minimum-norm fixed point of the pseudocontractive mapping.

1. Introduction

Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{H}$.

Recall that a mapping $\mathbb{T}:\mathbb{C}\to\mathbb{C}$ is said to be

(i) L-Lipschitz $\Leftrightarrow$ there exists a constant $L>0$ such that $\|\mathbb{T}u-\mathbb{T}v\|\le L\|u-v\|$ for all $u,v\in\mathbb{C}$; if $L\in(0,1)$, then $\mathbb{T}$ is said to be contractive; if $L=1$, then $\mathbb{T}$ is said to be nonexpansive;

(ii) pseudocontractive

$$\begin{aligned}&\Leftrightarrow \langle\mathbb{T}u-\mathbb{T}v,u-v\rangle\le\|u-v\|^2;\\&\Leftrightarrow \|\mathbb{T}u-\mathbb{T}v\|^2\le\|u-v\|^2+\|(I-\mathbb{T})u-(I-\mathbb{T})v)\|^2;\\&\Leftrightarrow \langle u-v,(I-\mathbb{T})u-(I-\mathbb{T})v\rangle\ge 0,\end{aligned}$$

for all $u,v\in\mathbb{C}$.

Interest in pseudocontractive mappings stems mainly from their firm connection with the class of nonlinear accretive operators. It is a classical result, see Deimling [1], that if $\mathbb{T}$ is an accretive operator, then the solutions of the equations $\mathbb{T}x=0$ correspond to the equilibrium points of some evolution systems. This explains the importance, from this point of view, of the improvement brought by the Ishikawa iteration which was introduced by Ishikawa [2] in 1974. The original result of Ishikawa is stated in the following.

Theorem 1. *Let $\mathbb{C}$ be a convex compact subset of a Hilbert space $\mathbb{H}$ and let $\mathbb{T}:\mathbb{C}\to\mathbb{C}$ be an L-Lipschitzian pseudocontractive mapping with $\mathrm{Fix}(\mathbb{T})\ne\emptyset$. For any $x_0\in\mathbb{C}$, define the sequence $\{x_n\}$ iteratively by*

$$\begin{aligned}y_n&=(1-\eta_n)x_n+\eta_n\mathbb{T}x_n,\\x_{n+1}&=(1-\xi_n)x_n+\xi_n\mathbb{T}y_n,\end{aligned}\tag{1}$$

for all $n\in\mathbb{N}$, where $\{\xi_n\}\subset[0,1]$ and $\{\eta_n\}\subset[0,1]$ satisfy the conditions: $\lim_{n\to\infty}\eta_n=0$ and $\sum_{n=1}^{\infty}\xi_n\eta_n=\infty$. Then the sequence $\{x_n\}$ generated by (1) *converges strongly to a fixed point of $\mathbb{T}$.*

The iteration (1) is now referred to as the Ishikawa iterative sequence. However, we note that C is compact subset. Now, we know that strong convergence has not been achieved without compactness assumption on the involved operation or the underlying spaces. A counter example can be found in Chidume and Mutangadura [3].

In order to obtain a strong convergence result for pseudocontractive mappings without the compactness assumption, in [4], Zhou coupled the Ishikawa algorithm with the hybrid technique and presented the following algorithm for Lipschitz pseudocontractive mappings:

$$\begin{aligned} y_n &= (1-\xi_n)x_n + \xi_n \mathbb{T}x_n, \\ z_n &= (1-\eta_n)x_n + \eta_n \mathbb{T}y_n, \\ \mathbb{C}_n &= \Big\{z \in \mathbb{C} : \|z_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad - \xi_n\eta_n\left(1 - 2\xi_n - \xi_n^2 L^2\right)\|x_n - \mathbb{T}x_n\|^2\Big\}, \\ \mathbb{Q}_n &= \{z \in \mathbb{C} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= \mathrm{proj}_{\mathbb{C}_n \cap \mathbb{Q}_n} x_0, \quad n \in \mathbb{N}. \end{aligned} \tag{2}$$

Zhou proved that the sequence $\{x_n\}$ generated by (2) converges strongly to the fixed point of $\mathbb{T}$. Further, in [5], Yao et al. introduced the hybrid Mann algorithm as follows.

Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\{\xi_n\}$ be a sequence in $(0,1)$. Let $x_0 \in \mathbb{H}$. For $\mathbb{C}_1 = \mathbb{C}$ and $x_1 = \mathrm{proj}_{\mathbb{C}_1} x_0$, define a sequence $\{x_n\}$ of $\mathbb{C}$ as follows:

$$\begin{aligned} y_n &= (1-\xi_n)x_n + \xi_n \mathbb{T}x_n, \\ \mathbb{C}_{n+1} &= \Big\{z \in \mathbb{C}_n : \|\xi_n (I - \mathbb{T}) y_n\|^2 \\ &\quad \le 2\xi_n \langle x_n - z, (I-\mathbb{T}) y_n\rangle\Big\}, \\ x_{n+1} &= \mathrm{proj}_{\mathbb{C}_{n+1}} x_0, \quad n \in \mathbb{N}. \end{aligned} \tag{3}$$

Note that, in iterations (2) and (3), we need to compute the half-spaces $\mathbb{C}_n$ (and/or $\mathbb{Q}_n$). Very recently, Zegeye et al. [6] further studied the convergence analysis of the Ishikawa iteration (1). They proved ingeniously the strong convergence of the Ishikawa iteration (1). However, we have to assume that the interior of $\mathrm{Fix}(\mathbb{T})$ is nonempty. This appears very restrictive since even in $\mathbb{R}$ with the usual norm, Lipschitz pseudocontractive maps with finite number of fixed points do not enjoy this condition that $\mathrm{intFix}(\mathbb{T}) \neq \emptyset$. For some related works, please refer to [7–19].

On the other hand, we notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a closed convex subset $\mathbb{C}$ of a Hilbert space $\mathbb{H}_1$ and a bounded linear operator $\mathbb{B} : \mathbb{H}_1 \to \mathbb{H}_2$, where $\mathbb{H}_2$ is another Hilbert space. The $\mathbb{C}$-constrained pseudoinverse of $\mathbb{B}$, $\mathbb{B}_{\mathbb{C}}^{\dagger}$, is then defined as the minimum-norm solution of the constrained minimization problem

$$\mathbb{B}_{\mathbb{C}}^{\dagger}(b) := \arg\min_{x \in \mathbb{C}} \|\mathbb{B}x - b\| \tag{4}$$

which is equivalent to the fixed point problem

$$u = \mathrm{proj}_{\mathbb{C}}\left(u - \mu\mathbb{B}^*(\mathbb{B}u - b)\right), \tag{5}$$

where $\mathbb{B}^*$ is the adjoint of $\mathbb{B}$, $\mu > 0$ is a constant, and $b \in \mathbb{H}_2$ is such that $\mathrm{proj}_{\overline{\mathbb{B}(\mathbb{C})}}(b) \in \mathbb{B}(\mathbb{C})$.

It is, therefore, an interesting problem to invent iterative algorithms that can generate sequences which converge strongly to the minimum-norm solution of a given fixed point problem. The purpose of this paper is to solve such a problem for pseudocontractions. More precisely, we will introduce an explicit projection algorithm with viscosity technique for finding the fixed points of a Lipschitzian pseudocontractive mapping. Strong convergence theorem is demonstrated. Consequently, as an application, we can find the minimum-norm fixed point of the pseudocontractive mapping.

2. Preliminaries

Recall that the metric projection $\mathrm{proj}_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ satisfies $\|u - \mathrm{proj}_{\mathbb{C}} u\| = \inf\{\|u - v\| : v \in \mathbb{C}\}$. The metric projection $\mathrm{proj}_{\mathbb{C}}$ is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is $\langle u - \mathrm{proj}_{\mathbb{C}} u, v - \mathrm{proj}_{\mathbb{C}} u\rangle \le 0$ for all $u \in \mathbb{H}$, $v \in \mathbb{C}$.

In the sequel we will use the following expressions:

(i) $\mathrm{Fix}(\mathbb{T})$ denotes the set of fixed points of $\mathbb{T}$;

(ii) $x_n \rightharpoonup x^{\dagger}$ denotes the weak convergence of x_n to $x^{\dagger}$;

(iii) $x_n \to x^{\dagger}$ denotes the strong convergence of x_n to $x^{\dagger}$.

The following lemmas will be useful for the next section.

Lemma 2 (see [20]). *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ be a nonexpansive mapping with $\mathrm{Fix}(\mathbb{T}) \neq \emptyset$. Then,*

$$\left\{\begin{matrix} \mathbb{C} \supset u_n \rightharpoonup u^{\ddagger} \\ (\mathbb{I} - \mathbb{T}) u_n \longrightarrow v \end{matrix}\right\} \Longrightarrow (\mathbb{I} - \mathbb{T}) u^{\ddagger} = v. \tag{6}$$

Lemma 3 (see [21]). *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Assume that a mapping $\mathbb{A} : \mathbb{C} \to \mathbb{H}$ is monotone and weakly continuous along segments (i.e., $\mathbb{A}(x + ty) \to \mathbb{A}(x)$ weakly, as $t \to 0$, whenever $x + ty \in \mathbb{C}$ for $x, y \in \mathbb{C}$). Then the variational inequality*

$$x^{\ddagger} \in C, \quad \left\langle \mathbb{A}x^{\ddagger}, x - x^{\ddagger}\right\rangle \ge 0, \quad \forall x \in \mathbb{C}, \tag{7}$$

is equivalent to the dual variational inequality

$$x^{\ddagger} \in \mathbb{C}, \quad \langle \mathbb{A}x, x - x^{\ddagger}\rangle \ge 0, \quad \forall x \in \mathbb{C}. \tag{8}$$

Lemma 4 (see [22]). *Assume that the sequence $\{a_n\}$ satisfies $a_n \ge 0$ and $a_{n+1} \le (1 - \nu_n)a_n + \varsigma_n\nu_n$ where $\{\nu_n\}$ is a sequence in $(0,1)$ and $\{\varsigma_n\}$ is a sequence such that $\sum_{n=1}^{\infty} \nu_n = \infty$ and $\limsup_{n\to\infty}\varsigma_n \le 0$ (or $\sum_{n=1}^{\infty} |\varsigma_n\nu_n| < \infty$). Then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

In order to prove our main result, we need the following proposition.

Proposition 5. *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{W} : \mathbb{C} \to \mathbb{C}$ be a nonexpansive mapping with $\mathrm{Fix}(\mathbb{W}) \neq \emptyset$. Let $\varrho : \mathbb{C} \to \mathbb{H}$ be a κ-contraction.*

For each $t \in (0,1)$, let the net $\{u_t\}$ be defined by

$$u_t = \mathbb{W}\,\mathrm{proj}_{\mathbb{C}}\left[t\varrho(u_t) + (1-t)u_t\right]. \tag{9}$$

Then, as $t \to 0^+$, the net $\{u_t\}$ converges strongly to a point $x^\ddagger \in \mathrm{Fix}(\mathbb{W})$ which solves the following variational inequality:

$$x^\ddagger \in \mathrm{Fix}(\mathbb{W}), \quad \left\langle (I-\varrho)x^\ddagger, z - x^\ddagger \right\rangle \geq 0, \quad z \in \mathrm{Fix}(\mathbb{W}). \tag{10}$$

Proof. For $t \in (0,1)$, define a mapping $\mathbb{W}_t : \mathbb{C} \to \mathbb{C}$ by

$$\mathbb{W}_t u = \mathbb{W}\,\mathrm{proj}_{\mathbb{C}}\left[t\varrho(u) + (1-t)u\right], \quad u \in \mathbb{C}. \tag{11}$$

For any $u, v \in \mathbb{C}$, we have

$$\begin{aligned}\|\mathbb{W}_t u - \mathbb{W}_t v\| &= \|\mathbb{W}\,\mathrm{proj}_{\mathbb{C}}\left[t\varrho(u) + (1-t)u\right] \\ &\quad -\mathbb{W}\,\mathrm{proj}_{\mathbb{C}}\left[t\varrho(v) + (1-t)v\right]\| \\ &\leq t\|\varrho(u) - \varrho(v)\| + (1-t)\|u - v\| \\ &\leq [1-(1-\kappa)t]\|u - v\|.\end{aligned} \tag{12}$$

Hence, $\mathbb{W}_t$ is a $1-(1-\kappa)t$-contraction on $\mathbb{C}$ with $u_t \in \mathbb{C}$ as its unique fixed point. So, $\{u_t\}$ is well defined.

Let $u \in \mathrm{Fix}(\mathbb{W})$. From (9), we have

$$\begin{aligned}\|u_t - u\| &= \|\mathbb{W}\,\mathrm{proj}_{\mathbb{C}}\left[t\varrho(u_t) + (1-t)u_t\right] - \mathbb{W}\,\mathrm{proj}_{\mathbb{C}} u\| \\ &\leq t\|\varrho(u_t) - \varrho(u)\| + t\|\varrho(u) - u\| + (1-t)\|u_t - u\| \\ &\leq [1-(1-\kappa)t]\|u_t - u\| + t\|\varrho(u) - u\|.\end{aligned} \tag{13}$$

It follows that

$$\|u_t - u\| \leq \frac{\|\varrho(u) - u\|}{1-\kappa}. \tag{14}$$

Thus, $\{u_t\}$ and $\{\varrho(u_t)\}$ are bounded.

Again from (9), we get

$$\begin{aligned}\|u_t - \mathbb{W}u_t\| &= \|\mathbb{W}\,\mathrm{proj}_{\mathbb{C}}\left[t\varrho(u_t) + (1-t)u_t\right] - \mathbb{W}\,\mathrm{proj}_{\mathbb{C}} u_t\| \\ &\leq t\|\varrho(u_t) - u_t\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0^+.\end{aligned} \tag{15}$$

Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0^+$ as $n \to \infty$. Put $u_n := u_{t_n}$. From (15), we have

$$\|u_n - \mathbb{W}u_n\| \longrightarrow 0. \tag{16}$$

From (9), we obtain

$$\begin{aligned}\|u_t - u\|^2 &= \|\mathbb{W}\,\mathrm{proj}_{\mathbb{C}}[t\varrho(u_t) + (1-t)u_t] - \mathbb{W}\,\mathrm{proj}_{\mathbb{C}} u\|^2 \\ &\leq \|u_t - u + t(\varrho(u_t) - u_t)\|^2 \\ &= \|u_t - u\|^2 + 2t\langle \varrho(u_t) - u_t, u_t - u\rangle \\ &\quad + t^2\|\varrho(u_t) - u_t\|^2 \\ &= \|u_t - u\|^2 + 2t\langle \varrho(u_t) - \varrho(u), u_t - u\rangle \\ &\quad + 2t\langle \varrho(u) - u, u_t - u\rangle \\ &\quad + 2t\langle u - u_t, u_t - u\rangle + t^2\|\varrho(u_t) - u_t\|^2 \\ &\leq [1-2(1-\kappa)t]\|u_t - u\|^2 \\ &\quad + 2t\langle \varrho(u) - u, u_t - u\rangle + t^2\|\varrho(u_t) - u_t\|^2.\end{aligned} \tag{17}$$

It follows that

$$\|u_t - u\|^2 \leq \frac{1}{1-\kappa}\langle \varrho(u) - u, u_t - u\rangle + tM, \tag{18}$$

where $M > 0$ is a constant such that

$$M > \frac{1}{2(1-\kappa)}\sup\left\{\|\varrho(u_t) - u_t\|^2 : t \in (0,1)\right\}. \tag{19}$$

In particular, we have

$$\|u_n - u\|^2 \leq \frac{1}{1-\kappa}\langle \varrho(u) - u, u_n - u\rangle + t_n M, \quad u \in \mathrm{Fix}(\mathbb{W}). \tag{20}$$

Noting that $\{u_n\}$ is bounded, without loss of generality, we assume that $u_n \rightharpoonup x^\ddagger$. It is obvious that $x^\ddagger \in \mathbb{C}$. From (16) and Lemma 2, we deduce $x^\ddagger \in \mathrm{Fix}(\mathbb{W})$. Substitute $x^\ddagger$ for u in (20) to get

$$\left\|u_n - x^\ddagger\right\|^2 \leq \frac{1}{1-\kappa}\left\langle \varrho\left(x^\ddagger\right) - x^\ddagger, x_n - x^\ddagger\right\rangle + t_n M. \tag{21}$$

Since $u_n \rightharpoonup x^\ddagger$, we deduce from (21) that $u_n \to x^\ddagger$. The net $\{u_t\}$ is, therefore, relatively compact, as $t \to 0^+$, in the norm topology.

In (20), letting $n \to \infty$, we get

$$\left\|x^\ddagger - u\right\|^2 \leq \frac{1}{1-\kappa}\left\langle \varrho(u) - u, x^\ddagger - u\right\rangle, \quad u \in \mathrm{Fix}(\mathbb{W}). \tag{22}$$

Therefore, $x^\ddagger$ solves the variational inequality

$$x^\ddagger \in \mathrm{Fix}(\mathbb{W}), \quad \left\langle (I-\varrho)u, u - x^\ddagger\right\rangle \geq 0, \quad u \in \mathrm{Fix}(\mathbb{W}). \tag{23}$$

By Lemma 3, (23) equals its dual variational inequality

$$x^\ddagger \in \mathrm{Fix}(\mathbb{W}), \quad \left\langle (\mathbb{I}-\varrho)x^\ddagger, u - x^\ddagger\right\rangle \geq 0, \quad u \in \mathrm{Fix}(\mathbb{W}). \tag{24}$$

This indicates that $x^\ddagger = (\mathrm{proj}_{\mathrm{Fix}(\mathbb{W})}\varrho)x^\ddagger$. That is, $x^\ddagger$ is the unique fixed point in $\mathrm{Fix}(\mathbb{W})$ of the contraction $\mathrm{proj}_{\mathrm{Fix}(\mathbb{W})}\varrho$. So, the entire net $\{u_t\}$ converges in norm to $x^\ddagger$ as $t \to 0^+$. This completes the proof. □

Corollary 6. *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{W} : \mathbb{C} \to \mathbb{C}$ be a nonexpansive mapping with $\mathrm{Fix}(\mathbb{W}) \neq \emptyset$.*

For each $t \in (0,1)$, let the net $\{u_t\}$ be defined by

$$u_t = \mathbb{W}\,\mathrm{proj}_{\mathbb{C}}\left[(1-t)u_t\right]. \tag{25}$$

Then, as $t \to 0^+$, the net $\{u_t\}$ converges strongly to the minimum-norm fixed point of $\mathbb{W}$.

Proof. As a matter of fact, in (9), we choose $\varrho = 0$, and then (9) reduces to (25). From Proposition 5, (24) is reduced to

$$0 \leq \left\langle x^\ddagger, u - x^\ddagger\right\rangle, \quad u \in \mathrm{Fix}(\mathbb{W}). \tag{26}$$

Equivalently,

$$\left\|x^{\ddagger}\right\|^{2} \leq \left\langle x^{\ddagger}, u\right\rangle, \quad u \in \operatorname{Fix}(\mathbb{W}). \tag{27}$$

This implies that

$$\left\|x^{\ddagger}\right\| \leq \|u\|, \quad u \in \operatorname{Fix}(\mathbb{W}). \tag{28}$$

Therefore, $x^{\ddagger}$ is the minimum-norm fixed point of $\mathbb{W}$. This completes the proof. □

Algorithm 7. Let $\mathbb{C}$ be a nonempty closed subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{T}: \mathbb{C} \to \mathbb{C}$ be a pseudocontraction and let $\varrho: \mathbb{C} \to \mathbb{H}$ be a κ-contraction. Let $\{\xi_n\} \subset [0,1]$ and $\{\eta_n\} \subset [0,1]$ be two real number sequences. For $x_0 \in \mathbb{C}$, we define a sequence $\{x_n\}$ iteratively by

$$x_{n+1} = \operatorname{proj}_{\mathbb{C}}\left[\xi_n \varrho(x_n) + (1-\xi_n-\eta_n)x_n + \eta_n \mathbb{T}x_n\right], \quad n \geq 0. \tag{29}$$

Theorem 8. *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{T}: \mathbb{C} \to \mathbb{C}$ be an L-Lipschitzian and pseudocontraction with $\operatorname{Fix}(\mathbb{T}) \neq \emptyset$ and $\varrho: \mathbb{C} \to \mathbb{H}$ a κ-contraction. Suppose the following conditions are satisfied:*

($C1$) $\lim_{n\to\infty}\xi_n = 0$ *and* $\sum_{n=0}^{\infty}\xi_n = \infty$;

($C2$) $\lim_{n\to\infty}(\xi_n/\eta_n) = \lim_{n\to\infty}(\eta_n^2/\xi_n) = 0$;

($C3$) $\lim_{n\to\infty}((\xi_n\eta_{n-1} - \xi_{n-1}\eta_n)/\xi_n^2\eta_{n-1}) = 0$.

Then the sequence $\{x_n\}$ generated by the algorithm (29) *converges strongly to $x^{\ddagger} = (\operatorname{proj}_{\operatorname{Fix}(\mathbb{T})}\varrho)x^{\ddagger}$.*

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. We will show this fact by induction. According to conditions (C1) and (C2), there exists a sufficiently large positive integer m such that

$$1 - \frac{2}{1-\kappa}(L+1)(L+3)\left(\xi_n + 2\eta_n + \frac{\eta_n^2}{\xi_n}\right) > 0, \quad n \geq m. \tag{30}$$

Fix a $p \in \operatorname{Fix}(\mathbb{T})$ and take a constant $M_1 > 0$ such that

$$\max\left\{\|x_0 - p\|, \|x_1 - p\|, \ldots, \|x_m - p\|, 2\|\varrho(p) - p\|\right\} \leq M_1. \tag{31}$$

Next, we show that $\|x_{m+1} - p\| \leq M_1$.
Set

$$y_m = \xi_m \varrho(x_m) + (1-\xi_m-\eta_m)x_m + \eta_m \mathbb{T}x_m; \tag{32}$$

thus, $x_{m+1} = \operatorname{proj}_{\mathbb{C}}[y_m]$.

Then we have

$$\langle x_{m+1} - y_m, x_{m+1} - p\rangle \leq 0. \tag{33}$$

Since $\mathbb{T}$ is pseudocontractive, $\mathbb{I} - \mathbb{T}$ is monotone. So, we have

$$\langle (\mathbb{I}-\mathbb{T})x_{m+1} - (\mathbb{I}-\mathbb{T})p, x_{m+1} - p\rangle \geq 0. \tag{34}$$

From (29), (33), and (34), we obtain

$$\begin{aligned}
\|x_{m+1} - p\|^2 &= \langle x_{m+1} - p, x_{m+1} - p\rangle \\
&= \langle x_{m+1} - y_m, x_{m+1} - p\rangle + \langle y_m - p, x_{m+1} - p\rangle \\
&\leq \langle y_m - p, x_{m+1} - p\rangle \\
&= \langle x_m - p, x_{m+1} - p\rangle \\
&\quad - \xi_m\langle x_m - \varrho(x_m), x_{m+1} - p\rangle \\
&\quad + \eta_m\langle \mathbb{T}x_m - x_m, x_{m+1} - p\rangle \\
&= \langle x_m - p, x_{m+1} - p\rangle \\
&\quad + \xi_m\langle x_{m+1} - x_m, x_{m+1} - p\rangle \\
&\quad + \xi_m\langle \varrho(x_m) - \varrho(p), x_{m+1} - p\rangle \\
&\quad + \xi_m\langle f(p) - p, x_{m+1} - p\rangle \\
&\quad - \xi_m\langle x_{m+1} - p, x_{m+1} - p\rangle \\
&\quad + \eta_m\langle \mathbb{T}x_m - \mathbb{T}x_{m+1}, x_{m+1} - p\rangle \\
&\quad + \eta_m\langle x_{m+1} - x_m, x_{m+1} - p\rangle \\
&\quad - \eta_m\langle x_{m+1} - \mathbb{T}x_{m+1}, x_{m+1} - p\rangle \\
&\leq \|x_m - p\|\|x_{m+1} - p\| + \xi_m\|x_{m+1} - x_m\| \\
&\quad \times \|x_{m+1} - p\| + \xi_m\kappa\|x_m - p\|\|x_{m+1} - p\| \\
&\quad + \xi_m\|\varrho(p) - p\|\|x_{m+1} - p\| - \xi_m\|x_{m+1} - p\|^2 \\
&\quad + \eta_m(\|\mathbb{T}x_m - \mathbb{T}x_{m+1}\| + \|x_{m+1} - x_m\|) \\
&\quad \times \|x_{m+1} - p\| \\
&\leq (1 + \xi_m\kappa)\|x_m - p\| \\
&\quad \times \|x_{m+1} - p\| + \xi_m\|\varrho(p) - p\| \\
&\quad \times \|x_{m+1} - p\| - \xi_m\|x_{m+1} - p\|^2 \\
&\quad + (L+1)(\xi_m + \eta_m)\|x_{m+1} - x_m\|\|x_{m+1} - p\|.
\end{aligned} \tag{35}$$

It follows that

$$\begin{aligned}
(1+\xi_m)\|x_{m+1} - p\| &\leq (1 + \xi_m\kappa)\|x_m - p\| \\
&\quad + \xi_m\|\varrho(p) - p\| \\
&\quad + (L+1)(\xi_m + \eta_m)\|x_{m+1} - x_m\|.
\end{aligned} \tag{36}$$

By (29), we have

$$\begin{aligned}
&\|x_{m+1}-x_m\| \\
&\quad= \|\mathrm{proj}_{\mathbb{C}}[\xi_m\varrho(x_m)+(1-\xi_m-\eta_m)x_m+\eta_m\mathbb{T}x_m] \\
&\qquad -\mathrm{proj}_{\mathbb{C}}[x_m]\| \\
&\quad\le \|\xi_m\varrho(x_m)+(1-\xi_m-\eta_m)x_m+\eta_m\mathbb{T}x_m-x_m\| \\
&\quad\le \xi_m(\|\varrho(x_m)-\varrho(p)\|+\|\varrho(p)-p\|+\|x_m-p\|) \\
&\qquad+\eta_m(\|\mathbb{T}x_m-p\|+\|x_m-p\|) \\
&\quad\le \xi_m(\|\varrho(p)-p\|+(1+\kappa)\|x_m-p\|) \\
&\qquad+(L+1)\eta_m\|x_m-p\| \\
&\quad\le (L+1+\kappa)(\xi_m+\eta_m)\|x_m-p\|+\xi_m\|\varrho(p)-p\| \\
&\quad\le (L+3)(\xi_m+\eta_m)M_1.
\end{aligned} \tag{37}$$

Substitute (37) into (36) to obtain

$$\begin{aligned}
&(1+\xi_m)\|x_{m+1}-p\| \\
&\quad\le (1+\xi_m\kappa)\|x_m-p\|+\xi_m\|\varrho(p)-p\| \\
&\qquad+(L+1)(L+3)(\xi_m+\eta_m)^2M_1 \\
&\quad\le \left(1+\frac{1+\kappa}{2}\xi_m\right)M_1+(L+1)(L+3)(\xi_m+\eta_m)^2M_1;
\end{aligned} \tag{38}$$

that is,

$$\begin{aligned}
&\|x_{m+1}-p\| \\
&\quad\le \left[1-\frac{((1-\kappa)\xi_m/2)-(L+1)(L+3)(\xi_m+\eta_m)^2}{1+\xi_m}\right]M_1 \\
&\quad= \Bigg\{1-\Bigg(\Bigg(\left(\frac{(1-\kappa)\xi_m}{2}\right)\Bigg[1-\frac{2}{1-\kappa}(L+1)(L+3) \\
&\qquad\qquad\times\left(\xi_m+2\eta_m+\left(\frac{\eta_m^2}{\xi_m}\right)\right)\Bigg]\Bigg) \\
&\qquad\times(1+\xi_m)^{-1}\Bigg)\Bigg\}M_1 \\
&\quad\le M_1.
\end{aligned} \tag{39}$$

By induction, we get

$$\|x_n-p\|\le M_1,\quad \forall n\ge 0, \tag{40}$$

which implies that $\{x_n\}$ is bounded and so is $\{\mathbb{T}x_n\}$. Now, we take a constant $M_2>0$ such that

$$M_2=\sup_n\{\|x_n\|+\|\mathbb{T}x_n-x_n\|\}. \tag{41}$$

Set $\mathbb{S}=(2\mathbb{I}-\mathbb{T})^{-1}$ (i.e., $\mathbb{S}$ is a resolvent of the monotone operator $\mathbb{I}-\mathbb{T}$). We then have that $\mathbb{S}$ is a nonexpansive self mapping of $\mathbb{C}$ and $\mathrm{Fix}(\mathbb{S})=\mathrm{Fix}(\mathbb{T})$.

By Proposition 5, we know that, whenever $\{\gamma_n\}\subset(0,1)$ and $\gamma_n\to 0^+$, the sequence $\{z_n\}$ defined by

$$z_n=\mathbb{S}\,\mathrm{proj}_{\mathbb{C}}[\gamma_n\varrho(z_n)+(1-\gamma_n)z_n] \tag{42}$$

converges strongly to the fixed point $x^\ddagger$ of $\mathbb{S}$ (and of $\mathbb{T}$ as $\mathrm{Fix}(\mathbb{S})=\mathrm{Fix}(\mathbb{T})$). Without loss of generality, we may assume that $\|z_n\|\le M_2$ for all n.

It suffices to prove that $\|x_{n+1}-z_n\|\to 0$ as $n\to\infty$ (for some $\gamma_n\to 0^+$). To this end, we rewrite (42) as

$$(2\mathbb{I}-\mathbb{T})z_n=\mathrm{proj}_{\mathbb{C}}[\gamma_n\varrho(z_n)+(1-\gamma_n)z_n],\quad n\ge 0. \tag{43}$$

By using the property of the metric projection, we have

$$\begin{aligned}
&\langle\gamma_n\varrho(z_n)+(1-\gamma_n)z_n-(2z_n-\mathbb{T}z_n),x_{n+1} \\
&\qquad-(2z_n-\mathbb{T}z_n)\rangle\le 0 \\
&\Longrightarrow\langle-\gamma_n(z_n-\varrho(z_n)),x_{n+1}-z_n-(z_n-\mathbb{T}z_n)\rangle \\
&\qquad+\langle\mathbb{T}z_n-z_n,x_{n+1}-z_n-(z_n-\mathbb{T}z_n)\rangle\le 0 \\
&\Longrightarrow\langle-\gamma_n(z_n-\varrho(z_n))+\mathbb{T}z_n-z_n,x_{n+1}-z_n\rangle \\
&\qquad+\|z_n-\mathbb{T}z_n\|^2\le\langle\gamma_n(z_n-\varrho(z_n)),\mathbb{T}z_n-z_n\rangle \\
&\Longrightarrow\langle-\gamma_n(z_n-\varrho(z_n))+\mathbb{T}z_n-z_n,x_{n+1}-z_n\rangle \\
&\quad\le\gamma_n\|z_n-\varrho(z_n)\|\|\mathbb{T}z_n-z_n\| \\
&\Longrightarrow\left\langle-(z_n-\varrho(z_n))+\frac{\mathbb{T}z_n-z_n}{\gamma_n},x_{n+1}-z_n\right\rangle \\
&\quad\le\|z_n-\varrho(z_n)\|\|\mathbb{T}z_n-z_n\|.
\end{aligned} \tag{44}$$

Note that

$$\begin{aligned}
\|z_n-\mathbb{T}z_n\|&=\|\mathrm{proj}_{\mathbb{C}}[\gamma_n\varrho(z_n)+(1-\gamma_n)z_n]-z_n\| \\
&\le\|\gamma_n\varrho(z_n)+(1-\gamma_n)z_n-z_n\| \\
&=\gamma_n\|z_n-\varrho(z_n)\|.
\end{aligned} \tag{45}$$

Hence, we get

$$\left\langle-(z_n-\varrho(z_n))+\frac{\mathbb{T}z_n-z_n}{\gamma_n},x_{n+1}-z_n\right\rangle\le\gamma_n\|z_n-\varrho(z_n)\|^2. \tag{46}$$

From (42), we have

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|\mathbb{S}\,\mathrm{proj}_{\mathbb{C}}[\gamma_{n+1}\varrho(z_{n+1})+(1-\gamma_{n+1})z_{n+1}] \\ &\quad -\mathbb{S}\,\mathrm{proj}_{\mathbb{C}}[\gamma_n\varrho(z_n)+(1-\gamma_n)z_n]\| \\ &\le \|\gamma_{n+1}\varrho(z_{n+1})+(1-\gamma_{n+1})z_{n+1} \\ &\quad -\gamma_n\varrho(z_n)-(1-\gamma_n)z_n\| \\ &= \|(1-\gamma_{n+1})(z_{n+1}-z_n)+(\gamma_n-\gamma_{n+1}) \\ &\quad \times(z_n-\varrho(z_n))+\gamma_{n+1}(\varrho(z_{n+1})-\varrho(z_n))\| \\ &\le [1-(1-\kappa)\gamma_{n+1}]\|z_{n+1}-z_n\| \\ &\quad +|\gamma_{n+1}-\gamma_n|\,\|z_n-\varrho(z_n)\|. \end{aligned} \tag{47}$$

It follows that

$$\|z_{n+1}-z_n\| \le \frac{|\gamma_{n+1}-\gamma_n|}{(1-\kappa)\gamma_{n+1}}\|z_n-\varrho(z_n)\|. \tag{48}$$

Set

$$\gamma_n := \frac{\xi_n}{\eta_n}. \tag{49}$$

By condition (C2), $\gamma_n \to 0^+$ and $\gamma_n \in (0,1)$, for n large enough. Hence, by (46) and (48), we have

$$\begin{aligned}&\left\langle -(z_n-\varrho(z_n))+\frac{\eta_n(\mathbb{T}z_n-z_n)}{\xi_n}, x_{n+1}-z_n\right\rangle \\ &\quad \le \frac{\xi_n}{\eta_n}\|z_n-\varrho(z_n)\|^2 \le \frac{\xi_n}{\eta_n}M_2^2, \\ &\|z_n-z_{n-1}\| \le \frac{|\xi_n\eta_{n-1}-\xi_{n-1}\eta_n|}{\xi_n\eta_{n-1}}M_2. \end{aligned} \tag{50}$$

By (29), we have

$$\begin{aligned}&\|x_{n+1}-x_n\| \\ &\quad = \|\mathrm{proj}_{\mathbb{C}}[\xi_n\varrho(x_n)+(1-\xi_n-\eta_n)x_n+\eta_n\mathbb{T}x_n] \\ &\qquad -\mathrm{proj}_{\mathbb{C}}x_n\| \\ &\quad \le \xi_n\|x_n-\varrho(x_n)\|+\eta_n\|\mathbb{T}x_n-x_n\| \\ &\quad \le (\xi_n+\eta_n)M_2. \end{aligned} \tag{51}$$

Next, we estimate $\|x_{n+1}-z_{n+1}\|$. Since $x_{n+1} = \mathrm{proj}_{\mathbb{C}}[y_n]$, $\langle x_{n+1}-y_n, x_{n+1}-z_n\rangle \le 0$. Then, we have

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &= \langle x_{n+1}-z_n, x_{n+1}-z_n\rangle \\ &= \langle x_{n+1}-y_n, x_{n+1}-z_n\rangle \\ &\quad +\langle y_n-z_n, x_{n+1}-z_n\rangle \\ &\le \langle y_n-z_n, x_{n+1}-z_n\rangle \\ &= \langle[\xi_n\varrho(x_n)+(1-\xi_n-\eta_n)x_n+\eta_n\mathbb{T}x_n] \\ &\quad -z_n, x_{n+1}-z_n\rangle \\ &= (1-\xi_n-\eta_n)\langle x_n-z_n, x_{n+1}-z_n\rangle \\ &\quad +\eta_n\langle\mathbb{T}x_n-\mathbb{T}x_{n+1}, x_{n+1}-z_n\rangle \\ &\quad +\eta_n\langle\mathbb{T}x_{n+1}-\mathbb{T}z_n, x_{n+1}-z_n\rangle \\ &\quad +\langle\xi_n(\varrho(x_n)-z_n) \\ &\qquad +\eta_n(\mathbb{T}z_n-z_n), x_{n+1}-z_n\rangle. \end{aligned} \tag{52}$$

It follows that

$$\begin{aligned}&\|x_{n+1}-z_n\|^2 \\ &\quad \le (1-\xi_n-\eta_n)\|x_n-z_n\|\,\|x_{n+1}-z_n\| \\ &\qquad +\eta_nL\|x_n-x_{n+1}\|\,\|x_{n+1}-z_n\| \\ &\qquad +\eta_n\|x_{n+1}-z_n\|^2 \\ &\qquad +\xi_n\left\langle\varrho(z_n)-z_n+\frac{\eta_n}{\xi_n}(\mathbb{T}z_n-z_n), x_{n+1}-z_n\right\rangle \\ &\quad \le \frac{1-\xi_n-\eta_n}{2}\left(\|x_n-z_n\|^2+\|x_{n+1}-z_n\|^2\right) \\ &\qquad +\frac{\eta_n^2}{2}\|x_{n+1}-z_n\|^2+\frac{L^2}{2}\|x_n-x_{n+1}\|^2 \\ &\qquad +\eta_n\|x_{n+1}-z_n\|^2+\frac{\xi_n^2}{\eta_n}\|z_n-\varrho(z_n)\|^2. \end{aligned} \tag{53}$$

Thus,

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &\le \frac{1-\xi_n-\eta_n}{1+\xi_n-\eta_n}\|x_n-z_n\|^2 \\ &\quad +\frac{L^2}{1+\xi_n-\eta_n}\|x_{n+1}-x_n\|^2 \\ &\quad +\frac{2\xi_n^2}{(1+\xi_n-\eta_n)\eta_n}\|z_n-\varrho(z_n)\|^2 \\ &\quad +\frac{\eta_n^2}{1+\xi_n-\eta_n}\|x_{n+1}-z_n\|^2 \\ &\le \left(1-\frac{2\xi_n}{1+\xi_n-\eta_n}\right)\|x_n-z_n\|^2 \\ &\quad +\frac{(\xi_n+\eta_n)^2}{1+\xi_n-\eta_n}L^2M_2^2+\frac{2\xi_n^2}{(1+\xi_n-\eta_n)\eta_n}M_2^2 \\ &\quad +\frac{\eta_n^2}{1+\xi_n-\eta_n}4M_2^2 \\ &\le \left(1-\frac{2\xi_n}{1+\xi_n-\eta_n}\right) \\ &\quad \times(\|x_n-z_{n-1}\|+\|z_n-z_{n-1}\|)^2 \end{aligned}$$

$$
\begin{aligned}
&+\left\{\frac{(\xi_n+\eta_n)^2}{1+\xi_n-\eta_n}+\frac{2\xi_n^2}{(1+\xi_n-\eta_n)\,\eta_n}\right.\\
&\qquad\left.+\frac{\eta_n^2}{1+\xi_n-\eta_n}\right\}M\\
&\leq\left(1-\frac{2\xi_n}{1+\xi_n-\eta_n}\right)\left\|x_n-z_{n-1}\right\|^2\\
&\quad+\frac{1}{1+\xi_n-\eta_n}\left\|z_n-z_{n-1}\right\|\\
&\quad\times\left(2\left\|x_n-z_{n-1}\right\|+\left\|z_n-z_{n-1}\right\|\right)\\
&\quad+\left\{\frac{(\xi_n+\eta_n)^2}{1+\xi_n-\eta_n}+\frac{2\xi_n^2}{(1+\xi_n-\eta_n)\,\eta_n}\right.\\
&\qquad\left.+\frac{\eta_n^2}{1+\xi_n-\eta_n}\right\}M\\
&\leq\left(1-\frac{2\xi_n}{1+\xi_n-\eta_n}\right)\left\|x_n-z_{n-1}\right\|^2\\
&\quad+\frac{1}{1+\xi_n-\eta_n}\frac{\left|\xi_n\eta_{n-1}-\xi_{n-1}\eta_n\right|}{\xi_n\eta_{n-1}}M\\
&\quad+\left\{\frac{(\xi_n+\eta_n)^2}{1+\xi_n-\eta_n}+\frac{2\xi_n^2}{(1+\xi_n-\eta_n)\,\eta_n}\right.\\
&\qquad\left.+\frac{\eta_n^2}{1+\xi_n-\eta_n}\right\}M,
\end{aligned}
\tag{54}
$$

where the finite constant $M>0$ is given by

$$
\begin{aligned}
M\ :=\max\Big\{&L^2M_2^2,4M_2^2,\\
&M_2\sup_n\left(2\left\|x_n-z_{n-1}\right\|+\left\|z_n-z_{n-1}\right\|\right)\Big\}.
\end{aligned}
\tag{55}
$$

Set

$$
\begin{aligned}
\delta_n&=\frac{2\xi_n}{1+\xi_n-\eta_n}\approx 2\xi_n\quad(\text{as } n\longrightarrow\infty),\\
\theta_n&=\left\{\frac{\xi_n\eta_{n-1}-\xi_{n-1}\eta_n}{2\xi_n^2\eta_{n-1}}\right.\\
&\qquad\left.+\frac{1}{2}\left(\xi_n+2\eta_n+\frac{\eta_n^2}{\xi_n}\right)+\frac{\xi_n}{\eta_n}+\frac{\eta_n^2}{2\xi_n}\right\}M.
\end{aligned}
\tag{56}
$$

Then (54) can be rewritten as

$$
\left\|x_{n+1}-z_n\right\|^2\leq\left(1-\delta_n\right)\left\|x_n-z_{n-1}\right\|^2+\delta_n\theta_n.
\tag{57}
$$

By the conditions (C1)–(C3), we deuce that

$$
\lim_{n\to\infty}\delta_n=0,\qquad\sum_{n=1}^{\infty}\delta_n=\infty,\qquad\lim_{n\to\infty}\theta_n=0.
\tag{58}
$$

From Lemma 4 and (57), we get $\|x_{n+1}-z_n\|^2\ \to\ 0$ as $n\ \to\ \infty$. This completes the proof. □

Algorithm 9. Let $\mathbb{C}$ be a nonempty closed subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{T}:\mathbb{C}\to\mathbb{C}$ be a pseudocontraction. Let $\{\xi_n\}\subset[0,1]$ and $\{\eta_n\}\subset[0,1]$ be two real number sequences. For $x_0\in\mathbb{C}$, we define a sequence $\{x_n\}$ iteratively by

$$
x_{n+1}=\operatorname{proj}_{\mathbb{C}}\left[\left(1-\xi_n-\eta_n\right)x_n+\eta_n\mathbb{T}x_n\right],\qquad n\geq 0.
\tag{59}
$$

Corollary 10. *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{T}:\mathbb{C}\to\mathbb{C}$ be an L-Lipschitzian and pseudocontraction with* $\mathrm{Fix}(\mathbb{T})\neq\emptyset$. *Suppose the following conditions are satisfied:*

(*C*1) $\lim_{n\to\infty}\xi_n=0$ *and* $\sum_{n=0}^{\infty}\xi_n=\infty$;

(*C*2) $\lim_{n\to\infty}(\xi_n/\eta_n)=\lim_{n\to\infty}(\eta_n^2/\xi_n)=0$;

(*C*3) $\lim_{n\to\infty}((\xi_n\eta_{n-1}-\xi_{n-1}\eta_n)/\xi_n^2\eta_{n-1})=0$.

Then the sequence $\{x_n\}$ generated by the algorithm (59) *converges strongly to the minimum-norm fixed point of* $\mathbb{T}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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