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Research Article

Positive Solutions for Coupled Nonlinear Fractional Differential Equations

Wenning Liu,[1] Xingjie Yan,[1] and Wei Qi[2]

[1] *Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221008, China*
[2] *Department of Physics, China University of Mining and Technology, Xuzhou, Jiangsu 221008, China*

Correspondence should be addressed to Xingjie Yan; yanxingjie04@gmail.com

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We consider the existence of positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary values. Assume the nonlinear term is superlinear in one equation and sublinear in the other equation. By constructing two cones K_1, K_2 and computing the fixed point index in product cone $K_1 \times K_2$, we obtain that the system has a pair of positive solutions. It is remarkable that it is established on the Cartesian product of two cones, in which the feature of two equations can be opposite.

1. Introduction

Fractional calculus is a very old concept dating back to 17th century; it involves fractional integration and fractional differentiation. At the first stage, fractional calculus theory is mainly focused on pure mathematical fields. In the last few decades, fractional differential equations and fractional integration equations have found many applications in various fields, such as science and engineering, physics, chemistry, biology, economics, and signal and image processing; for details, see [1–5]. In recent years, fractional differential equations have attracted increasing interests for their extensive applications, which leads to intensive development of the theory of fractional calculus. And more and more results about the existence and uniqueness of solutions appear. We can refer to [6] for the latest studies of fractional calculus.

The existence theory for initial value problems has been paid considerable attentions by many authors; see [3, 7, 8] and the references cited therein. Also the existence of positive solutions for boundary value problem of nonlinear fractional differential equation has attracted attentions from many researches; see [9–14]. By using the cone extension method Feng et al. [15] studied the existence of solutions for higher-order nonlinear fractional differential equation with integral boundary conditions:

$$\begin{aligned} D_{0+}^{\alpha} u(t) + g(t) f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \int_0^1 h(t) u(t)\, dt, \end{aligned} \tag{1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $n-1 < \alpha < n, n \geq 3$, $g \in \mathscr{C}((0,1),(0,+\infty))$ and g may be singular at $t = 0$ or $t = 1$, $h \in L^1(0,1)$ is nonnegative, and $f \in \mathscr{C}((0,1) \times (0,+\infty),(0,+\infty))$.

In recent years, many authors have established the existence and uniqueness for solutions of some systems of nonlinear fractional differential equations; readers can see [16–21]. For example, by using the alternative of Leray-Schauder theorem method Wang et al. [22] obtained the existence and uniqueness of positive solution of nonzero

boundary values problem for a coupled system of nonlinear fractional differential equations:

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= f(t, v(t)), \quad 0 < t < 1, \\ D_{0+}^{\beta} v(t) &= g(t, u(t)), \quad 0 < t < 1, \\ u(0) &= 0, \qquad u(1) = a u(\xi), \\ v(0) &= 0, \qquad v(1) = b v(\xi), \end{aligned} \tag{2}$$

where $1 < \alpha, \beta < 2$, $0 \le a \le b \le 1$, $0 < \xi < 1$, $f, g : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. By using the cone extension and cone compression Zhao et al. [23] studied the existence and nonexistence of positive solutions for a class of third order boundary value problem with integral boundary conditions in Banach spaces:

$$\begin{aligned} &x''' + f(t, x(t)) = 0, \quad t \in J, \\ &x(0) = 0, \qquad x''(0) = 0, \qquad x(1) = \int_0^1 g(t) x(t)\, dt. \end{aligned} \tag{3}$$

Or

$$\begin{aligned} &x''' + f(t, x(t)) = 0, \quad t \in J, \\ &x(0) = \int_0^1 g(t) x(t)\, dt, \qquad x''(0) = 0, \qquad x(1) = 0, \end{aligned} \tag{4}$$

where $J = (0,1)$, $f \in \mathscr{C}((0,1) \times P, P)$, $g \in L^1(0,1)$ is nonnegative and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative; P is a cone.

By using the cone extension and cone compression method Yang et al. [24] established sufficient conditions for the existence and nonexistence of positive solutions for a general class with integral boundary value problem for a coupled system of fractional equations:

$$\begin{aligned} D_{0+}^{\alpha} u(t) + a(t) f(t, v(t)) &= 0, \quad 0 < t < 1, \\ D_{0+}^{\beta} v(t) + b(t) g(t, u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) &= \int_0^1 \phi(t) u(t)\, dt, \\ v(0) = 0, \qquad v(1) &= \int_0^1 \varphi(t) v(t)\, dt, \end{aligned} \tag{5}$$

where $1 < \alpha, \beta \le 2, a, b \in \mathscr{C}((0,1), (0,+\infty))$, $f, g \in \mathscr{C}((0,1) \times (0,+\infty), (0,+\infty))$, and $\phi, \varphi \in L^1(0,1)$ are nonnegative and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative.

However, all these works are obtained by getting a fixed point of a completely continuous operator in a cone of product Banach spaces $X_1 \times X_2$. Thus, there is no essential difference between we consider the positive solution of coupled equations and positive solution of single equation. Furthermore, the authors cannot guarantee the obtained pair of positive solutions are all positive; maybe one is positive and another is zero. The features of nonlinear term are the same, which means the growth properties of nonlinear term are similar.

Motivated by the above mentioned works, in this paper, we consider the existence of positive solution to boundary value problem for a coupled system of nonlinear fractional differential equations as follows:

$$\begin{aligned} D_{0+}^{\alpha} u(t) + f(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\ D_{0+}^{\beta} v(t) + g(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) &= \int_0^1 \phi(t) u(t)\, dt, \\ v(0) = 0, \qquad v(1) &= \int_0^1 \varphi(t) v(t)\, dt, \end{aligned} \tag{6}$$

where $1 < \alpha, \beta \le 2$, $f, g \in \mathscr{C}((0,1) \times (0,+\infty) \times (0,+\infty), (0,+\infty))$, and $\phi, \varphi \in L^1(0,1)$ are nonnegative and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. f and g have different features. It is very difficult to deal with our problem if we directly use the usual method because the features of nonlinear term are different. In order to overcome the difficulties, by using the ideals in [25–27], we need to consider our problem on the Cartesian product of two cones in the space $\mathscr{C}(0,1)$; thus we can better exploit the feature of two equations. We choose a cone $K_1 \times K_2$ in the Cartesian product of two cones in $\mathscr{C}(0,1) \times \mathscr{C}(0,1)$. By computing the fixed point index in $K_1 \times K_2$, we obtain the existence of positive solution (u, v) of problem (6) such that $u > 0$, $v > 0$.

2. Preliminary

In this section, we review some basic facts which are used throughout this paper. We can see [10, 16, 25–29] for more details.

Definition 1. For a function $f : [a, +\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \tag{7}$$

and the right-hand side is pointwise defined on $[a, +\infty)$, where $a \in \mathbb{R}$ and Γ is the gamma function.

Definition 2. For a function $f : [a, +\infty) \to \mathbb{R}$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is given by

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \tag{8}$$

and the right-hand side is pointwise defined on $[a, +\infty)$, where $a \in \mathbb{R}$, $n = [\alpha] + 1$ and Γ is the gamma function.

Let E be a real Banach space with norm $\|\,\|$ and let $A : E \to E$ be an operator. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following: (i) if $x \in P$ and $\lambda \ge 0$, then $\lambda x \in P$; (ii) if $x \in P$ and $-x \in P$, then $x = 0$

are satisfied. u is said to be a fixed point of A, if $u = Au$. Ω is a bounded open subset of E with boundary $\partial\Omega$. Denote $i\ (A, P \cap \Omega, P)$ the fixed point index of A.

Lemma 3 (see [25–27]). *Let E be a Banach space and let $K_i \subset E$ $(i = 1, 2)$ be a closed convex cone in E. For $r_i > 0$ $(i = 1, 2)$, denote $K_{r_i} = \{u \in K_i \mid \|u\| < r_i\}$, $\partial K_i = \{u \in K_i \mid \|u\| = r_i\}$. Suppose $A_i : K_i \to K_i$ is completely continuous. If $u_i \neq A_i u_i$, $\forall u_i \in \partial K_i$, then*

$$i\left(A, K_{r_1} \times K_{r_2}, K_1 \times K_2\right) = i\left(A_1, K_{r_1}, K_1\right) \cdot i\left(A_2, K_{r_2}, K_2\right), \tag{9}$$

where $A(u, v) = (A_1 u, A_2 v)$ and for all $(u, v) \in (K_1 \times K_2)$.

In order to prove the existence of solution for problem (6), we can translate it into obtaining a fixed point of operator. Then we use the topological method to deal with it.

Theorem 4. *Let E be a Banach space and let $K_1, K_2 \subset E$ be a closed convex cone in E. Denote $K_{r_i} = \{u \in K_i \mid \|u\| < r_i\}$, $\partial K_i = \{u \in K_i \|u\| = r_i\}$, where $r > 0$, $i = 1, 2$. Let $D = (K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}})$. Suppose $T : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous and $T(u, v) = (T_1(u, v), T_2(u, v))$. For $\lambda \in [0, 1]$, let*

$$\begin{aligned} H_{\lambda,1}(u, v) &= \lambda T_1(u, v) + (1 - \lambda) T_1(u, 0), \\ H_{\lambda,2}(u, v) &= \lambda T_2(u, v) + (1 - \lambda) T_2(0, v). \end{aligned} \tag{10}$$

If $H_{\lambda,1}$, $H_{\lambda,2}$ satisfy the following conditions:

(i) *$H_{\lambda,1}$ is uniformly cone extension about $v \in K_{R_2}$ at $K_{R_1} \setminus \overline{K_{r_1}}$, that is to say,*

$$\begin{aligned} \left\|H_{\lambda,1}\right\| &< \|u\|, \quad \forall u \in \partial K_{r_1}, \\ \left\|H_{\lambda,1}\right\| &> \|u\|, \quad \forall u \in \partial K_{R_1}; \end{aligned} \tag{11}$$

(ii) *$H_{\lambda,2}$ is uniformly cone compression about $u \in K_{R_1}$ at $K_{R_2} \setminus \overline{K_{r_2}}$, that is to say,*

$$\begin{aligned} \left\|H_{\lambda,2}\right\| &> \|v\|, \quad \forall v \in \partial K_{r_2}, \\ \left\|H_{\lambda,2}\right\| &< \|v\|, \quad \forall v \in \partial K_{R_2}, \end{aligned} \tag{12}$$

then $i(T, D, K_1 \times K_2) = -1$. Thus the mapping T in the region D has a fixed point.

Proof. Following a similar procedure to the method given in [25–27], we present a proof. We suppose that $T_1(u, 0) = A_1 u$ and $T_2(0, v) = A_2 v$. Let $A(u, v) = (A_1 u, A_2 v)$. For $\lambda \in [0, 1]$, for any $(u, v) \in K_1 \times K_2$, set

$$H((u, v), \lambda) = \lambda T(u, v) + (1 - \lambda) A(u, v). \tag{13}$$

Since T is completely continuous, hence the mapping H is also completely continuous. By Lemma 3 and homotopy invariance of fixed point index (see [7, 28]), we have

$$\begin{aligned} &i(T, D, K_1 \times K_2) \\ &\quad = i\left(A_1, K_{R_1} \setminus \overline{K_{r_1}}, K_1\right) \cdot i\left(A_2, K_{R_2} \setminus \overline{K_{r_2}}, K_2\right). \end{aligned} \tag{14}$$

Next, we compute $i(T, D, K_1 \times K_2)$. By assumptions (i) and (ii), also using the properties of fixed point index (see [7, 28]), we have

$$\begin{aligned} i\left(A_1, K_{r_1}, K_1\right) &= i\left(A_2, K_{R_2}, K_2\right) = 1, \\ i\left(A_1, K_{R_1}, K_1\right) &= i\left(A_2, K_{r_2}, K_2\right) = 0. \end{aligned} \tag{15}$$

Thanks to (15), we obtain

$$\begin{aligned} &i(T, D, K_1 \times K_2) \\ &\quad = i\left(A_1, K_{R_1} \setminus \overline{K_{r_1}}, K_1\right) \cdot i\left(A_2, K_{R_2} \setminus \overline{K_{r_2}}, K_2\right) \\ &\quad = \left(i\left(A_1, K_{R_1}, K_1\right) - i\left(A_1, \overline{K_{r_1}}, K_1\right)\right) \\ &\qquad \cdot \left(i\left(A_2, K_{R_2}, K_2\right) - i\left(A_2, \overline{K_{r_2}}, K_2\right)\right) \\ &\quad = (0 - 1) \cdot (1 - 0) = -1. \end{aligned} \tag{16}$$

The proof is complete. □

Remark 5. If $H_{\lambda,1}$, $H_{\lambda,2}$ were all cone extension or cone compression, we can obtain the same conclusions.

The following lemmas are necessary to prove the existence of positive solution for problem (6).

Lemma 6 (see [15]). *Assume that $\int_0^1 \phi(t)u(t)dt \neq 1$. Then for any $\psi \in \mathscr{C}(0, 1)$, the solution of boundary value problem*

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + \psi(t) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = \int_0^1 \phi(t)\, u(t)\, dt, \end{aligned} \tag{17}$$

is given by $u(t) = \int_0^1 G_{1\alpha}(t, s)\psi(s)ds$, where $G_{1\alpha}(t, s) = G_{2\alpha}(t, s) + G_{3\alpha}(t, s)$,

$$\begin{aligned} G_{2\alpha}(t, s) &= \begin{cases} \dfrac{t^{\alpha-1}(1 - s)^{\alpha-1} - (t - s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1. \\ \dfrac{t^{\alpha-1}(1 - s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \\ G_{3\alpha}(t, s) &= \frac{t^{\alpha-1}}{1 - \int_0^1 \phi(t)\, t^{\alpha-1} dt} \int_0^1 \phi(t)\, G_{2\alpha}(t, s)\, dt. \end{aligned} \tag{18}$$

Lemma 7 (see [15]). *If $\int_0^1 \phi(t)t^{\alpha-1}dt \in [0, 1)$, then Green's function $G_{1\alpha}(t, s)$ has the following properties:*

(i) *$G_{1\alpha}(t, s) \ge 0$ is continuous for any $t, s \in [0, 1]$, $G_{1\alpha}(t, s) > 0$ for any $t, s \in (0, 1)$;*

(ii) *$G_{1\alpha}(t, s) \le G_{1\alpha}(s)$ for any $t, s \in [0, 1]$ and $\min_{t \in [\theta, 1-\theta]} G_{1\alpha}(t, s) \ge \gamma_\theta G_{1\alpha}(s)$, where $G_{1\alpha}(s) = G_{2\alpha}(s, s) + G_{3\alpha}(1, s)$, $\gamma_\theta = \theta^{\alpha-1}$, $\theta \in (0, 1/2)$.*

3. Main Results

In this section, we will discuss the existence of positive solutions for problem (6). For a coupled system, many researchers transformed the problem into computing the fixed point index of the composition operator on the single cone in the Cartesian product space and the nonlinear terms in two equations have similar features. However it is very difficult to deal with our problem directly. Denote $\mathscr{C}^+[0,1]$ the continuous function in $[0,1]$ with nonnegative value. In order to make full use of the different nature of the two equations, we construct single cone K_i and open set $K_{R_i} \setminus \overline{K_{r_i}}$, $i = 1,2$ in $\mathscr{C}^+[0,1]$. Thus we transform the problem into computing the fixed point index of operator that is defined in Cartesian product of two cones in open set $D = (K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}})$ of product cone $K_1 \times K_2$.

3.1. The Transformation of the Problem. Firstly, we construct the relevant cone K in Banach space $\mathscr{C}^+[0,1]$. Then we transform the problem into computing the fixed point index of Cartesian product of cone with the nature of analytical property of Green's function.

Following, we introduce notations:

$$\mathscr{C}^+[0,1] = \{u \in \mathscr{C}[0,1] \mid u(t) \geq 0, \forall t \in [0,1]\}, \tag{19}$$

with the norm $\|u\| = \max_{t\in[0,1]}\{|u(t)|\}$. Clearly, it is a Banach space. Define

$$\begin{aligned} K_1 = \Big\{ & u \in \mathscr{C}^+[0,1] \mid u(t) \geq \left(\frac{1}{2}\right)^{\alpha-1} \|u(t)\|, \\ & \forall t \in [\theta, 1-\theta], \theta \in \left(0, \frac{1}{2}\right) \Big\}, \\ K_2 = \Big\{ & u \in \mathscr{C}^+[0,1] \mid u(t) \geq \left(\frac{1}{2}\right)^{\beta-1} \|u(t)\|, \\ & \forall t \in [\theta, 1-\theta], \theta \in \left(0, \frac{1}{2}\right) \Big\}. \end{aligned} \tag{20}$$

For $\lambda \in [0,1], u, v \in \mathscr{C}^+[0,1]$, we define the operators

$$\begin{aligned} & T_{\lambda,1}(u,v) : \mathscr{C}^+[0,1] \times \mathscr{C}^+[0,1] \longrightarrow \mathscr{C}^+[0,1], \\ & T_{\lambda,2}(u,v) : \mathscr{C}^+[0,1] \times \mathscr{C}^+[0,1] \longrightarrow \mathscr{C}^+[0,1], \\ & T_\lambda(u,v) : \mathscr{C}^+[0,1] \times \mathscr{C}^+[0,1] \longrightarrow \mathscr{C}^+[0,1] \times \mathscr{C}^+[0,1], \end{aligned} \tag{21}$$

as follows:

$$\begin{aligned} T_{\lambda,1}(u,v) &= \int_0^1 G_{1\alpha}(t,s) \big(\lambda f(s,u(s),v(s)) \\ &\qquad + (1-\lambda) f(s,u(s),0)\big) ds, \\ T_{\lambda,2}(u,v) &= \int_0^1 G_{1\beta}(t,s) \big(\lambda g(s,u(s),v(s)) \\ &\qquad + (1-\lambda) g(s,0,v(s))\big) ds, \\ T_\lambda(u,v)(t) &= \big(T_{\lambda,1}(u,v)(t), T_{\lambda,2}(u,v)(t)\big). \end{aligned} \tag{22}$$

Then we translate that problem (6) possesses a pair of solutions (u,v) if and only if $T_\lambda(u,v)(t)$ have a fixed point in $K_1 \times K_2$.

Lemma 8. *The mapping $T_\lambda : K_1 \times K_2 \to K_1 \times K_2$ as mentioned above is completely continuous.*

Proof. For any $(u,v) \in K_1 \times K_2$, we want to prove $T_\lambda(u,v) \in K_1 \times K_2$, that is, to prove $T_{\lambda,1}(u,v) \in K_1$ and $T_{\lambda,2}(u,v) \in K_2$. By the preceding definition and the nature of Green's function, we have for $t \in [\theta, 1-\theta]$, $\theta \in (0, 1/2)$

$$\begin{aligned} & T_{\lambda,1}(u,v) \\ &= \int_0^1 G_{1\alpha}(t,s) \big(\lambda f(s,u(s),v(s)) \\ &\qquad + (1-\lambda) f(s,u(s),0)\big) ds \\ &\geq \int_0^1 \gamma_\theta G_{1\alpha}(s) \big(\lambda f(s,u(s),v(s)) \\ &\qquad + (1-\lambda) f(s,u(s),0)\big) ds \\ &= \gamma_\theta \int_0^1 G_{1\alpha}(s) \big(\lambda f(s,u(s),v(s)) \\ &\qquad + (1-\lambda) f(s,u(s),0)\big) ds \\ &\geq \left(\frac{1}{2}\right)^{\alpha-1} \int_0^1 G_{1\alpha}(t,s) \big(\lambda f(s,u(s),v(s)) \\ &\qquad + (1-\lambda) f(s,u(s),0)\big) ds \\ &\geq \left(\frac{1}{2}\right)^{\alpha-1} \|T_{\lambda,1}(u,v)\|. \end{aligned} \tag{23}$$

Similarly, $T_{\lambda,2}(u,v) \geq (1/2)^{\beta-1}\|T_{\lambda,2}(u,v)\|$, $t \in [\theta, 1-\theta]$, $\theta \in (0,1/2)$. Thus, $T_{\lambda,1}(u,v) \in K_1$, $T_{\lambda,2}(u,v) \in K_2$, and $T_\lambda(K_1 \times K_2) \subset K_1 \times K_2$.

It is easy to prove that $T_\lambda(u,v)$ is continuous in $K_1 \times K_2$.

So as Arzela-Ascoli theorem, we prove that $T_\lambda : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous. □

3.2. The Main Result and Its Proof. Now, we present our main result.

Theorem 9. *Assume that the nonlinear terms of the problem* (6) *f and g meet the following conditions:*

$$(H_1) \begin{cases} (a_1) \lim\limits_{u \to 0^+} \sup \max\limits_{t\in[0,1]} \dfrac{f(t,u,v)}{u} = 0 \\ \quad \textit{is established consistent about } v \in \mathbb{R}^+, \\ (b_1) \lim\limits_{u \to +\infty} \inf \min\limits_{t\in[0,1]} \dfrac{f(t,u,v)}{u} = +\infty \\ \quad \textit{is established consistent about } v \in \mathbb{R}^+, \end{cases}$$

$$(H_2)\begin{cases}(a_2)\ \lim\limits_{v\to 0^+}\inf\min\limits_{t\in[0,1]}\dfrac{g(t,u,v)}{v}=+\infty\\ \quad \textit{is established consistent about } u\in\mathbb{R}^+,\\ (b_2)\ \lim\limits_{v\to+\infty}\sup\max\limits_{t\in[0,1]}\dfrac{g(t,u,v)}{v}=0\\ \quad \textit{is established consistent about } u\in\mathbb{R}^+,\end{cases}$$

$$(H_3)\ \lim_{u\to+\infty}\sup\max_{t\in[0,1]}g(t,u,v)=h(v)$$

$$\textit{is established consistent about } v\in[0,M], \tag{24}$$

where $h(v)$ is continuous in $\mathbb{R}^+$ and M is a constant.

Then the problem (6) *has at least a pair of positive solutions.*

Proof. We select an appropriate open set $D=(K_{R_1}\setminus\overline{K_{r_1}})\times(K_{R_2}\setminus\overline{K_{r_2}})$ in the product of cone $K_1\times K_2$; then we will verify that the solution operator meets the conditions of Theorem 4.

In the following, we separate our proof into four steps.

Step 1. Select $r_1>0$, in such a fashion that $\|T_{\lambda,1}(u,v)\|<\|u\|$, where $\lambda\in[0,1]$, $(u,v)\in\partial K_{r_1}\times K_1$. Based on assuming $(H_1)(a_1)$, for $\varepsilon_1=(\int_0^1 G_{1\alpha}(s)ds)^{-1}>0$, there exists $r_1>0$, such that $f(t,u,v)<\varepsilon_1 u$, for all $t\in[0,1]$, $0\le u\le r_1$, $v\ge 0$. Then when $\lambda\in[0,1]$, $(u,v)\in\partial K_{r_1}\times K_1$, we have for $t\in[0,1]$

$$\begin{aligned}&\|T_{\lambda,1}(u,v)\|\\ &=\max_{t\in[0,1]}\left|\int_0^1 G_{1\alpha}(t,s)\left(\lambda f(s,u(s),v(s))\right.\right.\\ &\qquad\left.\left.+(1-\lambda)f(s,u(s),0)\right)ds\right|\\ &<\max_{t\in[0,1]}\int_0^1 G_{1\alpha}(s)\left|\lambda\varepsilon_1 u(s)+(1-\lambda)\varepsilon_1 u(s)\right|ds\\ &\le\max_{t\in[0,1]}\int_0^1 G_{1\alpha}(s)\varepsilon_1|u(s)|\,ds\\ &\le\varepsilon_1\int_0^1 G_{1\alpha}(s)\max_{t\in[0,1]}|u(s)|\,ds\\ &\le\varepsilon_1\int_0^1 G_{1\alpha}(s)\,ds\,\|u\|=\|u\|.\end{aligned}\tag{25}$$

That is to say, $\|T_{\lambda,1}(u,v)\|<\|u\|$, where $\lambda\in[0,1]$, $(u,v)\in\partial K_{r_1}\times K_1$.

Step 2. Select $R_1>r_1$; we claim that $\|T_{\lambda,1}(u,v)\|>\|u\|$, where $\lambda\in[0,1]$, $(u,v)\in\partial K_{R_1}\times K_1$. In view of the assumption $(H_1)(b_1)$, for $N_1=(\gamma_\theta\int_0^1 G_{1\alpha}(s)ds)^{-1}>0$, there exists $R_0>0$, such that $f(t,u,v)<N_1u$, for all $t\in[0,1]$, $u\ge R_0$, $v\ge 0$. Let $R_1>\max\{r_1,\gamma_\theta R_0\}$. Then when $\lambda\in[0,1]$, $(u,v)\in\partial K_{R_1}\times K_1$, we have $t\in[0,1]$

$$\begin{aligned}&\|T_{\lambda,1}(u,v)\|\\ &=\max_{t\in[0,1]}\left|\int_0^1 G_{1\alpha}(t,s)\left(\lambda f(s,u(s),v(s))\right.\right.\\ &\qquad\left.\left.+(1-\lambda)f(s,u(s),0)\right)ds\right|\\ &>\max_{t\in[0,1]}\int_0^1\gamma_\theta G_{1\alpha}(s)\left|\lambda N_1u(s)+(1-\lambda)N_1u(s)\right|ds\\ &\ge\max_{t\in[0,1]}\gamma_\theta\int_0^1 G_{1\alpha}(s)N_1|u(s)|\,ds\\ &\ge N_1\gamma_\theta\int_0^1 G_{1\alpha}(s)\max_{t\in(0,1)}|u(s)|\,ds\\ &\ge N_1\gamma_\theta\int_0^1 G_{1\alpha}(s)\,ds\,\|u\|=\|u\|.\end{aligned}\tag{26}$$

That is to say, $\|T_{\lambda,1}(u,v)\|>\|u\|$, where $\lambda\in[0,1]$, $(u,v)\in\partial K_{R_1}\times K_1$.

Step 3. Select $r_2>0$, such that $\|T_{\lambda,2}(u,v)\|>\|v\|$, where $\lambda\in[0,1]$, $(u,v)\in\partial K_{r_2}\times K_2$. In view of the assumption $(H_2)(a_2)$, for $N_2=(\gamma_\theta\int_0^1 G_{1\alpha}(s)ds)^{-1}>0$, there exists $r_2>0$, such that $g(t,u,v)>N_2v$, for all $t\in[0,1]$, $0\le v\le r_2$, $u\ge 0$. Hence when $\lambda\in[0,1]$, $(u,v)\in\partial K_{r_2}\times K_2$, we have for $t\in[0,1]$

$$\begin{aligned}&\|T_{\lambda,2}(u,v)\|\\ &=\max_{t\in[0,1]}\left|\int_0^1 G_{1\beta}(t,s)\left(\lambda g(s,u(s),v(s))\right.\right.\\ &\qquad\left.\left.+(1-\lambda)g(s,0,v(s))\right)ds\right|\\ &>\max_{t\in[0,1]}\int_0^1\gamma_\theta G_{1\beta}(s)\left|\lambda N_2v(s)+(1-\lambda)N_2v(s)\right|ds\\ &\ge\max_{t\in[0,1]}\gamma_\theta\int_0^1 G_{1\beta}(s)N_2|v(s)|\,ds\\ &\ge N_2\gamma_\theta\int_0^1 G_{1\beta}(s)\max_{t\in(0,1)}|v(s)|\,ds\\ &\ge N_2\gamma_\theta\int_0^1 G_{1\alpha}(s)\,ds\,\|v\|=\|v\|.\end{aligned}\tag{27}$$

That is to say, $\|T_{\lambda,2}(u,v)\|>\|v\|$, where $\lambda\in[0,1]$, $(u,v)\in\partial K_{r_2}\times K_2$.

Step 4. Select $R_2>r_2$, such that $\|T_{\lambda,2}(u,v)\|<\|v\|$, where $\lambda\in[0,1]$, $(u,v)\in\partial K_{R_2}\times K_2$. Based on the condition $(H_2)(b_2)$, for $\varepsilon_2=(\int_0^1 G_{1\alpha}(s)ds)^{-1}>0$, there exists $R'>0$, such that $g(t,u,v)<\varepsilon_2 v$, for all $t\in[0,1]$, $v\ge R'$, $u\ge 0$. Furthermore,

paying attention to the assumptions (H_3), we can select $R_2 > 0, R_2$ which is sufficiently large and

$$\max_{t\in[0,1]} \left\{ g(t,u,v) \mid 0 \le v \le R', u \ge 0 \right\} < \varepsilon_2 R_2. \tag{28}$$

Thus $g(t,u,v) < \varepsilon_2 R_2$, where $t \in [0,1]$, $0 \le v \le R_2$, $u \ge 0$. Hence when $\lambda \in [0,1]$, $(u,v) \in \partial K_{R_2} \times K_2$, we have for $t \in [0,1]$

$$\begin{aligned}
&\left\| T_{\lambda,2}(u,v) \right\| \\
&= \max_{t\in[0,1]} \left| \int_0^1 G_{1\beta}(t,s) \left(\lambda g(s,u(s),v(s)) \right.\right. \\
&\qquad \left.\left. + (1-\lambda) g(s,0,v(s)) \right) ds \right| \\
&< \max_{t\in[0,1]} \int_0^1 G_{1\beta}(s) \left(\lambda \varepsilon_2 R_2 + (1-\lambda) \varepsilon_2 R_2 \right) ds \\
&\le \max_{t\in[0,1]} \int_0^1 G_{1\beta}(s) \varepsilon_2 R_2 ds \\
&\le \varepsilon_2 R_2 \int_0^1 G_{1\beta}(s) \max_{t\in[0,1]} v(s)\, ds \\
&\le \varepsilon_2 R_2 \int_0^1 G_{1\alpha}(s)\, ds = \|v\|.
\end{aligned} \tag{29}$$

That is to say, $\|T_{\lambda,2}(u,v)\| < \|v\|$, where $\lambda \in [0,1]$, $(u,v) \in \partial K_{R_2} \times K_2$.

Combining with (25)–(29), we know the conditions of Theorem 4 are all established. Thus the completely continuous operator T has a fixed point in D; that is to say, at least a pair of positive solutions exist for the problem (6). The proof is complete. □

Remark 10. Binding (27), (29) and the assumption $(H_1)(a_1)$, we can get a nonnegative solution $(0,v)$ of the original equations (6). In fact, by (27) and (29), we know the fixed point index $i(T_{0,2}(u,v), K_{R_2} \setminus \overline{K_{r_2}}, K_2) = -1$; that is to say, the second equation of the original equations has a solution $(0,v)$. Further, by hypothesis $(H_1)(a_1)$, we have $f(t,0,v(t)) = 0$, so the solution of the second equation is also the solution of the first equation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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