Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 781594, 9 pages
http://dx.doi.org/10.1155/2014/781594

Research Article

Ultimate Bound of a 3D Chaotic System and Its Application in Chaos Synchronization

Jiezhi Wang,[1] Qing Zhang,[1] Zengqiang Chen,[1,2] and Hang Li[3]

[1] *College of Science, Civil Aviation University of China, Tianjin 300300, China*
[2] *Department of Automation, Nankai University, Tianjin 300071, China*
[3] *Economics and Management College, Civil Aviation University of China, Tianjin 300300, China*

Correspondence should be addressed to Jiezhi Wang; wjzh7845@gmail.com

Received 27 March 2014; Accepted 27 June 2014; Published 14 July 2014

Academic Editor: Josef Diblík

Two ellipsoidal ultimate boundary regions of a special three-dimensional (3D) chaotic system are proposed. To this chaotic system, the linear coefficient of the ith state variable in the ith state equation has the same sign; it also has two one-order terms and one quadratic cross-product term in each equation. A numerical solution and an analytical expression of the ultimate bounds are received. To get the analytical expression of the ultimate boundary region, a new result of one maximum optimization question is proved. The corresponding ultimate boundary regions are demonstrated through numerical simulations. Utilizing the bounds obtained, a linear controller is proposed to achieve the complete chaos synchronization. Numerical simulation exhibits the feasibility of the designed scheme.

1. Introduction

Bounded chaotic systems and their ultimate bounds are important for chaos synchronization and chaos control [1–3]. But it is generally difficult to obtain the ultimate bound of a chaotic system or the analytical expression of the bound even if the chaotic system has simple dynamic differential equations. The well-known Lorenz chaotic system was presented in 1963 [4]. It is a 3D autonomous system with only two quadratic terms. In 1987, a cylindrical bound and a spherical bound for the globally attractive and positive invariant sets of Lorenz system were proposed by Leonov et al. [5, 6]. Since then, several ultimate boundaries of Lorenz system have been obtained, like another cylindrical bound [7], the improved spherical bound [8], the ellipsoidal bounds [9–11], the butterfly bound [12], and so on [13–15]. References [10, 11] also discussed the ellipsoidal ultimate bounds of the unified Lorenz system [16]. The ultimate boundaries for other well-known chaotic attractors, such as Chen attractor [17], Lü attractor [18], and Qi attractor [19], were also proposed [20–22].

Since the research for the ultimate bounds set of chaotic systems is restricted by the region of the coefficients of the systems, in [20, 21], the ultimate boundary regions of the chaotic systems were researched only in several designated parameters regions. The ultimate boundaries of many existing chaotic systems are still not presented. So, it is also a challenging work to search the ultimate bounds of some new 3D chaotic systems [1, 2, 23–26] and hyperchaotic systems [27–29]. Recently, using the optimization idea and the Lyapunov method, which are often applied to estimate the boundaries of chaotic systems [1, 8, 10, 22, 27, 28], Wang et al. [30] constructed a special method to find the ultimate boundaries of a class of high dimensional autonomous quadratic chaotic systems. In the following parts, this method is called the unified method. Wang et al. [30] solved the ultimate boundary problem of more existing chaotic attractors and hyperchaotic attractors and got the numerical solutions of corresponding bounds. But the unified method is not applied successfully to every existing chaotic system.

In this paper, the following 3D chaotic system which was introduced by Tang et al. [31] in 2012 is considered:

$$\begin{aligned} \dot{x}_1 &= -ax_1 + bx_2 + x_2x_3, \\ \dot{x}_2 &= cx_1 - dx_2 - x_1x_3, \\ \dot{x}_3 &= ex_1 - fx_3 + gx_1x_2, \end{aligned} \tag{1}$$

where $x_1, x_2, x_3 \in R$ are state variables and a, b, c, d, e, f, and $g \in R^+$. Every state equation has two one-order terms and one quadratic cross-product term. System (1) has complex dynamic behaviors and several larger chaotic coefficient's regions. It has a typical chaotic attractor when $a = 25, b = 16$, $c = 40, d = 4, e = 5, f = 5$, and $g = 7$.

To system (1), the method used in [1, 8, 10, 22, 27, 28] to find the boundary of chaotic attractor does not seem very suitable. One can notice that the coefficients of the ith state variable x_i in the ith $(i = 1, 2, 3)$ equation have the same sign and they are negative. Under this special condition, the unified method [30] to find the boundary of chaotic attractor can be applied to system (1). In this paper, the unified method [30] is used to get the numerical solution of the ultimate bound of system (1) with $a > 0$, $b > 0$, $c > 0$, $d > 0$, $e > 0$, $f > 0$, and $g > 0$. Moreover, to get the analytical expression of the ellipsoidal ultimate boundary of system (1), a new conclusion about a designated maximum optimization question is proved. Utilizing this result, an analysis expression of the ellipsoidal ultimate boundary is given when the coefficients of the chaotic system $d = f$. The boundary is useful in the control or synchronization of chaos. Using the boundary set gained, one can realize the complete chaos synchronization.

The rest of the paper includes four sections. Section 2 introduces the unified approach [30] and proposes a new theorem about an interesting analytic solution of a maximum optimization problem. Utilizing the new theorem above and the unified method, Section 3 estimates the ellipsoidal ultimate boundary regions of system (1). Some numerical simulations about the boundary regions are exhibited. Section 4 applies the bound in chaos synchronization. Section 5 provides the conclusions.

2. Some Preliminaries and Notations

The unified method constructed in [30] to estimate the ultimate boundary of chaotic attractor is introduced firstly.

The considered autonomous system is described as

$$\dot{X} = f(X), \tag{2}$$

where $X = (x_1, x_2, \ldots, x_n)^T \in R^n$, $f: R^n \to R^n$. Let $X(t, t_0, X^0)$ be the solution satisfying $X(t, t_0, X^0) = X^0$ with the initial time t_0 and initial state X^0 and let $\Omega \in R^n$ be a compact set. The distance between $X(t, t_0, X^0)$ and Ω is defined by

$$\rho\left(X\left(t, t_0, X^0\right), \Omega\right) = \inf_{Y \in \Omega} \left\| X\left(t, t_0, X^0\right) - Y \right\|. \tag{3}$$

Denote $\Omega_\varepsilon = \{X \mid \rho(X, \Omega) < \varepsilon\}$. Obviously, $\Omega \in \Omega_\varepsilon$.

Definition 1 (see [10, 30]). Suppose that there exists a compact set $\Omega \in R^n$ satisfying

$$\lim_{t \to \infty} \rho(X(t), \Omega) = 0, \tag{4}$$

for all $X^0 \in R^n/\Omega$. It means that, for any $\varepsilon > 0$, there exists $\tau > t_0$ satisfying $X(t, t_0, X^0) \in \Omega_\varepsilon$ for all $t \geq \tau$. Then, the set Ω is called an ultimate bound of system (2).

Consider the HDQADS [30, 32], described by

$$\dot{X} = AX + \sum_{i=1}^{n} x_i B_i X + C, \tag{5}$$

where $X = (x_1, x_2, \ldots, x_n)^T \in R^n$, $A = (a_{ij})_{n\times n} \in R^{n\times n}$, $B_i = (b^i_{jk})_{n\times n} \in R^{n\times n}$, and $C = (c_1, c_2, \ldots, c_n)^T \in R^n$. Also, all elements of $B_1, B_2, \ldots, B_n$ satisfy $b^k_{ij} = b^j_{ik}$ $(i, j, k = 1, 2, \ldots, n)$.

Construct a general quadratic function candidate [30]

$$V(X) = (X + \mu)^T P (X + \mu), \tag{6}$$

where $X = (x_1, x_2, \ldots, x_n)^T \in R^n$, $P = P^T = (p_{ij})_{n\times n} \in R^{n\times n}$, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T \in R^n$ are real parameters to be determined.

Calculating the derivative of (6) along with system (5) [30], one can get

$$\dot{V}(X) = \sum_{i=1}^{n} x_i X^T \left(B_i^T P + P B_i\right) X + X^T Q X + M X + 2C^T P \mu, \tag{7}$$

where $Q = A^T P + PA + 2(B_1^T P\mu, B_2^T P\mu, \ldots, B_n^T P\mu)^T = Q^T$, $M = 2(\mu^T PA + C^T P)$.

Hereafter, the meaning of $P > 0$ is that the matrix P is positive definite and of $P < 0$ is that P is negative definite.

Lemma 2 (see [30]). *If there exists a $P \in R^{n\times n} > 0$ and a $\mu \in R^n$ such that*

$$\begin{gathered} Q < 0, \\ \sum_{i=1}^{n} x_i X^T \left(B_i^T P + P B_i\right) X = 0, \end{gathered} \tag{8}$$

for any $X = (x_1, x_2, \ldots, x_n)^T \in R^n$, then the boundness of system (5) is proved and the ultimate boundary region is

$$\Omega = \left\{X \in R^n \mid (X + \mu)^T P (X + \mu) \leq R_{\max}\right\}, \tag{9}$$

where $R_{\max} \in R$ which can be determined by solving the optimization problem:

$$\begin{aligned} \max \quad & V(X) = (X + \mu)^T P (X + \mu), \\ \text{s.t.} \quad & \dot{V}(X) = X^T Q X + M X + 2C^T P \mu = 0. \end{aligned} \tag{10}$$

The conditions (8) are sufficient but not necessary [30].

Since the symmetry of $P > 0$, then $V(X)$ can be transformed into a positive definite radially unbounded Lyapunov function $V(\widetilde{X})$ via $\widetilde{X} = X + \mu$.

For simplification, let $u = (u_1, u_2, \ldots, u_n) = 2\mu^T P$. One has $\mu^T = (1/2)uP^{-1}$, $\mu^T P\mu = (1/4)uP^{-1}u^T$. After a simple calculation, one can rewrite Lemma 2 as follows.

Lemma 3. *If there exists a real symmetric matrix $P > 0$ and a vector $u = (u_1, u_2, \ldots, u_n)$ such that*

$$Q = A^T P + PA + \left(B_1^T u^T, B_2^T u^T, \ldots, B_n^T u^T\right)^T < 0,$$
$$\sum_{i=1}^{n} x_i X^T \left(B_i^T P + PB_i^T\right) X = 0, \quad (11)$$

for any $X = (x_1, x_2, \ldots, x_n)^T \in R^n$, then the boundness of system (5) is proved and the ultimate boundary region is

$$\Omega = \left\{X \in R^n \mid 0 \le X^T PX + uX + \frac{1}{4}uP^{-1}u^T \le R_{\max}\right\}, \quad (12)$$

where $R_{\max} \in R$ which can be determined by solving the optimization problem:

$$\max \quad V(X) = X^T PX + uX + \frac{1}{4}uP^{-1}u^T,$$
$$\text{s.t.} \quad \dot{V}(X) = X^T QX + MX + uC = 0, \quad (13)$$

where $M = 2C^T P + uA$.

Theorem 4. *Denote the set*

$$\Gamma = \left\{(x_1, x_2, x_3)^T \in R^3 \mid \frac{x_1^2}{p^2} + \frac{(x_2 - m)^2}{m^2 + n^2} + \frac{(x_3 - n)^2}{m^2 + n^2} = 1, \quad p > 0,\ m \ne 0,\ n \ne 0\right\}, \quad (14)$$

and $G(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$, $(x_1, x_2, x_3) \in \Gamma$. Then

$$\max_{(x_1,x_2,x_3)\in\Gamma} G = \begin{cases} \dfrac{p^4}{p^2 - (m^2 + n^2)}, & p > \sqrt{2(m^2 + n^2)}, \\ 4\left(m^2 + n^2\right), & p \le \sqrt{2(m^2 + n^2)}, \end{cases} \quad (15)$$

$$\min_{(x_1,x_2,x_3)\in\Gamma} G = 0.$$

Proof. Let $\varphi(x_1, x_2, x_3) = (x_1^2/p^2) + (x_2 - m)^2/(m^2 + n^2) + ((x_3 - n)^2/(m^2 + n^2)) - 1$. Notice that $(\partial\varphi/\partial x_1, \partial\varphi/\partial x_2, \partial\varphi/\partial x_3) = (2x_1/p^2, 2(x_2 - m)/(m^2 + n^2), 2(x_3 - n)/(m^2 + n^2)) = (0, 0, 0)$ if and only if $(x_1, x_2, x_3) = (0, m, n)(\overline{\in}\Gamma)$.

Now, define

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + \lambda\left(\frac{x_1^2}{p^2} + \frac{(x_2 - m)^2}{m^2 + n^2} + \frac{(x_3 - n)^2}{m^2 + n^2} - 1\right). \quad (16)$$

Let

$$\frac{1}{2}F_{x_1} = x_1\left(1 + \frac{\lambda}{p^2}\right) = 0, \quad (17)$$

$$\frac{1}{2}F_{x_2} = x_2 + \frac{\lambda(x_2 - m)}{m^2 + n^2} = 0, \quad (18)$$

$$\frac{1}{2}F_{x_3} = x_3 + \frac{\lambda(x_3 - n)}{m^2 + n^2} = 0, \quad (19)$$

$$F_\lambda = \frac{x_1^2}{p^2} + \frac{(x_2 - m)^2}{m^2 + n^2} + \frac{(x_3 - n)^2}{m^2 + n^2} - 1 = 0. \quad (20)$$

From (17), $x_1 = 0$ or $\lambda = -p^2$. From (18), $x_2 = m\lambda/(m^2 + n^2 + \lambda)$, $x_2 - m = -m(m^2 + n^2)/(m^2 + n^2 + \lambda)$. From (19), $x_3 = n\lambda/(m^2 + n^2 + \lambda)$, $x_3 - n = -n(m^2 + n^2)/(m^2 + n^2 + \lambda)$. From (18), (19), and $mn \ne 0$, one gets $x_3 = (n/m)x_2$. From (20), $x_1^2 = p^2(1 - (x_2 - m)^2/(m^2 + n^2) - (x_3 - n)^2/(m^2 + n^2))$.

(i) When $x_1 = 0$, substituting $x_3 = (n/m)x_2$ into (20), one obtains $0 + (x_2 - m)^2/(m^2 + n^2) + (x_3 - n)^2/(m^2 + n^2) = (x_2 - m)^2/(m^2 + n^2) + ((n/m)x_2 - n)^2/(m^2 + n^2) = 1$; that is, $x_2(x_2 - 2m) = 0$. Then, one gets $x_2 = 0$ or $x_2 = 2m$ and two equilibria $(0, 0, 0)$ and $(0, 2m, 2n)$. Since $mn \ne 0$, obviously,

$$G(0, 2m, 2n) = 4\left(m^2 + n^2\right) > G(0, 0, 0) = 0. \quad (21)$$

(ii) When $\lambda = -p^2$ and $p > \sqrt{2(m^2 + n^2)}$, (18)–(20) have the following solutions: $\widehat{x}_1 = \pm p^2\sqrt{p^2 - 2(m^2 + n^2)}/((m^2 + n^2) - p^2)$, $\widehat{x}_2 = -mp^2/(m^2 + n^2 - p^2)$, $\widehat{x}_3 = -np^2/(m^2 + n^2 - p^2)$, and

$$G(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \frac{p^4}{p^2 - (m^2 + n^2)}. \quad (22)$$

Notice that $\widehat{x}_1$ is not able to be zero. In fact, if $\widehat{x}_1 = 0$, by $p > \sqrt{2(m^2 + n^2)}$, one has $p = 0$; this is a contradiction.

When $p > \sqrt{2(m^2 + n^2)}$, one has

$$\begin{aligned} G(0, 2m, 2n) - G(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) &= 4\left(m^2 + n^2\right) - \frac{p^4}{p^2 - (m^2 + n^2)} \\ &= \frac{\left(2\left(m^2 + n^2\right) - p^2\right)^2}{(m^2 + n^2) - p^2} < 0. \end{aligned} \quad (23)$$

Since Γ is a closed set and G is continuous on Γ, the extreme values of G can be attained on Γ. Then, from (i) and (ii), one can achieve

$$\max_{(x_1,x_2,x_3)\in\Gamma} G = \begin{cases} G(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \dfrac{p^4}{p^2 - (m^2 + n^2)}, & p > \sqrt{2(m^2 + n^2)}, \\ G(0, 2m, 2n) = 4\left(m^2 + n^2\right), & p \le \sqrt{2(m^2 + n^2)}, \end{cases}$$
$$\min_{(x_1,x_2,x_3)\in\Gamma} G = 0. \quad (24)$$

The proof is complete. □

3. The Ultimate Bound Set of Chaotic System (1)

In the following, Lemma 3 and Theorem 4 are applied to estimate the ultimate bounds of the 3D chaotic system (1).

Rewrite system (1) into the form of system (5); then, one has

$$A = \begin{bmatrix} -a & b & 0 \\ c & -d & 0 \\ e & 0 & -f \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{g}{2} & 0 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{g}{2} & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$C = 0. \tag{25}$$

Let $P = (p_{ij})_{3\times 3}$, $p_{ij} = p_{ji}$ $(i, j = 1, 2, 3)$, $u = (u_1, u_2, u_3)$. According to (11) in Lemma 2, calculate

$$\begin{aligned}\sum_{i=1}^{3} x_i X^T \left(B_i^T P + P B_i^T\right) X &= 2x_2\left(g x_1^2 + x_3^2\right) p_{13} + 2x_3\left(x_2^2 - x_1^2\right) p_{12} \\ &\quad + 2x_1\left(9x_2^2 - x_3^2\right) p_{23} + 2x_1x_2x_3\left(p_{11} - p_{22} + p_{33}\right).\end{aligned} \tag{26}$$

Since

$$\sum_{i=1}^{3} x_i X^T \left(B_i^T P + P B_i\right) X = 0, \tag{27}$$

holds for any $x_i \in R$ $(i = 1, 2, 3)$, letting

$$p_{12} = p_{13} = p_{23} = 0, \qquad p_{22} = p_{11} + p_{33}, \tag{28}$$

one gets

$$\begin{aligned} P &= \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}, \\ M &= uA + 2C^T P \\ &= \left[-u_1 a + u_2 c + u_3 e, u_1 b - u_2 d, -u_3 f\right], \\ Q &= A^T P + PA + \left[B_1^T u^T, B_2^T u^T, B_3^T u^T\right]^T \\ &= \begin{bmatrix} -2ap_{11} & bp_{11} + cp_{22} + \frac{g}{2}u_3 & ep_{33} - \frac{1}{2}u_2 \\ bp_{11} + cp_{22} + \frac{g}{2}u_3 & -2dp_{22} & \frac{1}{2}u_1 \\ ep_{33} - \frac{1}{2}u_2 & \frac{1}{2}u_1 & -2fp_{33} \end{bmatrix}. \end{aligned} \tag{29}$$

For simplifying Q, let

$$bp_{11} + cp_{22} + \frac{g}{2}u_3 = 0, \qquad ep_{33} - \frac{1}{2}u_3 = 0; \tag{30}$$

that is,

$$u_3 = -\frac{2}{g}\left(bp_{11} + cp_{22}\right), \qquad u_2 = 2ep_{33}; \tag{31}$$

then

$$u = \left(0, 2ep_{33}, -\frac{2}{g}\left(bp_{11} + cp_{22}\right)\right). \tag{32}$$

So, one has

$$\begin{aligned} P &= \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}, \\ M &= \Big[2ep_{33}c - \frac{2}{g}\left(bp_{11} + cp_{22}\right)e, -2ep_{33}d, \\ &\qquad \frac{2}{g}\left(bp_{11} + cp_{22}\right)f\Big], \\ Q &= \begin{bmatrix} -2ap_{11} & 0 & 0 \\ 0 & -2dp_{22} & 0 \\ 0 & 0 & -2fp_{33} \end{bmatrix}. \end{aligned} \tag{33}$$

From Lemma 3, the next theorem is achieved.

Theorem 5. *Suppose that $a > 0$, $b > 0$, $c > 0$, $d > 0$, $e > 0$, $f > 0$, $g > 0$ p_{11}, $p_{33} \in R^+$, and $p_{22} = p_{11} + p_{33}$. Denote*

$$\begin{aligned}\Omega = \Bigg\{X \in R^3 \mid p_{11}x_1^2 + p_{22}\left(x_2 + \frac{ep_{33}}{p_{22}}\right)^2 \\ + p_{33}\left(x_3 - \frac{bp_{11} + cp_{22}}{gp_{33}}\right)^2 \le R_{\max}\Bigg\},\end{aligned} \tag{34}$$

where $X = (x_1, x_2, x_3)^T$. Then, Ω is the ultimate bound set of system (1). $R_{\max}$ can be found by calculating the maximum optimization question:

$$\begin{aligned} \max \quad & V = p_{11}x_1^2 + p_{22}\left(x_2 + \frac{ep_{33}}{p_{22}}\right)^2 \\ & \quad + p_{33}\left(x_3 - \frac{bp_{11} + cp_{22}}{gp_{33}}\right)^2 \\ \text{s.t.} \quad & 2ap_{11}x_1^2 + 2dp_{22}x_2^2 + 2fp_{33}x_3^2 - \frac{2f}{g}\left(bp_{11} + cp_{22}\right)x_3 \\ & \quad + \left(\frac{2e}{g}\left(bp_{11} + cp_{22}\right) - 2ep_{33}c\right)x_1 + 2ep_{33}dx_2 = 0. \end{aligned} \tag{35}$$

Proof. Since $p_{11}, p_{33} \in R^+$, $p_{22} = p_{11} + p_{33} \in R^+$, one gets $P > 0$, $Q > 0$. According to (13), one obtains the Lyapunov-like quadratic function

$$\begin{aligned} V(X) &= X^T P X + uX + \frac{1}{4} u P^{-1} u^T \\ &= p_{11} x_1^2 + p_{22}\left(x_2 + \frac{e p_{33}}{p_{22}}\right)^2 \\ &\quad + p_{33}\left(x_3 - \frac{b p_{11} + c p_{22}}{g p_{33}}\right)^2 \end{aligned} \tag{36}$$

and its derivative along with system (1)

$$\begin{aligned} \dot{V}(X) &= X^T Q X + MX + uC \\ &= 2a p_{11} x_1^2 + 2d p_{22} x_2^2 + 2f p_{33} x_3^2 - \frac{2f}{g}(b p_{11} + c p_{22}) x_3 \\ &\quad + \left(\frac{2e}{g}(b p_{11} + c p_{22}) - 2e p_{33} c\right) x_1 + 2e p_{33} d x_2. \end{aligned} \tag{37}$$

From Lemma 3, the above conclusion holds. □

Remark 6. It is generally difficult to get the analytic solution of the optimization problem (35). But, by using Lingo, it is very easy to solve the optimization problem (35) numerically for the fixed system parameters. For example, obviously, one has $P > 0$, $Q > 0$ for $p_{11} = 1.2$, $p_{22} = 1.7$, $p_{33} = 0.5$, $a = 25$, $b = 16$, $c = 40$, $d = 4$, $e = 5$, $f = 5$, and $g = 7$. With the appointed parameters, utilizing Lingo to deal with the optimization problem (35), one gets the corresponding ultimate boundary region of system (1) as follows:

$$\begin{aligned} \Omega = \Big\{ X \in R^3 \mid 1.2 x_1^2 + 1.7\left(x_2 + \frac{5}{11}\right)^2 \\ +0.5\left(x_3 - \frac{600}{7}\right)^2 \le 319.4716 \Big\}. \end{aligned} \tag{38}$$

Figure 1 exhibits the ultimate boundary set of the chaotic strange attractor of system (1) under $p_{11} = 1.2$, $p_{22} = 1.7$, and $p_{33} = 0.5$.

Furthermore, to simplify the constraint condition of the maximum optimization problem (35), let the coefficient of x_1 be equal to 0. That is, $(2e/g)(b p_{11} + c p_{22}) - 2e p_{33} c = 0$. Then, if $d = f$, one can solve the maximum problem (35) analytically.

Theorem 7. *Suppose that $a > d/2$, $b > 0$, $c > 0$, $d = f > 0$, $e > 0$, $g > 1$, and $p_{ii} \in R^+$ $(i = 1, 2, 3)$, $p_{22} = p_{11} + p_{33}$, and*

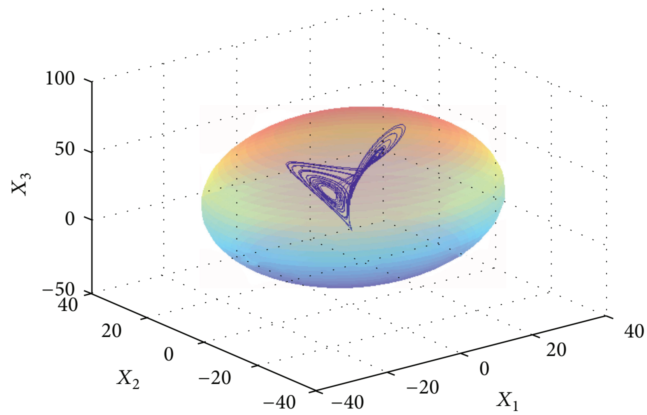

FIGURE 1: The chaotic attractor of system (1) with $a = 25$, $b = 16$, $c = 40$, $d = 4$, $e = 5$, $f = 5$, and $g = 7$ and its ultimate bound with $p_{11} = 1.2$, $p_{22} = 1.7$, and $p_{33} = 0.5$.

$(2e/g)(b p_{11} + c p_{22}) - 2e p_{33} c = 0$. Then, system (1) possesses following ultimate bound:

$$\begin{aligned} \Omega = \Big\{ X \in R^3 \mid (cg - c) x_1^2 + (cg + b)\left(x_2 + \frac{e(b+c)}{b + cg}\right)^2 \\ + (b+c)(x_3 - c)^2 \\ \le \frac{(be^2 + bc^2 + ce^2 + c^3 g)(b+c)}{b + cg} \Big\}. \end{aligned} \tag{39}$$

Proof. When

$$\begin{gathered} b > 0, \quad c > 0, \quad g > 1, \quad p_{33} \in R^+, \\ p_{22} = p_{11} + p_{33}, \\ \frac{2e}{g}(b p_{11} + c p_{22}) - 2e p_{33} c = 0, \end{gathered} \tag{40}$$

one has

$$p_{11} = \frac{cg - c}{b + c} p_{33} \in R^+, \tag{41}$$

$$p_{22} = \frac{cg + b}{b + c} p_{33} \in R^+, \tag{42}$$

$$\frac{e p_{33}}{p_{22}} = \frac{e(b+c)}{b + cg}, \tag{43}$$

$$\frac{b p_{11} + c p_{22}}{g p_{33}} = c. \tag{44}$$

According to (44) and Theorem 5, one obtains

$$\begin{aligned} \Omega = \Big\{ X \in R^3 \mid p_{11} x_1^2 + p_{22}\left(x_2 + \frac{e p_{33}}{p_{22}}\right)^2 + p_{33}(x_3 - c)^2 \\ \le R_{\max} \Big\}, \end{aligned} \tag{45}$$

and the following maximum problem

$$\begin{aligned} \max \quad & V = p_{11}x_1^2 + p_{22}\left(x_2 + \frac{ep_{33}}{p_{22}}\right)^2 + p_{33}(x_3 - c)^2 \\ \text{s.t.} \quad & 2ap_{11}x_1^2 + 2dp_{22}x_2^2 + 2fp_{33}x_3^2 + 2dep_{33}x_2 \\ & - 2fcp_{33}x_3 = 0. \end{aligned} \tag{46}$$

The above optimization problem is rewritten by

$$\begin{aligned} \max \quad & V = \left(\sqrt{p_{11}}x_1\right)^2 + \left(\sqrt{p_{22}}x_2 + \frac{ep_{33}}{\sqrt{p_{22}}}\right)^2 \\ & + \left(\sqrt{p_{33}}x_3 - c\sqrt{p_{33}}\right)^2 \\ \text{s.t.} \quad & a\left(\sqrt{p_{11}}x_1\right)^2 + d\left(\sqrt{p_{22}}x_2 + \frac{ep_{33}}{2\sqrt{p_{22}}}\right)^2 \\ & + f\left(\sqrt{p_{33}}x_3 - \frac{c\sqrt{p_{33}}}{2}\right)^2 \\ & = d\left(\frac{ep_{33}}{2\sqrt{p_{22}}}\right)^2 + f\left(\frac{c\sqrt{p_{33}}}{2}\right)^2. \end{aligned} \tag{47}$$

Denote $m = ep_{33}/2\sqrt{p_{22}}$, $n = -c\sqrt{p_{33}}/2$, $\overline{x}_1 = \sqrt{p_{11}}x_1$, $\overline{x}_2 = \sqrt{p_{22}}x_2 + 2m$, and $\overline{x}_3 = \sqrt{p_{22}}x_3 + 2n$. By $d = f > 0$, the corresponding maximum problem is described by

$$\begin{aligned} \max \quad & V = \overline{x}_1^2 + \overline{x}_2^2 + \overline{x}_3^2 \\ \text{s.t.} \quad & \frac{\overline{x}_1^2}{d(m^2+n^2)/a} + \frac{(\overline{x}_2 - m)^2}{m^2+n^2} + \frac{(\overline{x}_3 - n)^2}{m^2+n^2} = 1. \end{aligned} \tag{48}$$

Set $p^2 = d(m^2 + n^2)/a$. Since $a > d/2$, then $p = \sqrt{d(m^2+n^2)/a} < \sqrt{2(m^2+n^2)}$. Then, with the new result in Theorem 4 and (42) in problem (46), V has the maximum $R_{\max} = 4(m^2 + n^2) = (be^2 + bc^2 + ce^2 + c^3g)p_{33}/(b + cg)$. According to Lemma 3, Theorem 5, (41)–(43), and $p_{33} \in R^+$, system (1) gets the ellipsoidal ultimate boundary region as follows:

$$\begin{aligned} \Omega = \Bigg\{ & X \in R^3 \mid p_{11}x_1^2 + p_{22}\left(x_2 + \frac{ep_{33}}{p_{22}}\right)^2 + p_{33}(x_3 - c)^2 \\ & \leq \frac{\left(be^2 + bc^2 + ce^2 + c^3g\right)p_{33}}{b + cg} \Bigg\} \end{aligned}$$

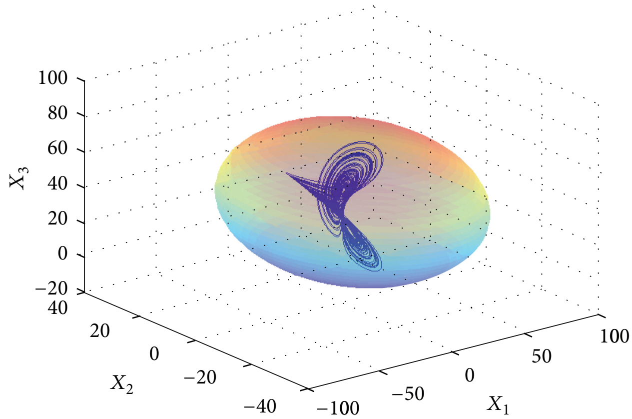

Figure 2: The chaotic attractor of system (1) with $a = 25$, $b = 16$, $c = 40$, $d = 4$, $e = 5$, $f = 4$, and $g = 1.5$ and its analytic ellipsoidal ultimate boundary region.

$$\begin{aligned} = \Bigg\{ & X \in R^3 \mid \frac{cg - c}{b + c}p_{33}x_1^2 + \frac{cg + b}{b + c}p_{33}\left(x_2 + \frac{e(b+c)}{b + cg}\right)^2 \\ & + p_{33}(x_3 - c)^2 \\ & \leq \frac{\left(be^2 + bc^2 + ce^2 + c^3g\right)p_{33}}{b + cg} \Bigg\} \\ = \Bigg\{ & X \in R^3 \mid (cg - c)x_1^2 + (cg + b)\left(x_2 + \frac{e(b+c)}{b+cg}\right)^2 \\ & + (b + c)(x_3 - c)^2 \\ & \leq \frac{\left(be^2 + bc^2 + ce^2 + c^3g\right)(b + c)}{b + cg} \Bigg\}. \end{aligned} \tag{49}$$

The proof is complete. □

Remark 8. If one lets $d = f$, system (1) still possesses a large range of chaos. Through Theorem 7, the analytic expression of the ultimate bound can be acquired easily. For example, when $a = 25$, $b = 16$, $c = 40$, $d = f = 4$, $e = 5$, and $g = 1.5$, the corresponding ellipsoidal ultimate boundary set of (1) is gained as

$$\begin{aligned} \Omega = \Bigg\{ & X \in R^3 \mid 20x_1^2 + 76\left(x_2 + \frac{70}{19}\right)^2 + 56(x_3 - 40)^2 \\ & \leq \frac{1722000}{19} \Bigg\}, \end{aligned} \tag{50}$$

which is demonstrated clearly in Figure 2.

4. Application in Chaos Synchronization

Consider two nonlinear autonomous systems

$$\dot{X} = g(t, X), \quad (51)$$

$$\dot{Y} = h(t, Y) + U(t, X, Y), \quad (52)$$

where $X = (x_1, x_2, \ldots, x_n)^T$, $Y = (y_1, y_2, \ldots, y_n)^T \in R^n$, $g, h \in C^r[R_+ \times R^n, R^n]$, $U = (U_1, U_2, \ldots, U_n)^T \in C^r[R_+ \times R^n \times R^n, R^n]$, and $r \geq 1$. R_+ means the nonnegative real set. Let (51) be the drive system and let (52) be the response system. $U(t, X, Y)$ means the controller function. $X^0 = X(t_0)$, $Y^0 = Y(t_0) \in R^n$ are the initial values of (51), (52).

Definition 9. The driver system (51) and the response system (52) are called to achieve global complete synchronization, if $\lim_{t\to\infty}\|Y(t) - X(t)\| = 0$ for any initial values X^0, Y^0.

Next, let system (1) be the driver system. Design the controller $U_i = -k_i(y_i - x_i)(i = 1, 2, 3)$. So, the response system to system (1) is described as follows:

$$\begin{aligned}
\dot{y}_1 &= -ay_1 + by_2 + y_2y_3 - k_1(y_1 - x_1), \\
\dot{y}_2 &= cy_1 - dy_2 - y_1y_3 - k_2(y_2 - x_2), \\
\dot{y}_3 &= ey_1 - fy_3 + gy_1y_2 - k_3(y_3 - x_3),
\end{aligned} \quad (53)$$

where y_1, y_2, $y_3 \in R$ are state variables and $k_i \in R^+$ $(i = 1, 2, 3)$ are all controller parameters which can be adjusted.

Theorem 10. *The driver system* (1) *and the response system* (53) *are globally complete synchronization when*

$$k_1 > \frac{g(p_{11}(b-2a)+p_{22}c+(p_{11}+p_{22})M_3+p_{22}M_2)+p_{33}e}{2gp_{11}}, \quad (54)$$

$$k_2 > \frac{p_{11}b + p_{22}(c-2d) + (p_{11}+p_{22})M_3 + p_{33}M_1}{2p_{22}}, \quad (55)$$

$$k_3 > \frac{g(p_{22}M_2 + p_{33}M_1) + p_{33}(e-2f)}{2p_{33}}, \quad (56)$$

where $M_1 = \sqrt{R/p_{11}}$, $M_2 = \sqrt{R/p_{22}} + ep_{33}/p_{22}$, $M_3 = \sqrt{R/p_{33}} + (bp_{11} + cp_{22})/gp_{33}$, $R = R_{\max}$, $p_{11} \in R^+$, $p_{33} \in R^+$, *and* $p_{22} = p_{11} + p_{33}$.

Proof. Let $R = R_{\max}$. From Theorem 5, one has $|x_1| \leq \sqrt{R/p_{11}} = M_1$, $|x_2| \leq \sqrt{R/p_{22}} + ep_{33}/p_{22} = M_2$, and $|x_3| \leq \sqrt{R/p_{33}} + (bp_{11} + cp_{22})/gp_{33} = M_3$. Let the state errors be $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, and $e_3 = y_3 - x_3$, then the error dynamics of system (1) and system (53) is

$$\begin{aligned}
\dot{e}_1 &= \dot{y}_1 - \dot{x}_1 = -(a + k_1)e_1 + be_2 + e_2e_3 + e_2x_3 + e_3x_2, \\
\dot{e}_2 &= \dot{y}_2 - \dot{x}_2 = ce_1 - (d + k_2)e_2 - e_1e_3 - e_3x_1 - e_1x_3, \\
\dot{e}_3 &= \dot{y}_3 - \dot{x}_3 = ee_1 - (f + k_3)e_3 + ge_1e_2 + ge_2x_1 + ge_1x_2.
\end{aligned} \quad (57)$$

Noticing the formula (28), one has $p_{22} = p_{11} + p_{33}$. Let $V(e) = (1/2)(p_{11}e_1^2 + p_{22}e_2^2 + (p_{33}/g)e_3^2)$; then its time derivative along the orbit of system (57) is

$$\begin{aligned}
\dot{V}(e) &= p_{11}e_1\dot{e}_1 + p_{22}e_2\dot{e}_2 + \frac{p_{33}}{g}e_3\dot{e}_3 \\
&= -p_{11}(a + k_1)e_1^2 - p_{22}(d + k_2)e_2^2 - \frac{p_{33}}{g}(f + k_3)e_3^2 \\
&\quad + (p_{11}b + p_{22}c + (p_{11} + p_{22})x_3)e_1e_2 \\
&\quad + \left(\frac{p_{33}}{g}e + (p_{11} + p_{33})x_2\right)e_1e_3 + p_{33}x_1e_2e_3 \\
&\quad + (p_{11} - p_{22} + p_{33})e_1e_2e_3 \\
&\leq -p_{11}(a + k_1)e_1^2 - p_{22}(d + k_2)e_2^2 - \frac{p_{33}}{g}(f + k_3)e_3^2 \\
&\quad + (p_{11}b + p_{22}c + (p_{11} + p_{22})M_3)|e_1||e_2| \\
&\quad + \left(\frac{p_{33}}{g}e + (p_{11} + p_{33})M_2\right)|e_1||e_3| + p_{33}M_1|e_2||e_3| + 0 \\
&\leq -p_{11}(a + k_1)e_1^2 - p_{22}(d + k_2)e_2^2 - \frac{p_{33}}{g}(f + k_3)e_3^2 \\
&\quad + (p_{11}b + p_{22}c + (p_{11} + p_{22})M_3)\frac{e_1^2 + e_2^2}{2} \\
&\quad + \left(\frac{p_{33}}{g}e + (p_{11} + p_{33})M_2\right)\frac{e_1^2 + e_3^2}{2} + p_{33}M_1\frac{e_2^2 + e_3^2}{2} \\
&= -p_{11}\Big(k_1 - \Big(\big(g(p_{11}(b - 2a) + p_{22}c + (p_{11} + p_{22})M_3 \\
&\qquad + p_{22}M_2) + p_{33}e\big) \times (2gp_{11})^{-1}\Big)\Big)e_1^2 \\
&\quad - p_{22} \\
&\quad \times \left(k_2 - \frac{p_{11}b + p_{22}(c - 2d) + (p_{11} + p_{22})M_3 + p_{33}M_1}{2p_{22}}\right) \\
&\quad \times e_2^2 \\
&\quad - \frac{p_{33}}{g}\left(k_3 - \frac{g(p_{22}M_2 + p_{33}M_1) + p_{33}(e - 2f)}{2p_{33}}\right)e_3^2 \\
&= -E^TKE,
\end{aligned} \quad (58)$$

where $E = [|e_1|, |e_2|, |e_3|]^T$,

$$K = \begin{bmatrix} -p_{11}(k_1 - k_1') & 0 & 0 \\ 0 & -p_{22}(k_2 - k_2') & 0 \\ 0 & 0 & -p_{33}(k_3 - k_3') \end{bmatrix}, \quad (59)$$

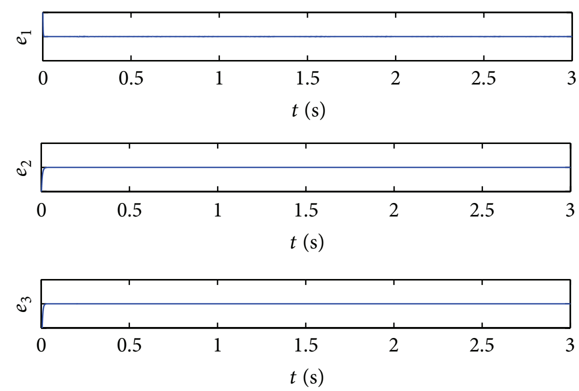

Figure 3: Synchronization error of the response system (53) and the diver system (1).

with

$$k_1' = \frac{g\left(p_{11}(b-2a)+p_{22}c+\left(p_{11}+p_{22}\right)M_3+p_{22}M_2\right)+p_{33}e}{2gp_{11}}, \tag{60}$$

$$k_2' = \frac{p_{11}b+p_{22}(c-2d)+\left(p_{11}+p_{22}\right)M_3+p_{33}M_1}{2p_{22}}, \tag{61}$$

$$k_3' = \frac{g\left(p_{22}M_2+p_{33}M_1\right)+p_{33}(e-2f)}{2p_{33}}. \tag{62}$$

When $k_i > k_1'$ $(i = 1, 2, 3)$, $K > 0$. One can draw that the origin of the error system (57) is asymptotically stable, which implies that the driver system (1) and the response system (53) achieve globally complete synchronization. □

Remark 11. The numerical simulations are studied by MATLAB 7.6.0. Take $(-1, -0.5, 5)$ and $(1, -3, -4)$ as the values of the initial condition of system (1) and system (53), respectively. When $a = 25$, $b = 16$, $c = 40$, $d = 4$, $e = 5$, $f = 5$, $g = 7$, $p_{11} = 1.2$, $p_{22} = 1.7$, and $p_{33} = 0.5$, from Remark 6, one gets $R = 319.4716$, $M_1 = 16.3164$, $M_2 = 15.1791$, and $M_3 = 50.1916$. By Theorem 10, one can choose the three feedback control coefficients as $k_1 = 600$, $k_2 = 190$, and $k_3 = 236$. Figure 3 proves that the response system realizes synchronization with the driver system through a short time.

5. Conclusion

In this paper, the ultimate boundary regions of a special 3D chaotic system are studied through a unified method for the ultimate boundary set estimating of chaotic systems. In this unified way, to get the analytical expression of the ultimate boundary region, the key is to calculate the analytical solution of the maximum optimization problem. Furthermore, an interesting result about the analytic solution of the corresponding maximum optimization problem is proposed to obtain the analytic ellipsoidal ultimate boundary regions of the chaotic system. The ultimate bounds which are useful in chaos synchronization are demonstrated through numerical simulations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China through Grant no. 11102226, the Fundamental Research Funds for the Central Universities through Grant nos. ZXH2010D011, ZXH2012B003, and ZXH2012K002, and the Scientific Research Foundation of Civil Aviation University of China through Grant no. 07QD05X.

References

[1] Y. Shu, H. Xu, and Y. Zhao, "Estimating the ultimate bound and positively invariant set for a new chaotic system and its application in chaos synchronization," *Chaos, Solitons and Fractals*, vol. 42, no. 5, pp. 2852–2857, 2009.

[2] F. Zhang, Y. Shu, and H. Yang, "Bounds for a new chaotic system and its application in chaos synchronization," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1501–1508, 2011.

[3] Y. Li, Y. Zhao, and Z. A. Yao, "Chaotic control and generalized synchronization for a hyperchaotic Lorenz-Stenflo system," *Abstract and Applied Analysis*, vol. 2013, Article ID 515106, 18 pages, 2013.

[4] E. N. Lorenz, "Deterministic nonperiodic flow," *Journal of the Atmospheric Sciences*, vol. 20, pp. 130–141, 1963.

[5] G. A. Leonov and V. Reitmann, *Attraktoreingrenzung fur Nichtlineare System*, Tenbner, Leipzing , Germany, 1987.

[6] G. A. Leonov, A. L. Bunin, and N. Koksch, "Attraktor localization of the Lorenz system," *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 67, pp. 649–656, 1987.

[7] A. Y. Pogromsky, G. Santoboni, and H. Nijmeijer, "An ultimate bound on the trajectories of the Lorenz system and its applications," *Nonlinearity*, vol. 16, no. 5, pp. 1597–1605, 2003.

[8] D. Li, J. Lu, X. Wu, and G. Chen, "Estimating the bounds for the Lorenz family of chaotic systems," *Chaos, Solitons and Fractals*, vol. 23, no. 2, pp. 529–534, 2005.

[9] X. Liao, Y. Fu, and S. Xie, "On the new results of global attractive set and positive invariant set of the Lorenz chaotic system and the applications to chaos control and synchronization," *Science in China F: Information Sciences*, vol. 48, no. 3, pp. 304–321, 2005.

[10] D. Li, J. Lu, X. Wu, and G. Chen, "Estimating the ultimate bound and positively invariant set for the Lorenz system and a unified chaotic system," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 844–853, 2006.

[11] X. Liao, Y. Fu, S. Xie, and P. Yu, "Globally exponentially attractive sets of the family of Lorenz systems," *Science in China F: Information Sciences*, vol. 51, no. 3, pp. 283–292, 2008.

[12] M. Suzuki, N. Sakamoto, and T. Yasukochi, "A butterfly-shaped localization set for the Lorenz attractor," *Physics Letters A*, vol. 372, no. 15, pp. 2614–2617, 2008.

[13] P. Yu and X. Liao, "Globally attractive and positive invariant set of the Lorenz system," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 16, no. 3, pp. 757–764, 2006.

[14] P. Yu and X. Liao, "New estimations for globally attractive and positive invariant set of the family of the Lorenz systems," *International Journal of Bifurcation and Chaos*, vol. 16, no. 11, pp. 3383–3390, 2006.

[15] G. A. Leonov, "Bound for attractors and the existance of homoclinic orbits in the Lorenz system," *Journal of Applied Mathematics and Mechanics*, vol. 65, no. 1, pp. 19–32, 2001.

[16] J. LÜ, G. Chen, and D. Cheng, "Bridge the gap between the Lorenz system and the Chen system," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 12, no. 12, pp. 2917–2926, 2002.

[17] G. Chen and T. Ueta, "Yet another chaotic attractor," *International Journal of Bifurcation and Chaos*, vol. 9, no. 7, pp. 1465–1466, 1999.

[18] J. Lü and G. Chen, "A new chaotic attractor coined," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 12, no. 3, pp. 659–661, 2002.

[19] G. Qi, G. Chen, S. Du, Z. Chen, and Z. Yuan, "Analysis of a new chaotic system," *Physica A*, vol. 352, no. 2–4, pp. 295–308, 2005.

[20] W.-X. Qin and G. Chen, "On the boundedness of solutions of the Chen system," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 445–451, 2007.

[21] F. Zhang, C. Mu, and X. Li, "On the boundness of some solutions of the Lü system," *International Journal of Bifurcation and Chaos*, vol. 22, no. 1, Article ID 1250015, 5 pages, 2012.

[22] H. Xu, Y. Shu, and G. Yuan, "The ultimate bound of a chaotic system and its application in chaos synchronization," *Journal of Chongqing Technology and Business University*, vol. 25, no. 6, pp. 564–568, 2008 (Chinese).

[23] C. Mu, F. Zhang, Y. Shu, and S. Zhou, "On the boundedness of solutions to the Lorenz-like family of chaotic systems," *Nonlinear Dynamics*, vol. 67, no. 2, pp. 987–996, 2012.

[24] P. Yu and X. X. Liao, "On the study of globally exponentially attractive set of a general chaotic system," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 8, pp. 1495–1507, 2008.

[25] F. Xu, P. Yu, and X. Liao, "Global analysis on *n*-scroll chaotic attractors of modified Chua's circuit," *International Journal of Bifurcation and Chaos*, vol. 19, no. 1, pp. 135–137, 2009.

[26] G. A. Leonov, N. V. Kuznetsov, and V. I. Vagaitsev, "Localization of hidden Chua's attractors," *Physics Letters A*, vol. 375, no. 23, pp. 2230–2233, 2011.

[27] D. Li, X. Wu, and J. Lu, "Estimating the ultimate bound and positively invariant set for the hyperchaotic Lorenz-Haken system," *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1290–1296, 2009.

[28] P. Wang, D. Li, and Q. Hu, "Bounds of the hyper-chaotic Lorenz-Stenflo system," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 9, pp. 2514–2520, 2010.

[29] J. Zhang and W. Tang, "A novel bounded 4D chaotic system," *Nonlinear Dynamics*, vol. 67, no. 4, pp. 2455–2465, 2012.

[30] P. Wang, D. Li, X. Wu, and J. Lü, "Ultimate bound estimation of a class of high dimensional quadratic autonomous dynamical systems," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 21, no. 9, pp. 2679–2694, 2011.

[31] L. R. Tang, B. Fan, and Z. M. Kang, "A chaos synchronization method based on amplitude," *Acta Physica Sinica*, vol. 61, no. 8, Article ID 080508, 5 pages, 2012, (Chinese).

[32] J. Lü, G. Chen, and D. Cheng, "A new chaotic system and beyond: the generalized Lorenz-like system," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 14, no. 5, pp. 1507–1537, 2004.