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Research Article

Symmetric Spaces and Fixed Points of Generalized Contractions

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Some fixed point results in semi-metric spaces as well as in symmetric spaces are proved. Applications of our results to probabilistic spaces are also presented.

1. Introduction

There have been a number of generalizations of metric space. Two of them are the notions of symmetric spaces and semi-metric spaces introduced and studied by Wilson [1]. For historical remarks about these spaces see [2]. Fixed point theory of various classes of maps in a metric space and its generalizations has been studied by a number of authors; see, for example, [3–9] and the references cited therein. In 1976, Cicchese proved the first fixed point theorem for contractions in semi-metric spaces. Further fixed point results for this class of spaces were obtained by Jachymski et al. [10], Hicks and Rhoades [11], Aamri and El Moutawakil [12], Aamri et al. [13], Zhu et al. [14], Miheţ [15], Imdad et al. [16], Aliouche [17], and Radenović and Kadelburg [18]. For more information on fixed point theory in symmetric spaces and semi-metric spaces, we refer the reader to [2].

In this paper we prove some fixed point results in semi-metric spaces and symmetric spaces. We also present applications of our results to probabilistic spaces. Our results generalize earlier results obtained by Aranđelović and Kečkić [2], Browder [19], Walter [20], and Maiti et al. [21].

2. Preliminary Notes

A symmetric space is a pair (X, d) consisting of a nonempty set X and a function $d : X \times X \to [0, \infty)$ such that for all x, y in X the following conditions hold:

(W1) $d(x, y) = 0$ if and only if $x = y$;

(W2) $d(x, y) = d(y, x)$.

Let (X, d) be symmetric space. The *open ball* with center $x \in X$ and radius $r > 0$ is defined by

$$B(x, r) = \{y \in X : d(x, y) < r\}. \quad (1)$$

Also if A is a subset of X, then

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\} \quad (2)$$

denotes *the diameter of* A.

Many properties and notions in symmetric spaces are similar to those in metric spaces (but not all, because of the absence of the triangle inequality). For example, a sequence $\{x_n\} \subseteq X$ is said to be d-Cauchy sequence if given $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$, for all $m, n \geqslant N$.

In every symmetric space (X, d) one may introduce the topology τ_d by defining the family of closed sets as follows: a set $A \subseteq X$ is closed if and only if for each $x \in X$, $d(x, A) = 0$ implies $x \in A$, where

$$d(x, A) = \inf\{d(x, a) : a \in A\}. \quad (3)$$

The following conditions can be used as partial replacements for the triangle inequality's absence in the symmetric space (X, d) :

(W3) $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$;

(W4) $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$;

(W) $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$ imply $\lim_{n\to\infty} d(x_n, z_n) = 0$;

(JMS) $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$ imply $\lim_{n\to\infty} d(x_n, z_n) \neq \infty$;

(CC) $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$;

(SC) $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\overline{\lim}_{n\to\infty} d(x_n, y) \leqslant d(x, y)$;

(MT) there exists $s \geqslant 1$ such that for any $x, y, z \in X$

$$d(x,z) \leqslant s\left(d(x,y) + d(y,z)\right). \tag{4}$$

The properties (W3) and (W4) were induced by Wilson [1], (W) by Miheţ [15], (JMS) by Jachymski et al. [10], (CC) by Cho et al. [22] and earlier by Borges [23] (as 1-continuity property), (MT) by Czerwik [24] (see also [25]), and (SC) by Aranđelović and Kečkić [2].

Next statement gives the characterization of symmetric space which satisfies the property (JMS).

Proposition 1 (Jachymski et al. [10]). *Let (X, d) be a symmetric space. Then the following conditions are equivalent.*

(i) *(X, d) satisfies property (JMS).*

(ii) *There exists $\delta, \eta > 0$ such that for any $x, y, z \in X$,*

$$d(x,z) + d(z,y) < \delta \text{ implies that } d(x,y) < \eta. \tag{5}$$

(iii) *There exists $r > 0$ such that*

$$\sup\{\operatorname{diam}(B(x,r)) : x \in X\} < \infty. \tag{6}$$

The convergence of a sequence $\{x_n\}$ in the topology τ_d need not imply $d(x_n, x) \to 0$, although the converse is true (see Proposition 2).

The following two propositions have been well known for a long time, but for the convenience of the reader we will state them without proofs, which can also be found in [2].

Proposition 2. *If (X, d) is a symmetric space, then the family $\{B(x, r) : r > 0\}$ forms a local basis at x. Also, if $d(x_n, x) \to 0$, then $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) in the topology τ_d.*

Definition 3. A topological space (X, τ) is semimetrizable if there is a symmetric function $d : X \times X \to \mathbf{R}$ such that $\tau_d = \tau$ and that the mapping

$$X \supseteq A \longmapsto c(A) = \{x \in X : d(x, A) = 0\} \tag{7}$$

is the closure operator in τ_d. In terms of d it can be expressed by saying that the operator c is idempotent. In this case we say that (X, d) is *semi-metric space*; d is said to be *semi-metric* function on X (or admissible semi-metric for (X, τ)).

It is worth mentioning that this basis need not consist of open sets. Moreover, in [26], a semimetrizable space (X, τ) was constructed with the property that, for any d that generates τ, there exist $x \in X$ and $r > 0$ such that $B(x, r)$ is not open.

Proposition 4. *Let (X, d) be a symmetric space. Then (X, d) is a semi-metric space if and only if the following conditions hold.*

(1) *(X, τ_d) is first countable.*

(2) *For any sequence $\{x_n\} \subseteq X$, $d(x_n, x) \to 0$ is equivalent to $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) in the topology τ_d.*

Example 5. Let $X = \mathbb{N}$. Define $d : X \times X \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 2^{1/2^{\min\{x,y\}}} - 1 & \text{if } |x-y| = 1 \\ 3 & \text{if } |x-y| \geqslant 2 \\ 0 & \text{if } x = y. \end{cases} \tag{8}$$

Then (X, d) is a d-Cauchy complete semi-metric.

A symmetric space (X, d) is said to be d-Cauchy complete if every d-Cauchy sequence converges to some $x \in X$ in the topology τ_d, and it is said to be *d-weakly complete* if every decreasing sequence $\{F_n\}$ of nonempty closed subsets, such that there exists a sequence $\{x_n\}$, $x_n \in F_n$ with $F_n \subseteq B(x_n, 2^{-n})$ has a nonempty intersection.

Next statement was proved in [27] (see also [28]).

Proposition 6 (Galvin and Shore [27]). *Let (X, d) be a semi-metric space. Then the following are equivalent:*

(1) *(X, τ_d) is d-weakly complete;*

(2) *every d-Cauchy sequence in X has a convergent subsequence;*

(3) *every decreasing sequence $\{F_n\}$ of nonempty closed subsets of X such that $\operatorname{diam}(F_n) \leqslant 2^{-n}$ for each n has a nonempty intersection.*

Let X be a nonempty set and $f : X \to X$. Then $z \in X$ is called a fixed point of f if $z = f(z)$. Let $x \in X$. The sequence $\{x_n\}$ defined by $x_n = f^n(x)$ is called the sequence of *Picard iterates* of f at point x. This sequence $\{x_n\}$ is also called the *orbit* of f at point x. We will denote it by $O(x)$ and use $O(x, y)$ to denote set $O(x) \cup O(y)$.

Let Φ denote the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(a) φ is monotone nondecreasing;

(b) $\lim \varphi^n(t) = 0$ for any $t > 0$.

The function $\varphi \in \Phi$ is known as the comparison function (see [29]). As a consequence of the above properties, we have the following (see [29]).

Lemma 7. *If $\varphi \in \Phi$ then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.*

Definition 8. If (X, d) is a metric space and $f : X \to X$, then f is called a

(1) *contraction* if there exists real number $\alpha \in (0, 1)$ such that

$$d(f(x), f(y)) \leqslant \alpha d(x, y) \quad \forall x, y \in X; \tag{9}$$

(2) *φ-contraction* if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that for any $x, y \in X$

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)); \tag{10}$$

(3) *generalized φ-contraction* if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that for any $x, y \in X$

$$d(f(x), f(y)) \leqslant \varphi(d_f(x, y)), \tag{11}$$

where

$$d_f(x, y) = \max\{d(x, y), d(x, f(x)), d(y, f(y))\}. \tag{12}$$

Lemma 9 (Aranđelović and Kečkić [2]). *Let X be a nonempty set, let $f : X \to X$, and let n be a fixed positive integer such that the iterate f^n has a unique fixed point z. Then*

(1) *z is a unique fixed point of f;*

(2) *if X is a topological space and any sequence of Picard iterates defined by f^n converges to z, then the sequence of Picard iterates defined by f always converges to z.*

3. Some Topological Results

Proposition 10. *Let (X, d) be a symmetric space satisfying (W). Then it satisfies (W4) and (JMS).*

Proof. The implication (W)⇒(W4) is straightforward (see [15]).

Now let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in X such that

$$\lim_{n\to\infty} d(x_n, y_n) = 0, \quad \lim_{n\to\infty} d(y_n, z_n) = 0. \tag{13}$$

From (W) it follows that

$$\lim_{n\to\infty} d(x_n, z_n) \neq \infty. \tag{14}$$

So, (X, d) satisfies (JMS). □

A semi-metric space in which all balls $B(x, r)$, $x \in X$ and $r > 0$, are open will be called a semi-metric space with open balls.

Proposition 11. *Let (X, d) be a compact semi-metric space with open balls and $K \subseteq X$ a nonempty compact set. Then K is bounded.*

Proof. (X, τ_d) is first countable [2, Proposition 3] and T_1-space [2, page 5161]. Also, (X, d) satisfies the property (SC) because all $B(x, r)$ are open sets [2, Theorem 1]. K is countably compact because it is compact [30, Theorem 11.9]. It is sequentially compact, as a first countable countably compact set [30, Problem 10.7].

Suppose that K is not bounded. Let $x_0 \in K$. For each positive integer n there exists $x_n \in K$ such that $d(x_n, x_0) > n$. Then there exists $x_* \in K$ and an increasing sequence of positive integers n_k such that, in the topology τ_d,

$$\lim_{k\to\infty} x_{n_k} = x_*, \tag{15}$$

because K is sequentially compact. So, we get that

$$\overline{\lim_{k\to\infty}} d(x_{n_k}, x_0) \leqslant d(x_*, x_0) \tag{16}$$

which is a contradiction because

$$\lim_{k\to\infty} d(x_{n_k}, x_0) = \infty. \tag{17}$$

□

4. Bounded Semi-Metric Spaces: Fixed Point Results

In this section, we obtain generalizations of fixed point results of Browder [19] and Walter [20] (see also [7]).

Theorem 12. *Let (X, d) be a bounded semi-metric and d-Cauchy complete space satisfying (W4). Suppose $f : X \to X$ satisfies that, for $x \in X$, there exists $\nu(x) \in \mathbb{N}$ such that, for any $\nu \geqslant \nu(x)$ and $y \in X$,*

$$d(f^\nu(x), f^\nu(y)) \leqslant \varphi(\operatorname{diam}(O(x, y))) \tag{18}$$

with $\varphi \in \Phi$. Then there exists $z \in X$ such that $\lim_{p\to\infty} f^p(x) = z$ in the topology τ_d (or equivalently $d(f^p(x), z) \to 0$ as $p \to \infty$), $\forall x \in X$.

Proof. Let $\mu = \max\{\nu(x), \nu(y)\}$ and $\nu \geqslant \mu$.

Each element $\alpha \in O(f^\mu(x), f^\mu(y))$ has one of forms $f^{\nu+\rho}(x)$ or $f^{\nu+\rho}(y)$, with $\rho \geqslant 0$. Now let $\alpha, \beta \in O(f^\mu(x), f^\mu(y))$ and consider the case with $\alpha = f^{\nu+\rho}(x)$ and $\beta = f^{\nu+\rho}(y)$; then

$$\begin{aligned} d(\alpha, \beta) &= d(f^{\nu+\rho}(x), f^{\nu+\rho}(y)) \\ &\leqslant \varphi(\operatorname{diam}(O(f^\rho(x), f^\rho(y)))) \\ &\leqslant \varphi(\operatorname{diam}(O(x, y))). \end{aligned} \tag{19}$$

Other cases will lead to the same inequality:

$$\operatorname{diam}(O(f^\mu(x), f^\mu(y))) \leqslant \varphi(\operatorname{diam}(O(x, y))). \tag{20}$$

Now, define the sequences $\{E_n\}_{n=0}^\infty \subset X$ and $\{p(n)\}_{n=0}^\infty$ by $p(0) = 0$, $p(n+1) = p(n) + \max\{\nu(f^{p(n)}(x)), \nu(f^{p(n)}(y))\}$, and $E_n = O(f^{p(n)}(x), f^{p(n)}(y))$, $n = 0, 1, 2, \ldots$.

We want to prove that

$$\operatorname{diam}(E_{n+1}) \leqslant \varphi(\operatorname{diam}(E_n)), \quad n = 0, 1, 2, \ldots. \tag{21}$$

By (20), we get that (21) is valid for $n = 0$.

Now, let n be arbitrary and set $\gamma = f^{p(n)}(x)$, $\xi = f^{p(n)}(y)$, and $\eta = \max\{\nu(f^{p(n)}(x)), \nu(f^{p(n)}(y))\}$; then

$$\begin{aligned} &\operatorname{diam}(O(f^{\eta}(x), f^{\eta}(y))) \\ &\quad\leqslant \varphi(\operatorname{diam}(O(\gamma, \xi))) \\ &\quad= \varphi\left(\operatorname{diam}\left(O\left(f^{p(n)}(x), f^{p(n)}(y)\right)\right)\right) \\ &\quad= \varphi(\operatorname{diam}(E_n)). \end{aligned} \tag{22}$$

But $f^{\eta}(\gamma) = f^{p(n)+\eta}(x) = f^{p(n+1)}(x)$ and $f^{\eta}(\xi) = f^{p(n)+\eta}(y) = f^{p(n+1)}(y)$. Thus $\operatorname{diam}(O(f^{\eta}(\gamma), f^{\eta}(\xi))) = \operatorname{diam}(O(f^{p(n+1)}(x), f^{p(n+1)}(y))) = \operatorname{diam}(E_{n+1})$.

Therefore (21) holds for all $n = 0, 1, 2, \ldots$.

Now, by (21) and the monotonicity of φ, we get that $\operatorname{diam}(E_{n+1}) \leqslant \varphi^{(n+1)}(\operatorname{diam}(E_0)) = \varphi^{(n+1)}(\operatorname{diam}(O(x, y))) \to 0$ as $n \to \infty$ that is, $\lim_{n\to\infty} \operatorname{diam}(E_n) = \lim_{n\to\infty} \operatorname{diam}(O(f^{p(n)}(x), f^{p(n)}(y))) = 0$ which is equivalent to $\lim_{p\to\infty} \operatorname{diam}(O(f^{p}(x), f^{p}(y))) = 0$. But

$$\begin{aligned} &\lim_{p\to\infty} \operatorname{diam}(O(f^{p}(x))) \\ &\quad\leqslant \lim_{p\to\infty} \operatorname{diam}(O(f^{p}(x), f^{p}(y))) = 0, \end{aligned} \tag{23}$$

which implies

$$\lim_{p\to\infty} \operatorname{diam}(O(f^{p}(x))) = 0. \tag{24}$$

Similarly

$$\lim_{p\to\infty} \operatorname{diam}(O(f^{p}(y))) = 0. \tag{25}$$

Hence, $\{f^{p}(x)\}$ and $\{f^{p}(y)\}$ are d-Cauchy sequences, and by the d-completeness of X, there exists $z, w \in X$ such that $\lim_{p\to\infty} f^{p}(x) = z$ and $\lim_{p\to\infty} f^{p}(y) = w$ in the topology τ_d.

Since $\lim_{p\to\infty} d(f^{p}(x), z) = 0$ and $\lim_{p\to\infty} d(f^{p}(x), f^{p}(y)) = 0$, (W4) implies that $\lim_{p\to\infty} d(f^{p}(y), z) = 0$. But $\lim_{p\to\infty} f^{p}(y) = w$ in the topology τ_d and so $\lim_{p\to\infty} d(f^{p}(y), w) = 0$. Since (W4) implies (W3), we have $z = w$. Since y is arbitrary in X, $\lim_{p\to\infty} f^{p}(x) = z$ in the topology τ_d, $\forall x \in X$. □

Corollary 13. *If, in addition to the hypothesis of Theorem 12, one assumes that f is τ_d-continuous (i.e, $x_n \to x$ in the topology τ_d implies $f(x_n) \to f(x)$ in the topology τ_d) then f has a fixed point.*

Proof. Since $\lim_{p\to\infty} f^{p}(x) = z$ in the topology τ_d, by the τ_d-continuity of f, $\lim_{p\to\infty} f^{p+1}(x) = f(z)$ in the topology τ_d. Therefore, since (W4) implies (W3), $f(z) = z$. Hence, $z \in X$ is a fixed point. □

Theorem 14. *Let (X, d) be a bounded semi-metric and d-Cauchy complete space satisfying (W4), (CC) and (JMS). Suppose that f is a self-map on X, and for $x, y \in X$.*

$$d(f(x), f(y)) \leqslant \varphi(\operatorname{diam}(O(x, y))). \tag{26}$$

Then f has a unique fixed point and $\lim_{p\to\infty} f^{p}(x) = z$ in the topology τ_d (or equivalently $d(f^{p}(x), z) \to 0$ as $p \to \infty$), $\forall x \in X$.

Proof. By Theorem 12, there exists $z \in X$ such that $\lim_{p\to\infty} d(f^{p}(x), z) = 0$ for all $x \in X$.

Next, assume that $z \neq f(z)$; that is, $\operatorname{diam}(O(z)) = \beta > 0$.

Thus, it is possible to choose two sequences $\{i(p)\}, \{j(p)\}$ such that

$$\lim_{p\to\infty} d\left(f^{i(p)}(z), f^{j(p)}(z)\right) = \beta. \tag{27}$$

So one can pick $\delta > 0$ with a corresponding $\eta > 0$, such that $\eta \leqslant \beta/2$.

Since there exists $p_0 \in \mathbb{N}$ such that

$$d(f^{n}(z), z) \leqslant \frac{\delta}{2}, \quad d(f^{m}(z), (z)) \leqslant \frac{\delta}{2}, \quad \forall n, m \geqslant p_0, \tag{28}$$

therefore $d(f^{n}(z), z) + d(f^{m}(z), z) \leqslant \delta$, by Proposition 1(ii)

$$d(f^{n}(z), f^{m}(z)) \leqslant \eta \leqslant \frac{\beta}{2}, \quad \forall n, m \geqslant p_0. \tag{29}$$

So, $i(p) \equiv i$ for infinitely many p, with $0 \leqslant i \leqslant p_0$; thus there exists a sequence $\{r(p)\} \subseteq \{j(p)\}$ such that $\lim_{p\to\infty} d(f^{i}(z), f^{r(p)}(z)) = \beta$. So, either $r(p) \equiv j$ for infinitely many p (i.e., $d(f^{i}(z), f^{j}(z)) = \beta$) or there exists a sequence $\{s(p)\} \subseteq \{r(p)\}$ with $s(p) \to \infty$ as $p \to \infty$ which implies $d(f^{i}(z), z) = \beta$.

In both cases, one can conclude that there exists $i, j \geqslant 0$ such that $d(f^{i}(z), f^{j}(z)) = \beta$.

If $d(z, f^{j}(z)) = \beta$, since $\lim_{p\to\infty} d(f^{p}(x), z) = 0$ and by (CC) of d, we get

$$\begin{aligned} \beta &= d\left(z, f^{j}(z)\right) \\ &= \lim_{p\to\infty} d\left(f^{p}(z), f^{j}(z)\right) \\ &\leqslant \lim_{p\to\infty} \varphi\left(\operatorname{diam}\left(O\left(f^{p-1}(z), f^{j-1}(z)\right)\right)\right) \\ &\leqslant \lim_{p\to\infty} \varphi(\operatorname{diam}(O(z))) = \varphi(\beta), \end{aligned} \tag{30}$$

which is a contradiction with $\varphi(\beta) < \beta$, for $\beta > 0$.

On the other hand, if $i, j \geqslant 1$, by (26),

$$\begin{aligned} \beta &= d\left(f^{i}(z), f^{j}(z)\right) \\ &\leqslant \varphi\left(\operatorname{diam}\left(O\left(f^{i-1}(z), f^{j-1}(z)\right)\right)\right) \\ &\leqslant \varphi(\beta) \end{aligned} \tag{31}$$

which is also a contradiction. Hence $\beta = 0$; that is, $f(z) = z$. □

Corollary 15. *Let (X, d) be a bounded semi-metric and d-Cauchy complete space satisfying (W) and (CC). Suppose that f is a self-map on X, and for $x, y \in X$*

$$d(f(x), f(y)) \leqslant \varphi(\operatorname{diam}(O(x, y))). \tag{32}$$

Then f has a unique fixed point and $\lim_{p\to\infty} f^p(x) = z$ in the topology τ_d (or equivalently $d(f^p(x), z) \to 0$ as $p \to \infty$), for all $x \in X$.

5. Symmetric Spaces: Fixed Point Results

In this section, we extend results attributed to Maiti et al. [21, Theorem 4] and Aranđelović and Kečkić [2, Theorem 3].

Theorem 16. *Let (X, d) be a d-Cauchy complete symmetric space satisfying (W3) and (JMS). Let $f : X \to X$ be a τ_d-continuous map such that*

$$d(f(x), f(y)) \leqslant \varphi(d_f(x, y)), \tag{33}$$

for all $x, y \in X$, and $\varphi \in \Phi$. Then f has a unique fixed point $z \in X$ and for each $x \in X$, the sequence of Picard iterates defined by f at x converges to z in the topology τ_d.

Proof. Define $d^* : X \times X$ as follows: $d^*(x, y) = 0$ for $x = y$ and $d^*(x, y) = d_f(x, y)$ otherwise. Then the space (X, d^*) is a symmetric space. Also, we have $d(x, y) \leqslant d^*(x, y)$ for any $x, y \in X$. So, if $\{x_n\} \subseteq X$ is an arbitrary d^*-Cauchy sequence in (X, d^*), then $\{x_n\}$ is a d-Cauchy sequence in (X, d).

Let $x, y \in X$. From

$$\begin{aligned} &d(f^2(x), f(x)) \leqslant \varphi(d(x, f(x))), \\ &d(f^2(y), f(y)) \leqslant \varphi(d(y, f(y))), \\ &d(f(x), f(y)) \leqslant \varphi(d_f(x, y)), \end{aligned} \tag{34}$$

it follows that

$$d^*(f(x), f(y)) \leqslant \varphi(d^*(x, y)). \tag{35}$$

So f is a φ-contraction on (X, d^*).

Let δ, η be defined as in (ii) of Proposition 1. Then there exists the least positive integer $j \geqslant 1$ such that $\varphi^j(\eta) \leqslant \delta/2$.

Let $g = f^j$. We have that g is continuous (in τ_d). Then

$$\begin{aligned} d^*(g(x), g(y)) &= d^*(f(f^{j-1}(x)), f(f^{j-1}(y))) \\ &\leqslant \varphi(d^*(f^{j-1}(x), f^{j-1}(y))) \\ &\leqslant \varphi^j(d^*(x, y)). \end{aligned} \tag{36}$$

So g is a φ^j-contraction on (X, d^*).

Let $x \in X$ and $\psi = \varphi^j$. Then $\psi \in \Phi$ and

$$d^*(g^{m+n}(x), g^n(x)) \leqslant \psi^n(d^*(x, g^m(x))) \quad \text{for any } m, n \in \mathbb{N}. \tag{37}$$

So

$$d^*(g^{n+1}(x), g^n(x)) \leqslant \psi^n(d^*(x, g(x))) \tag{38}$$

which implies that

$$d^*(g^{n+1}(x), g^n(x)) \longrightarrow 0. \tag{39}$$

Then there exists $k \in \mathbb{N}$ such that

$$d^*(g^k(x), g^{k+1}(x)) \leqslant \min\left\{\frac{\delta}{2}, \eta\right\}. \tag{40}$$

We will prove that, for all $n \in \mathbb{N}$,

$$d^*(g^k(x), g^{k+n}(x)) \leqslant \eta. \tag{41}$$

By definition of k, we get that (41) is valid for $n = 1$. Now, assume that (41) is satisfied for some $n \in \mathbb{N}$. From

$$\begin{aligned} &d^*(g^k(x), g^{k+1}(x)) \leqslant \frac{\delta}{2}, \\ &d^*(g^{k+1}(x), g^{k+n+1}(x)) \\ &\quad \leqslant \psi(d^*(g^k(x), g^{k+n}(x))) \\ &\quad \leqslant \psi(\eta) \leqslant \frac{\delta}{2}, \end{aligned} \tag{42}$$

it follows that

$$d^*(g^k(x), g^{k+1}(x)) + d^*(g^{k+1}(x), g^{k+n+1}(x)) \leqslant \delta, \tag{43}$$

which by Proposition 1 implies that

$$d^*(g^k(x), g^{k+n+1}(x)) \leqslant \eta. \tag{44}$$

So, by induction we get that (41) is satisfied for any $n \geqslant 1$. Thus

$$d^*(g^{k+n}(x), g^{k+n+m}(x)) \leqslant \psi^n(\eta), \quad \text{for any } m, n \in \mathbb{N}. \tag{45}$$

Hence $\{g^n(x)\}$ is a d^*-Cauchy sequence in (X, d^*), which implies that $\{g^n(x)\}$ is a d-Cauchy sequence in (X, d). It follows that there exists $z \in X$ such that $\lim_{n\to\infty} g^n(x) = z$ (in the topology τ_d) because (X, d) is d-Cauchy complete. Then $\lim_{n\to\infty} g^{n+1}(x) = g(z)$ (in the topology τ_d) because g is τ_d-continuous. Now we get that $g(z) = z$ because (X, d) satisfies (W3).

If y is another fixed point of f, then for all n we have

$$d^*(y, z) = d^*(g^n(y), g^n(z)) \leqslant \psi^n(d^*(y, z)) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{46}$$

So z is a unique fixed point of g. By Lemma 9 we get that z is a unique fixed point of f.

From

$$\begin{aligned} d^*(z, g^{n+1}(x)) \leqslant \varphi(\max\{&d(z, g(z)), d(z, g^n(x)), \\ &d(g^n(x), g^{n+1}(x))\}) \end{aligned} \tag{47}$$

it follows that for each $x \in X$ the sequence of Picard iterates defined by $g = f^j$ at x converges, in the topology τ_{d^*}, to z, which implies their convergence in the topology τ_d. So, by Lemma 9, we obtain that for each $x \in X$ the sequence of Picard iterates defined by f at x converges, in the topology τ_d, to z. □

Remark 17. The next example of [10] illustrates that the continuity of f in Theorem 16 can not be omitted.

Example 18. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and let d be defined as follows:

$$d(0,1) = d(1,0) = 1;$$
$$d(1,1/n) = d(1/n,1) = 2/3 \text{ for } n \geqslant 2;$$
$$d(1,1) = 0; \text{ otherwise } d(x,y) = |x-y|.$$

Let $f : X \to X$ given by

$$f(x) = \begin{cases} \dfrac{x}{4}, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases} \tag{48}$$

Then (X,d) is a bounded d-Cauchy complete semi-metric space and

$$d(f(x), f(y)) \leqslant \varphi\left(d_f(x,y)\right) \tag{49}$$

for all $x, y \in X$, (see [10, Example 3]). (X,d) satisfies (W3) and (JMS).

But f does not have a fixed point in X. Note that f is not continuous.

6. Applications

We now present applications of our results to probabilistic spaces. We begin with some essential definitions.

Definition 19. Let X be a set and $\mathcal{F}$ a mapping of $X \times X$ into a collection $\mathcal{L}$ of all distribution functions F (a distribution function F is a nondecreasing and left continuous mapping of reals into $[0,1]$ with $\inf\{F(x)\} = 0$ and $\sup\{F(x)\} = 1$). Consider the following conditions:

(I) $F_{x,y}(0) = 0$ for all $x, y \in X$, where $F_{x,y}$ denotes the value of $\mathcal{F}$ at $(x,y) \in X \times X$.

(II) $F_{x,y} = H$ if and only if $x = y$, where H denotes the distribution function defined by $H(x) = 0$ if $x \leqslant 0$ and $H(x) = 1$ if $x > 0$.

(III) $F_{x,y} = F_{y,x}$.

(IV) If $F_{x,y}(\varepsilon) = 1$ and $F_{y,z}(\delta) = 1$, then $F_{x,z}(\varepsilon + \delta) = 1$.

If $\mathcal{F}$ satisfies (I) and (II), then it is called a *PPM-structure* on X and the pair $(X,\mathfrak{F})$ is called a *PPM-space*. $\mathfrak{F}$ satisfying (III) is said to be symmetric. A symmetric PPM-space satisfying (IV) is a probabilistic metric space (or briefly *PM-space*).

The topology $\tau_{\mathfrak{F}}$ in $(X,\mathfrak{F})$ is generated by the family

$$\mathcal{U} = \{U_x(\varepsilon,\lambda) : x \in X, \varepsilon > 0, \lambda > 0\}, \tag{50}$$

where the set

$$U_x(\varepsilon,\lambda) = \left\{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\right\} \tag{51}$$

is called (ε,λ)-*neighborhood* of $x \in X$. A sequence $\{x_n\}$ is said to be a Cauchy sequence if, for every given $\varepsilon, \lambda > 0$, there exists a positive integer $n_0 = n_0(\varepsilon,\lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ for all $m, n \geqslant n_0$. A T_1 topology $\tau_{\mathfrak{F}}$ on X is defined as follows: $U \in \tau_{\mathfrak{F}}$ if, for any $x \in U$, there exists $\varepsilon > 0$ such that $U_x(\varepsilon,\varepsilon) \subset U$. If $U_x(\varepsilon,\varepsilon) \in \tau_{\mathfrak{F}}$, then $\tau_{\mathfrak{F}}$ is said to be *topological*.

The space $(X,\mathfrak{F})$ is called $\mathfrak{F}$-complete if for every Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that $\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1$ for all $\varepsilon > 0$.

Remark 20. (1) The condition (W) is equivalent to

$$\lim_{n\to\infty} F_{x_n,y_n}(\varepsilon) = 1, \qquad \lim_{n\to\infty} F_{y_n,z_n}(\varepsilon) = 1, \qquad \text{imply } \lim_{n\to\infty} F_{x_n,z_n}(\varepsilon) = 1. \tag{P}$$

(2) The condition (W4) is equivalent to

$$\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1, \qquad \lim_{n\to\infty} F_{x_n,y_n}(\varepsilon) = 1, \qquad \text{imply } \lim_{n\to\infty} F_{y_n,x}(\varepsilon) = 1. \tag{P4}$$

The following lemma was proved in [11].

Lemma 21 (Hicks and Rhoades [11]). *Let $(X,\mathfrak{F})$ be a symmetric PPM-space. Set*

$$d(x,y) = \begin{cases} 0, & \textit{if } y \notin U_x(\varepsilon,\varepsilon), \forall \varepsilon > 0 \\ \sup\{\varepsilon : y \notin U_x(\varepsilon,\varepsilon), \ \varepsilon > 0\}, & \textit{otherwise}. \end{cases} \tag{52}$$

Then d is a bounded compatible symmetric for X.

Lemma 22 (Hicks and Rhoades [11]). *Let $(X,\mathfrak{F})$ be a symmetric PPM-space. Define d as in* (52). *Then*

(1) *$d(x,y) < t$ if and only if $F_{x,y}(t) > 1 - t$;*

(2) *d is compatible symmetric for $\tau_{\mathfrak{F}}$;*

(3) *$(X,\mathfrak{F})$ is complete if and only if (X,d) is d-Cauchy complete symmetric space;*

(4) *if $\tau_{\mathfrak{F}}$ is topological, d is semi-metric.*

$f : (X,\mathcal{F}) \to (X,\mathcal{F})$ is $\mathcal{F}$-continuous if $F_{x_n,x}(t) \to 1$ for all $t > 0$ implies $F_{f(x_n),f(x)}(t) \to 1$. This is equivalent to the continuity of $f : (X,d) \to (X,d)$, where d is as in Lemma 21.

Let Φ' denote the set of all functions $\varphi \in \Phi$ satisfying

$$\lim_{\epsilon\to 0} \varphi(t + \epsilon) = \varphi(t) \tag{53}$$

for all $t > 0$.

Theorem 23. *Let $(X,\mathfrak{F})$ be a complete symmetric PPM-space that satisfies* (P4), *where $\tau_{\mathfrak{F}}$ is a topological. Suppose $f : X \to$*

X is $\mathcal{F}$-continuous and satisfies that for $x \in X$ there exists $\nu(x) \in \mathbb{N}$ such that for any $\nu \geqslant \nu(x)$ and $y \in X$

$$F_{u,v}(t) > 1 - t \quad \text{implies } F_{f^{\nu}(x),f^{\nu}(y)}(\varphi(t)) > 1 - \varphi(t), \quad u, v \in O(x, y) \tag{54}$$

for every $t > 0$ with $\varphi \in \Phi'$. Then f has a fixed point.

Proof. Define d as in (52). According to Lemmas 21 and 22, (X, d) is a bounded d-Cauchy complete semi-metric space satisfying (W4). Now assume that (54) is satisfied. Let $\varepsilon > 0$ be given and let $t = d(u, v) + \varepsilon$. Then $d(u, v) < t$ gives $F_{u,v}(t) > 1 - t$ and so $F_{f^{\nu}(x),f^{\nu}(y)}(\varphi(t)) > 1 - \varphi(t)$ and so $d(f^{\nu}(x), f^{\nu}(y)) < \varphi(t) = \varphi(d(u, v) + \varepsilon)$. Since ε was arbitrary, we have that

$$\begin{aligned} d(f^{\nu}(x), f^{\nu}(y)) &\leqslant \varphi(d(u, v)) \\ &\leqslant \varphi\left(\sup_{u,v \in O(x,y)} d(u, v)\right) \\ &= \varphi(\operatorname{diam}(O(x, y))). \end{aligned} \tag{55}$$

By Corollary 13, f has a fixed point. □

Theorem 24. *Let $(X, \mathfrak{F})$ be a complete symmetric PPM-space that satisfies (P). Let $f : X \to X$ be $\mathcal{F}$-continuous such that*

$$F_{x,y}(t) > 1 - t \quad \text{implies } F_{f(x),f(y)}(\varphi(t)) > 1 - \varphi(t) \tag{56}$$

for all $x, y \in X$, and $\varphi \in \Phi'$. Then f has a unique fixed point.

Proof. Define d as in (52). According to Lemma 21, d is a bounded compatible symmetric for $\tau_{\mathfrak{F}}$ and (X, d) is d-Cauchy complete symmetric space satisfying (W3) and (JMS). Suppose (56) is satisfied and let $\varepsilon > 0$ be given. Let $t = \max\{d(x, y) + \varepsilon, d(x, f(x)), d(y, f(y))\}$. Then $d(x, y) + \varepsilon \leqslant t$ and so $d(x, y) \leqslant t - \varepsilon < t$, which implies $F_{x,y}(t) > 1 - t$. This further implies that $F_{f(x),f(y)}(\varphi(t)) > 1 - \varphi(t)$ and so

$$\begin{aligned} d(f(x), f(y)) & \\ &< \varphi(t) = \varphi(\max\{d(x, y) + \varepsilon, d(x, f(x)), d(y, f(y))\}) \\ &= \max\{\varphi(d(x, y) + \varepsilon), \varphi(d(x, f(x))), \\ &\qquad \varphi(d(y, f(y)))\}. \end{aligned} \tag{57}$$

Since ε was arbitrary, we have that

$$\begin{aligned} d(f(x), f(y)) & \\ &\leqslant \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y)))\} \\ &= \varphi(\max\{d(x, y), d(x, f(x)), d(y, f(y))\}). \end{aligned} \tag{58}$$

Now Theorem 16 guarantees that f has a unique fixed point $z \in X$. □

7. Some Open Problems

Problem 25 (see [2]). Let (X, d) be a symmetric space which satisfies the property (MT). Is it a semi-metric space (not necessarily with open balls)?

Problem 26. Does Theorem 16 hold if $d_f(x, y)$ is replaced with

$$\begin{aligned} D_f(x, y) = \max\{&d(x, y), d(x, f(x)), d(y, f(y)), \\ &d(x, f(y)), d(f(x), y)\}? \end{aligned} \tag{59}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] W. A. Wilson, "On semi-metric spaces," *American Journal of Mathematics*, vol. 53, no. 2, pp. 361–373, 1931.

[2] I. D. Aranđelović and D. J. Kečkić, "Symmetric spaces approach to some fixed point results," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 13, pp. 5157–5168, 2012.

[3] M. A. Alghamdi, N. Shahzad, and O. Valero, "On fixed point theory in partial metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 175, 25 pages, 2012.

[4] R. P. Agarwal, M. A. Alghamdi, and N. Shahzad, "Fixed point theory for cyclic generalized contractions in partial metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 40, 11 pages, 2012.

[5] R. P. Agarwal, D. O'Regan, and N. Shahzad, "Fixed point theory for generalized contractive maps of Meir-Keeler type," *Mathematische Nachrichten*, vol. 276, pp. 3–22, 2004.

[6] Z. Kadelburg, S. Radenovic, and N. Shahzad, "A note on various classes of compatible-type pairs of mappings and common fixed point theorems," *Abstract and Applied Analysis*, vol. 2013, Article ID 697151, 6 pages, 2013.

[7] W. A. Kirk, "Contraction mappings and extensions," in *Handbook of Metric Fixed Point Theory*, pp. 1–34, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[8] W. A. Kirk and N. Shahzad, "Generalized metrics and Caristi's theorem," *Fixed Point Theory and Applications*, vol. 2013, article 129, 9 pages, 2013.

[9] Sh. Rezapour, R. H. Haghi, and N. Shahzad, "Some notes on fixed points of quasi-contraction maps," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 498–502, 2010.

[10] J. Jachymski, J. Matkowski, and T. Świątkowski, "Nonlinear contractions on semimetric spaces," *Journal of Applied Analysis*, vol. 1, no. 2, pp. 125–134, 1995.

[11] T. L. Hicks and B. E. Rhoades, "Fixed point theory in symmetric spaces with applications to probabilistic spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 36, no. 3, pp. 331–344, 1999.

[12] M. Aamri and D. El Moutawakil, "Common fixed points under contractive conditions in symmetric spaces," *Applied Mathematics E-Notes*, vol. 3, pp. 156–162, 2003.

[13] M. Aamri, A. Bassou, and D. EL Moutawakil, "Common fixed points for weakly compatible maps in symmetric spaces with application to probabilistic spaces," *Applied Mathematics E-Notes*, vol. 5, pp. 171–175, 2005.

[14] J. Zhu, Y. J. Cho, and S. M. Kang, "Equivalent contractive conditions in symmetric spaces," *Computers & Mathematics with Applications*, vol. 50, no. 10–12, pp. 1621–1628, 2005.

[15] D. Miheţ, "A note on a paper of Hicks and Rhoades," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1411–1413, 2006.

[16] M. Imdad, J. Ali, and L. Khan, "Coincidence and fixed points in symmetric spaces under strict contractions," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 1, pp. 352–360, 2006, Erratum in *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, p. 752, 2007.

[17] A. Aliouche, "A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 796–802, 2006.

[18] S. Radenović and Z. Kadelburg, "Quasi-contractions on symmetric and cone symmetric spaces," *Banach Journal of Mathematical Analysis*, vol. 5, no. 1, pp. 38–50, 2011.

[19] F. E. Browder, "Remarks on fixed point theorems of contractive type," *Nonlinear Analysis*, vol. 3, no. 5, pp. 657–661, 1979.

[20] W. Walter, "Remarks on a paper by F. Browder about contraction," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 5, no. 1, pp. 21–25, 1981.

[21] M. Maiti, J. Achari, and T. K. Pal, "Remarks on some fixed point theorems," *Publications de l'Institut Mathématique*, vol. 21, pp. 115–118, 1977.

[22] S. H. Cho, G. Y. Lee, and J. S. Bae, "On coincidence and fixed-point theorems in symmetric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 562130, 9 pages, 2008.

[23] C. J. R. Borges, "On continuously semimetrizable and stratifiable spaces," *Proceedings of the American Mathematical Society*, vol. 24, pp. 193–196, 1970.

[24] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[25] M. Jovanović, Z. Kadelburg, and S. Radenović, "Common fixed point results in metric-type spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 978121, 15 pages, 2010.

[26] R. W. Heath, "A regular semi-metric space for which there is no semi-metric under which all spheres are open," *Proceedings of the American Mathematical Society*, vol. 12, pp. 810–811, 1961.

[27] F. Galvin and S. D. Shore, "Completeness in semimetric spaces," *Pacific Journal of Mathematics*, vol. 113, no. 1, pp. 67–75, 1984.

[28] L. F. McAuley, "A relation between perfect separability, completeness, and normality in semi-metric spaces," *Pacific Journal of Mathematics*, vol. 6, pp. 315–326, 1956.

[29] I. A. Rus, A. Petrusel, and G. Petrusel, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, Romania, 2008.

[30] S. Lipschutz, *General Topology*, McGraw-Hill, New York, NY, USA, 1965.