Research Article

On a Fractional SPDE Driven by Fractional Noise and a Pure Jump Lévy Noise in $\mathbb{R}^d$

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We study a stochastic partial differential equation in the whole space $x \in \mathbb{R}^d$, with arbitrary dimension $d \geq 1$, driven by fractional noise and a pure jump Lévy space-time white noise. Our equation involves a fractional derivative operator. Under some suitable assumptions, we establish the existence and uniqueness of the global mild solution via fixed point principle.

1. Introduction

Let $d \geq 1$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\delta = (\delta_1, \ldots, \delta_d)$, and $\mathbb{D}_\delta^\alpha$ be the nonlocal fractional differential operator defined by

$$\mathbb{D}_\delta^\alpha = \sum_{i=1}^d D_{\delta_i}^{\alpha_i}, \tag{1}$$

where $D_{\delta_i}^{\alpha_i}$ denotes the fractional differential derivative with respect to the ith coordinate defined via its Fourier transform $\mathscr{F}$ by

$$\mathscr{F}\left(D_\delta^\alpha \varphi\right)(\xi) = -|\xi|^\alpha \exp\left(-i\delta \frac{\pi}{2}\, \mathrm{sgn}(\xi)\right) \mathscr{F}(\varphi)(\xi). \tag{2}$$

In this paper, we are concerned with the following jump type stochastic partial differential equation (SPDE for abbreviation) with fractional noise:

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \mathbb{D}_\delta^\alpha u(t,x) + b(t,x,u(t,x)) \\
&\quad + \sigma(t,x,u(t,x))\, \dot{L}(t,x) + B^H(dt,dx), \\
u(0,x) &= u_0(x),
\end{aligned} \tag{3}$$

where $(t,x) \in [0,T] \times \mathbb{R}^d$, $\dot{L}$ is a pure jump Lévy space-time white noise on $[0,T] \times \mathbb{R}^d$ defined on a complete

probability space $(\Omega, \mathscr{F}, P)$, and $B^H(dt,dx)$ denotes the fractional noise on $[0,T] \times \mathbb{R}^d$ with multiparameter $H = (h_0, h_1, \ldots, h_d)$ for $h_i \in (0,1)$ and $i \in \{0,1,2,\ldots,d\}$ defined on a complete probability space $(\Omega, \mathscr{F}, P)$ (see Section 2 for precise definitions). Actually, we understand this equation as Walsh [1] sense, and so we can rewrite (3) as follows:

$$\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t, x-y)\, u_0(y)\, dy \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t-s, x-y)\, B^H(dy,ds) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t-s, x-y)\, b(s,y,u(s,y))\, dy\, ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t-s, x-y)\, \sigma(s,y,u(s,y))\, \dot{L}(y,s)\, dy\, ds,
\end{aligned} \tag{4}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, where $\mathbb{G}_{\alpha,\delta}(\cdot, *)$ denotes the Green function associated with (3).

In the present paper, we are interested in the study of (3) with respect to d-dimensional nonlocal fractional differential operator $\mathbb{D}_\delta^\alpha$. Such operator is initially introduced by Debbi

and Dozzi [2] for $d = 1$ and is generalized by Boulanba et al. [3] for the multidimensional space $\mathbb{R}^d$. This operator is a generalization of various well-known operators, such as the Laplacian operator, the inverse of the generalized Riesz-Feller potential, and the Riemann-Liouville differential operator. In probabilistic terms, replacing the Laplacian by its fractional power (which is an integrodifferential operator) leads to interesting and largely open questions of extensions of results for Brownian motion driven stochastic equations to those driven by Lévy stable processes. In the physical literature, such fractal anomalous diffusions have been recently enthusiastically embraced by a slew of investigators in the context of hydrodynamics, acoustics, trapping effects in surface diffusion, statistical mechanics, relaxation phenomena, and biology. We refer the readers to [4–9] and references therein for more information about such fractional differential operator.

On the other hand, many researchers are interested in studying SPDES driven by a fractional noise. The heat equations with a multiparameter fractional noise of Hurst parameter $H = (h_0, \ldots, h_d)$ on $[0, \infty] \times \mathbb{R}^d$ were introduced by Hu [10], and he showed the existence and uniqueness of the solutions to the equation, via chaos expansion. Wei [11] considered a class of four-order stochastic partial differential equations driven by the multiparameter fractional noise; the author studied the regularity of the solution and the existence of the density of the law of the solution. Hu and Nualart [12] studied the d-dimensional stochastic heat equation with a multiplicative Gaussian noise which is white in space and has the covariance of a fractional Brownian motion with Hurst parameter $h \in (0, 1)$ in time. More works for the fields can be found in Balan and Tudor [13, 14], Bo et al. [15, 16], Jiang et al. [17–19], and Shi and Wang [20] and the references therein. In the meanwhile, there also have been some works on SPDES involving Lévy space-time white noise (e.g., Albeverio et al. [21], Shi and Wang [22], Truman and Wu [23, 24], and Wu and Xie [25]). Løkka et al. [26] studied the stochastic partial differential equations driven by a d-parameter pure jump Lévy white noise. As an example they used this theory to solve the stochastic Poisson equation with respect to Lévy white noise for any dimension d. We notice that the fixed point principle and Picard iteration scheme work in Albeverio et al. [21] and Truman and Wu [23, 24], since Burkhölder-Davis-Gundy (B-D-G) inequality can be applied to estimate stochastic integral with respect to compensated Poisson random measure in L^2-sense. Unfortunately, the usual B-D-G inequality cannot work in estimating stochastic integral with respect to compensated Poisson random measure in L^p-sense ($p > 2$). Hence a new version of B-D-G inequality will be adopted for L^p ($p \geq 2$)-estimates on the solution to (3) (see Bo and Wang [27]).

Motivated by the above results, in this paper, we study a stochastic fractional partial differential equation in the whole space $x \in \mathbb{R}^d$, with arbitrary dimension $d \geq 1$, driven by fractional noise and a pure jump Lévy space-time white noise. The main subject of this paper is to establish the existence and uniqueness of the solution of (3) via fixed point principle.

The rest of the paper is organized as follows. In Section 2, we begin by making some notations and by recalling some basic preliminaries about fractional noises, Lévy space-time white noise, and Green function which will be needed later. In Section 3, we will prove the existence and uniqueness of the mild solution to (4) in L^p ($p \geq 2$) sense under some approximate conditions. Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by C, which may not be the same in each occurrence. Sometimes we will emphasize the dependence of these constants upon parameters.

2. Preliminaries

In this section, we will present the definitions and some results of the multiparameter fractional noises, Lévy space-time white noise, and Green function.

2.1. Multiparameter Fractional Noises. Recall that a fractional Brownian motion with Hurst parameter $h \in (0, 1)$ on $[0, T]$ is a centered Gaussian process (B_t) with covariance

$$E\left(B_t B_s\right) = \frac{1}{2}\left(t^{2h} + s^{2h} - |t - s|^{2h}\right). \tag{5}$$

Following Hu [10] and Wei [11], we introduce a multiparameter fractional Brownian field.

Definition 1. A multiparameter fractional Brownian field $B^H = \{B^H(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ with multiparameter $H = (h_0, h_1, \ldots, h_d)$ for $h_i \in (0, 1)$ and $i \in \{0, 1, 2, \ldots, d\}$ is a centered Gaussian field defined on some probability space $(\Omega, \mathcal{F}, P)$ with covariance

$$
\begin{aligned}
E&\left[B^H(t, x) B^H(s, y)\right] \\
&= \frac{1}{2^{d+1}}\left(t^{2h_0} + s^{2h_0} - |t - s|^{2h_0}\right) \\
&\quad \times \prod_{i=1}^{d}\left(x_i^{2h_i} + y_i^{2h_i} - |x_i - y_i|^{2h_i}\right),
\end{aligned}
\tag{6}
$$

for all $t, s \in [0, T]$ and $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

Throughout the paper, we limit our consideration on the multiparameter fractional Brownian field with parameter $H = (h_0, h_1, \ldots, h_d) = (h, h, \ldots, h)$ and $h \in (1/2, 1)$.

Let

$$\Psi_h(t, s, x, y) := [h(2h + 1)]^{d+1}|t - s|^{2h-2}\prod_{i=1}^{d}|x_i - y_i|^{2h-2},$$

$$0 \leq s \leq T, \quad x, y \in \mathbb{R}^d. \tag{7}$$

Introduce the following Hilbert space:

$$L^2_\Psi := \left\{ f : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}; \right.$$

$$\|f\|^2_\Psi = \int_{[0,T]^2} \int_{\mathbb{R}^{2d}} \Psi_h(t, s, x, y)$$

$$\left. \times f(t, x) f(s, y) dy\, dx\, ds\, dt < \infty \right\}, \tag{8}$$

where $dx = dx_1 dx_2, \ldots, dx_d$ and $dy = dy_1 dy_2, \ldots, dy_d$. Let $f \in L^2_\Psi$; one can define the following stochastic integral:

$$\left\{ \int_0^t \int_{\mathbb{R}^d} f(s, x) dB^H(t, x); t \in [0, T] \right\}, \tag{9}$$

(see, e.g., Hu [10]); it is easy to check that this integral has the usual properties as a stochastic integral.

Proposition 2. *Let* $f, g \in L^2_\Psi$. *Then*

(1) $E[\int_0^T \int_{\mathbb{R}^d} f(s, x) dB^H(s, x)] = 0$,

(2) $E[\int_0^T \int_{\mathbb{R}^d} f(s, x) dB^H(s, x) \int_0^T \int_{\mathbb{R}^d} g(s, x) dB^H(s, x)] = \langle f, g \rangle_{L^2_\Psi}$.

The following embedding proposition which is an extension to the results in Mémin et al. [28] had been proved in Wei [11].

Lemma 3. *Consider the following:*
$L^h([0, T] \times \mathbb{R}^d) \subset L^2_\Psi \ (h > 1/2)$.

2.2. Lévy Space-Time White Noise. Let $(E_i, \mathscr{E}_i, \mu_i)$, $i = 1, 2$, be two σ-finite measurable spaces. We call $N : (E_1, \mathscr{E}_1, \mu_1) \times (E_2, \mathscr{E}_2, \mu_2) \times (\Omega, \mathscr{F}, P) \to \mathbb{N} \cup \{0\} \cup \{\infty\}$ a Poisson noise on $(E_1, \mathscr{E}_1, \mu_1)$, if, for all $A \in \mathscr{E}_1, B \in \mathscr{E}_2$, and $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$,

$$P(N(A, B) = n) = \frac{e^{-\mu_1(A)\mu_2(B)}[\mu_1(A)\mu_2(B)]^n}{n!}. \tag{10}$$

In particular, if $(E_1, \mathscr{E}_1, \mu_1) = ([0, \infty) \times \mathbb{R}^d, \mathscr{B}([0, \infty) \times \mathbb{R}^d), dt \times dx)$, then define a compensated random martingale measure by

$$M(B, A, t) = N([0, t] \times A, B) - \mu_1([0, t] \times A) \mu_2(B), \tag{11}$$

by assuming that $\mu_1([0, t] \times A)\mu_2(B) < \infty$ for all $(t, A, B) \in [0, \infty) \times \mathscr{B}(\mathbb{R}^d) \times \mathscr{E}_2$. Further, let $\phi : E_1 \times E_2 \times \Omega \to \mathbb{R}$ be an $(\mathscr{F}_t)_{t \geq 0}$-predictable function satisfying

$$\mathbb{E} \left[\int_0^t \int_A \int_B |\phi(t, x, y)|^2 \mu_2(dy)\, dx\, ds \right] < \infty, \tag{12}$$

for all $t > 0$ and $(A, B) \in \mathscr{E}_1 \times \mathscr{E}_2$. We can define a stochastic integral process

$$\left(R_t = \int_0^{t+} \int_A \int_B \phi(s, x, y) M(dy, dx, ds) \right) \tag{13}$$

which is a square integrable $(P, (\mathscr{F}_t)_{t \geq 0})$-martingale. It is well known that a (pure jump) Lévy space-time white noise possesses the following structure:

$$\dot{L}(x, t) = \int_{U_0} h_1(t, x, y) \dot{M}(dy, x, t)$$

$$+ \int_{E_2 \setminus U_0} h_2(t, x, y) \dot{N}(dy, x, t), \tag{14}$$

for some $U_0 \in E_2$ such that $\mu_2(E_2 \setminus U_0) < \infty$ and $\int_{U_0} z^2 \mu_2(dz) < +\infty$. Here $h_1, h_2 : [0, \infty) \times \mathbb{R} \times E_2 \to \mathbb{R}$ are some measurable functions; $\dot{M}$ and $\dot{N}$ are the Radon-Nikodym derivatives given by

$$\dot{M}(dy, x, t) = \frac{M(dy, dx, dt)}{dt \times dx},$$

$$\dot{N}(dy, x, t) = \frac{N(dt \times dx, dy)}{dt \times dx}, \tag{15}$$

with $(t, x, y) \in [0, \infty) \times \mathbb{R}^d \times E_2$.

The following B-D-G inequality is given by Bo and Wang [27], which is useful to estimate the higher order moments of mild solution to (4).

Lemma 4. *Let* $\phi : [0, \infty) \times \mathbb{R}^d \times E_2 \times \Omega \to \mathbb{R}$ *be* $(\mathscr{F}_t)_{t \geq 0}$-*predictable and satisfy* (13). *Define the integral process by*

$$\left\{ X_t = \int_0^{t+} \int_{\mathbb{R}^d} \int_{E_2} \phi(s, y, z) M(dz, dy, ds), \ t \geq 0 \right\}. \tag{16}$$

Then, for any $T > 0$ *and* $p > 1$, *there exists a constant* $C_{p,T} > 0$ *such that*

$$\sup_{t \in [0,T]} \mathbb{E}\left[|X_t|^p \right]$$

$$\leq C_{p,T} \left[\int_0^T \int_{\mathbb{R}^d} \int_{E_2} \left(\mathbb{E}\left[|\phi(s, y, z)|^p \right] \right)^{2/p} \mu_2(dz)\, dy\, ds \right]^{p/2}. \tag{17}$$

2.3. Green Function. In this subsection, we will introduce the nonlocal fractional differential operator defined by

$$\mathbb{D}_\delta^\alpha = \sum_{i=1}^d D_{\delta_i}^{\alpha_i}, \tag{18}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\delta = (\delta_1, \ldots, \delta_d)$, and $D_{\delta_i}^{\alpha_i}$ denotes the fractional differential derivative with respect to the ith coordinate defined via its Fourier transform $\mathscr{F}$ by

$$\mathscr{F}(D_\delta^\alpha \varphi)(\xi) = -|\xi|^\alpha \exp\left(-i\delta \frac{\pi}{2} \operatorname{sgn}(\xi) \right) \mathscr{F}(\varphi)(\xi). \tag{19}$$

In this paper, we will assume that $|\delta_i| \leq \min\{\alpha_i - [\alpha_i]_2, 2 + [\alpha_i]_2 - \alpha_i\}$, $i = 1, \ldots, d$, $[\alpha_i]_2$ is the largest even integer less than or equal to α_i (even part of α_i), and $\delta_i \in 2\mathbb{N} + 1$.

In one space dimension, the operator D_δ^α is a closed, densely defined operator on $L^2(\mathbb{R})$ and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. This operator is a generalization of various well-known operators, such as the Laplacian operator (when $\alpha = 2$), the inverse of the generalized Riesz-Feller potential (when $\alpha > 2$), and the Riemann-Liouville differential operator (when $|\delta| = 2 + [\alpha]_2$ or $|\delta| = \alpha - [\alpha]$). It is self-adjoint only when $\delta = 0$ and, in this case, it coincides with the fractional power of the Laplacian. We refer the readers to Debbi [29], Debbi and Dozzi [2], and Komatsu [30] for more details about this operator.

According to Komatsu [30], D_δ^α can be represented, for $1 < \alpha < 2$, by

$$D_\delta^\alpha \varphi(x)$$
$$= \int_\mathbb{R} \frac{\varphi(x+y) - \varphi(x) - y\varphi'(x)}{|y|^{1+\alpha}} \quad (20)$$
$$\times \left(\kappa_-^\delta 1_{(-\infty,0)}(y) + \kappa_+^\delta 1_{(-0,+\infty)}(y) \right) dy$$

and, for $0 < \alpha < 1$, by

$$D_\delta^\alpha \varphi(x)$$
$$= \int_\mathbb{R} \frac{\varphi(x+y) - \varphi(x)}{|y|^{1+\alpha}} \quad (21)$$
$$\times \left(\kappa_-^\delta 1_{(-\infty,0)}(y) + \kappa_+^\delta 1_{(-0,+\infty)}(y) \right) dy,$$

where κ_-^δ and κ_+^δ are two nonnegative constants satisfying $\kappa_-^\delta + \kappa_+^\delta > 0$ and φ is a smooth function for which the integral exists, and φ' is its derivative. This representation identifies it as the infinitesimal generator for a nonsymmetric α-stable Lévy process.

Let $G_{\alpha,\delta}(t,x)$ be the fundamental solution of the following Cauchy problem:

$$\frac{\partial u}{\partial t}(t,x) = D_\delta^\alpha(t,x),$$
$$u(0,x) = \delta_0(x), \quad t > 0, \ x \in \mathbb{R}, \quad (22)$$

where $\delta_0(\cdot)$ is the Dirac distribution. By Fourier transform, we see that $D_\delta^\alpha(t,x)$ is given by $\mathbb{F}$:

$$G_{\alpha,\delta}(t,x)$$
$$= \frac{1}{2\pi} \int_\mathbb{R} \exp\left(-izx - t|z|^\alpha \exp\left(-i\delta\frac{\pi}{2} \operatorname{sgn}(z) \right) \right) dz. \quad (23)$$

The relevant parameters α called the index of stability and δ (related to the asymmetry) improperly referred to as the skewness are real numbers satisfying $|\delta| \le \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$, and $\delta = 0$ when $\delta \in 2\mathbb{N} + 1$.

Let us list some known facts on $G_{\alpha,\delta}(t,x)$ which will be used later on (see, e.g., Debbi [29] and Debbi and Dozzi [2]).

Lemma 5. *Let $\alpha \in (0,\infty)/\{\mathbb{N}\}$; one has the following.*

(1) *The function $G_{\alpha,\delta}(t,\cdot)$ is not in general symmetric relatively to x and it is not everywhere positive.*

(2) *For any $s,t \in (0,\infty)$ and $x \in \mathbb{R}$,*

$$\frac{\partial^n}{\partial x^n} G_{\alpha,\delta}(t,x) = (s)^{-(n+1)/\alpha} G_{\alpha,\delta}\left(s^{-1}t, s^{-1/\alpha}x \right), \quad (24)$$

or equivalently

$$\frac{\partial^n}{\partial x^n} G_{\alpha,\delta}(t,x) = (t)^{-(n+1)/\alpha} G_{\alpha,\delta}\left(1, (t)^{-1/\alpha}x \right). \quad (25)$$

(3) $G_\alpha(s,\cdot) * G_{\alpha,\delta}(t,\cdot) = G_{\alpha,\delta}(s+t,\cdot)$ *for any $s,t \in (0,\infty)$.*

(4) *For $n \ge 1$, there exist some constants C and $C_n > 0$ such that, for all $x \in \mathbb{R}$,*

$$|G_{\alpha,\delta}(1,x)| \le C \frac{1}{1+|x|^{1+\alpha}},$$
$$\left| \frac{\partial^n}{\partial x^n} G_{\alpha,\delta}(1,x) \right| \le C_n \frac{|x|^{\alpha+n-1}}{\left(1+|x|^{\alpha+n}\right)^2}. \quad (26)$$

(5) $\int_0^T \int_\mathbb{R} |G_{\alpha,\delta}(t,x)|^\lambda dt\, dx < \infty$ *if and only if $1/\alpha < \lambda < \alpha$.*

For $d \ge 1$ and any multi-index $\alpha = (\alpha_1,\ldots,\alpha_d)$ and $\delta = (\delta_1,\ldots,\delta_d)$, let $\mathbb{G}_{\alpha,\delta}(t,x)$ be the Green function of the deterministic equation

$$\frac{\partial u}{\partial t}(t,x) = \mathbb{D}_\delta^\alpha(t,x),$$
$$u(0,x) = \delta_0(x), \quad t > 0, \ x \in \mathbb{R}^d. \quad (27)$$

Clearly

$$\mathbb{G}_{\alpha,\delta}(t,x)$$
$$= \prod_{i=1}^d G_{\alpha_i,\delta_i}(t,x_i)$$
$$= \frac{1}{(2\pi)^d}$$
$$\times \int_{\mathbb{R}^d} \exp\left(-i\langle \xi,x \rangle - t\sum_{i=1}^d |\xi_i|^{\alpha_i} \exp\left(-i\delta\frac{\pi}{2} \operatorname{sgn}(\xi_i) \right) \right) d\xi. \quad (28)$$

3. Existence and Uniqueness

In this section, we are going to prove the existence and uniqueness of the global mild solution to (3). Recall (4) and (14). Then, for all $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$
\begin{aligned}
&u(t, x) \\
&= \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t, x - y) u_0(y) \, dy \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t - s, x - y) B^H(dy, ds) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t - s, x - y) b(s, y, u(s, y)) \, dy \, ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t - s, x - y) \sigma(s, y, u(s, y)) \psi(s, y) \, dy \, ds \\
&\quad + \int_0^{t+} \int_{\mathbb{R}^d} \int_{E_2} \mathbb{G}_{\alpha,\delta}(t - s, x - y) \\
&\qquad\qquad \times \sigma(s, y, u(s, y)) h(s, y, z) M(dz, dy, ds),
\end{aligned} \tag{29}
$$

with the mappings

$$
\psi(t, y) = \int_{E_2 \setminus U_0} h_2(t, y, z) \mu_2(dz), \tag{30}
$$

$$
h(t, y, z) = h_1(t, y, z) 1_{U_0}(z) + h_2(t, y, z) 1_{E_2 \setminus U_0}(z),
$$

with indicator $1_A(\cdot)$ of the set $A \in \mathscr{E}_2$.

In the following, we will show that such a mild solution indeed exists and is unique, which is stated as follows.

Theorem 6. *Let* $1 + \sum_{i=1}^d (1/\alpha_i - 1/h\alpha_i) > 0$. *Suppose the following conditions hold.*

(1) b, σ *are uniformly Lipschitzian; that is, there exists a constant* $C > 0$ *such that for* $(t, x) \in [0, T] \times \mathbb{R}^d$ *and* $u, v \in \mathbb{R}$

$$
|b(t, x, u) - b(t, x, v)| + |\sigma(t, x, u) - \sigma(t, x, v)| \le C |u - v|. \tag{31}
$$

(2) b *is linear growth; that is, there exists a constant* $C > 0$ *such that for* $u \in \mathbb{R}$

$$
|b(t, x, u)| \le C(1 + |u|). \tag{32}
$$

(3) *For* $p \in (2(1 + \sum_{i=1}^d (1/\alpha_i))/(\sum_{i=1}^d (1/\alpha_i) - 1), +\infty)$ *with* $\alpha > 1$,

$$
\sup_{t \in [0, T]} \|\psi(t, \cdot)\|_p^p < \infty, \tag{33}
$$

$$
\sup_{t \in [0, T]} \left\| \int_{E_2} |h(t, \cdot, z)|^2 \mu_2(dz) \right\|_{p/2}^{p/2} < \infty. \tag{34}
$$

Then, for all $\mathscr{F}_0$-*measurable* $u_0 : \mathbb{R}^d \times \Omega \to \mathbb{R}$ *satisfying* $\mathbb{E}[\|u_0(\cdot)\|_p^p] < \infty$, *there exists a unique mild solution* $(u(t, x))_{(t,x) \in [0,T] \times \mathbb{R}^d}$ *to* (3) *and, for all* $p \in (2(1 + \sum_{i=1}^d (1/\alpha_i))/(\sum_{i=1}^d (1/\alpha_i) - 1), +\infty)$,

$$
\sup_{t \in [0, T]} \mathbb{E}\left[\|u(t, \cdot)\|_p^p \right] < \infty. \tag{35}
$$

In order to prove the above theorem, we need the following lemmas.

Lemma 7. *Suppose* $p \in [1, \infty)$, $\rho \in [1, p]$, *and* $r \in [1, \infty)$ *such that*

$$
\frac{1}{r} = \frac{1}{p} - \frac{1}{\rho} + 1 \in [0, 1]. \tag{36}
$$

Let $\mathbb{G}_{\alpha,\delta} = \mathbb{G}_{\alpha,\delta}(t, x - y)$ *be the Green kernel,* $H = \mathbb{G}_{\alpha,\delta}$, *or* $\mathbb{G}_{\alpha,\delta}^2$ *with* $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. *Define an operator* $\mathscr{A}$ *by*

$$
\mathscr{A}(v)(t, x) = \int_0^t \int_{\mathbb{R}^d} H(t - s, x - y) v(s, y) \, dy \, ds, \tag{37}
$$

with $v \in L^1([0, T]; L^p())$. *Then* $\mathscr{A} : L^1([0, T]; L^p()) \to L^\infty([0, T]; L^p())$ *is bounded linear operator and satisfies the following.*

(a) *If* $H = \mathbb{G}_{\alpha,\delta}$, *then there exists a constant* $C > 0$ *such that for all* $r \in [1, 1 + \min \alpha_i)$

$$
\begin{aligned}
&\|\mathscr{A}(v)(t, \cdot)\|_p \\
&\le C \int_0^t (t - s)^{\sum_{i=1}^d ((r-1)/\alpha_i)} \|v(x, \cdot)\|_\rho \, ds, \quad \forall t \in [0, T].
\end{aligned} \tag{38}
$$

(b) *If* $\mathbb{G}_{\alpha,\delta}^2$, *then there exists a constant* $C > 0$ *such that for all* $r \in [1, (1 + \min \alpha_i)/2)$

$$
\begin{aligned}
&\|\mathscr{A}(v)(t, \cdot)\|_p \\
&\le C \int_0^t (t - s)^{\sum_{i=1}^d ((r-2)/\alpha_i)} \|v(x, \cdot)\|_\rho \, ds, \quad \forall t \in [0, T].
\end{aligned} \tag{39}
$$

Proof. We only need to prove the case (a), since the proof of (b) is similar. Together with Minkowski's inequality, (3) of Lemma 5, and Young inequality, one can get

$$
\begin{aligned}
&\|\mathscr{A}(v)(t, \cdot)\|_p \\
&= \left\| \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t - s, \cdot - y) v(s, y) \, dy \, ds \right\|_p
\end{aligned}
$$

$$\leq \int_0^t \left\| \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}\left(t-s, \cdot - y\right) v(s,y) dy \right\|_p ds$$

$$\leq C \int_0^t (t-s)^{-\sum_{i=1}^d (1/\alpha_i)}$$

$$\times \left\| \int_{\mathbb{R}^d} \prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1, (t-s)^{-1/\alpha_i}(\cdot - y_i)\right) |v(s,y)| \, dy \right\|_p ds$$

$$\leq C \int_0^t (t-s)^{-\sum_{i=1}^d (1/\alpha_i)}$$

$$\times \left\| \left(\prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1, (t-s)^{-1/\alpha_i}(\cdot)\right) \right) * |v(s,\cdot)| \right\|_p ds$$

$$\leq C \int_0^t (t-s)^{-\sum_{i=1}^d (1/\alpha_i)}$$

$$\times \left\| \prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1, (t-s)^{-1/\alpha_i}(\cdot)\right) \right\|_r \cdot \|v(s,\cdot)\|_\rho ds$$

$$\leq C \int_0^t (t-s)^{-\sum_{i=1}^d (1/\alpha_i)} \|v(s,\cdot)\|_\rho ds, \tag{40}$$

where we have used the fact that for $r \in [1, 1 + \min \alpha_i)$

$$\left\| \prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1, (t-s)^{-1/\alpha_i}(\cdot)\right) \right\|_r$$

$$= \left[\prod_{i=1}^d \left(\int_{\mathbb{R}} \left| G_{\alpha_i,\delta_i}\left(1, (t-s)^{-1/\alpha_i} y_i\right) \right|^r dy_i \right) \right]^{1/r} \tag{41}$$

$$= (t-s)^{\sum_{i=1}^d (r/\alpha_i)} \left[\prod_{i=1}^d \left(\int_{\mathbb{R}} \left| G_{\alpha_i,\delta_i}\left(1, z_i\right) \right|^r dz_i \right) \right]^{1/r}$$

$$\leq C(t-s)^{\sum_{i=1}^d (r/\alpha_i)}.$$

The proof of this lemma is completed. □

In the following, we mainly adopt the fixed point principle to prove Theorem 6. Let $\mathbb{B}$ be the space of all $L^p(\mathbb{R})$-valued $\mathscr{F}_t$-adapted processes $u(t,\cdot)_{0 \leq t \leq T} : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ with the norm

$$\|a\|_{\mathbb{B}} := \left[\sup_{0 \leq t \leq T} e^{-\eta t} \mathbb{E}\left[\|u(t,\cdot)\|_p^p \right] \right]^{1/p}, \quad \eta > 0. \tag{42}$$

Then $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a Banach space. Now, for $u \in \mathbb{B}$, let us define an operator $\mathscr{T}$ by

$$\mathscr{T}(t,x) = \sum_{i=1}^5 \mathscr{T}_i(u)(t,x), \tag{43}$$

where

$$\mathscr{T}_1(u)(t,x) = \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t, x-y) u_0(y) \, dy,$$

$$\mathscr{T}_2(u)(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t-s, x-y) b(s,y,u(s,y)) \, dy \, ds,$$

$$\mathscr{T}_3(u)(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t-s, x-y) B^H(dy, ds),$$

$$\mathscr{T}_4(u)(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t-s, x-y) \sigma(s,y,u(s,y))$$
$$\times \psi(s,y) \, dy \, ds,$$

$$\mathscr{T}_5(u)(t,x) = \int_0^t \int_{\mathbb{R}^d} \int_{E_2} \mathbb{G}_{\alpha,\delta}(t-s, x-y) \sigma(s,y,u(s,y))$$
$$\times h(s,y,z) M(dz, dy, ds). \tag{44}$$

According to (43), we have the following.

Proposition 8. *Suppose* $1 + \sum_{i=1}^d (1/\alpha_i - 1/h\alpha_i) > 0$ *and the assumptions (1), (2), and (3) of Theorem 6 are satisfied. Then, for each* $p > 2(1 + \sum_{i=1}^d (1/\alpha_i)) / \sum_{i=1}^d (1/\alpha_i) - 1$ *and* $u \in \mathbb{B}$*, it holds that* $\mathscr{T} \in \mathbb{B}$*.*

Proof. From Lemma 3 and Young inequality, it follows that

$$\left\| \mathscr{T}_1(u)(t,x) \right\|_p$$

$$= \left\| \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t, x-y) u_0(y) dy \right\|_p$$

$$= t^{-\sum_{i=1}^d (1/\alpha_i)} \left\| \int_{\mathbb{R}^d} \prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1, t^{-1/\alpha_i}(\cdot - y_i)\right) u_0(y) dy \right\|_p$$

$$\leq t^{-\sum_{i=1}^d (1/\alpha_i)} \left\| \left[\prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1, t^{-1/\alpha_i}\cdot\right) \right] * u_0(\cdot)(\cdot) \right\|_p$$

$$\leq t^{-\sum_{i=1}^d (1/\alpha_i)} \left\| \prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1, t^{-1/\alpha_i}\cdot\right) \right\|_1 \cdot \|u_0(\cdot)\|_p$$

$$\leq C \|u_0(\cdot)\|_p < \infty, \tag{45}$$

where we have used the fact that $\mathbb{E}[\|u_0(\cdot)\|_p^p] < \infty$. Now we deal with $\mathscr{T}_2(u)$. Applying (a) of Lemma 7 with $1/r = 1/p - 1/p + 1 = 1$ and condition (1) of Theorem 6, we conclude that

$$\mathbb{E}\left[\left\| \mathscr{T}_2(u)(t,x) \right\|_p^p \right]$$

$$\leq C\mathbb{E}\left[\int_0^t (t-s)^{\sum_{i=1}^d ((r-1)/\alpha_i)} \|b(s,\cdot,u(s,\cdot))\|_p ds \right]^p$$

$$\le C\mathbb{E}\left[\int_0^t \left(1 + \|u(s,\cdot)\|_p\right) ds\right]^p$$

$$\le C_{p,T}\left[1 + \sup_{0\le t\le T} \mathbb{E}\|u(s,\cdot)\|_p^p\right]$$

$$\le C_{p,T}\left[1 + \|u(s,\cdot)\|_{\mathbb{B}}^p\right] < \infty,\tag{46}$$

since $u \in \mathbb{B}$.

Now, let us consider $\mathcal{T}_3(u)(t,x)$. By Lemma 3, we deduce that

$$\mathbb{E}\|\mathcal{T}_3(u)(t,x)\|_p^p$$

$$= \int_{\mathbb{R}^d} \mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} \mathbb{G}_{\alpha,\delta}(t-s,x-y)\,B^H(dy,ds)\right|^p dx$$

$$= \int_{\mathbb{R}^d} \mathbb{E}\left|\int_0^t \int_{\mathbb{R}^d} \left(K_H^*\mathbb{G}_{\alpha,\delta}(t-\cdot,x-\cdot)\right)(s,y)\,W(dy,ds)\right|^p dx$$

$$\le C_p \int_{\mathbb{R}^d} \|\mathbb{G}_{\alpha,\delta}(t-\cdot,x-\cdot)\|_{L^{1/h}([0,T]\times\mathbb{R}^d)}^p dx,\tag{47}$$

where $W(s,y)$ is a space-time white noise on $[0,T]\times\mathbb{R}^d$.

Note that

$$\|\mathbb{G}_{\alpha,\delta}(t-\cdot,x-*)\|_{L^{1/h}([0,T]\times\mathbb{R}^d)}^p$$

$$= \left[\int_0^t \int_{\mathbb{R}^d} |\mathbb{G}_{\alpha,\delta}(t-s,x-y)|^{1/h}\,dy\,ds\right]^{ph}$$

$$= \left[\int_0^t (t-s)^{-\sum_{i=1}^d(1/h\alpha_i)}\right.$$

$$\left.\times \int_{\mathbb{R}^d}\left|\prod_{i=1}^d G_{\alpha_i,\delta_i}\left(1,(t-s)^{-1/\alpha_i}(x_i-y_i)\right)\right|^{1/h} dy\,ds\right]^{ph}$$

$$= \left[\int_0^t (t-s)^{\sum_{i=1}^d(1/\alpha_i-1/h\alpha_i)}\right.$$

$$\left.\times \prod_{i=1}^d\left(\int_{\mathbb{R}} |G_{\alpha_i,\delta_i}(1,z_i)|^{1/h}\,dz_i\right) ds\right]^{ph}$$

$$\le C_{\alpha,H}\left[\int_0^t (t-s)^{\sum_{i=1}^d(1/\alpha_i-1/h\alpha_i)}\,ds\right]^{ph} < \infty,\tag{48}$$

under the assumption $1 + \sum_{i=1}^d(1/\alpha_i - 1/h\alpha_i) > 0$. So we have $\mathcal{T}_3(u)(t,x) \in \mathbb{B}$ for $p \ge 2$.

Combining (a) of Lemma 7 with $1/r = 1/p - 2/p + 1 = 1 - 1/p \in (0,1]$ and assumption (33), we obtain that, for $p \in [2,\infty)$,

$$\mathbb{E}\|\mathcal{T}_4(u)(t,x)\|_p^p$$

$$\le C\mathbb{E}\left[\int_0^t (t-s)^{-(\sum_{i=1}^d(1/\alpha_i))(1-1/r)}\|1 + |u(s,\cdot)|\|_p \cdot \|\psi\|_p ds\right]^p$$

$$\le C_p\left[1 + \|u(\cdot)\|_{\mathbb{B}}^p\right]\sup_{0\le t\le T}\|\psi(t,\cdot)\|_p^p$$

$$\cdot\left[\int_0^t (t-s)^{-(\sum_{i=1}^d(1/\alpha_i))(1-1/r)(p/(p-1))} ds\right]^{p-1}$$

$$\le C_{p,T}\sup_{0\le t\le T}\|\psi(t,\cdot)\|_p^p\left(1 + \|u(\cdot)\|_{\mathbb{B}}^p\right).\tag{49}$$

with $1 - (1/(\sum_{i=1}^d(1/\alpha_i))(p-1)) > 0$; that is, $p > 1 + (1/\sum_{i=1}^d(1/\alpha_i))$. Finally, let us estimate $\mathcal{T}_5(u)(t,x)$. This is a key step in the proof of Proposition 8. Together with the condition (34), (b) of Lemma 7 with $1/r = 1/p - 4/p + 1 = 1 - 2/p \in (0,1]$ and Lemma 4, we conclude that, for $p > 2(1 + \sum_{i=1}^d(1/\alpha_i))/(\sum_{i=1}^d(1/\alpha_i) - 1)$,

$$\mathbb{E}\|\mathcal{T}_5(u)(t,\cdot)\|_p^p$$

$$= \mathbb{E}\left[\left\|\int_0^{t+}\int_{\mathbb{R}^d}\int_{E_2} \mathbb{G}_{\alpha,\delta}(t-s,\cdot-y)\,\sigma(s,y,u(s,y))\right.\right.$$

$$\left.\left.\times h(s,y,z)\,M(dy,dz,ds)\right\|_p^p\right]$$

$$\le C_p\int_{\mathbb{R}^d}\left[\int_0^t\int_{\mathbb{R}^d}\int_{E_2} |\mathbb{G}_{\alpha,\delta}(t-s,x-y)\,h(s,y,z)|^2\right.$$

$$\times\left(\mathbb{E}\left[1 + |u(s,y)|^p\right]\right)^{2/p}$$

$$\left.\times \mu_2(dz)\,dy\,ds\right]^{p/2} dx$$

$$= C_p\left\|\int_0^t\int_{\mathbb{R}^d} |\mathbb{G}_{\alpha,\delta}(t-s,x-y)|^2\right.$$

$$\times\left[\int_{E_2} |h(s,y,z)|^2\mu_2(dz)\right]$$

$$\left.\times[1 + \mathbb{E}|u(s,\cdot)|^p]^{2/p}\,dy\,ds\right\|^{p/2}$$

$$\leq C_p \left[\int_0^t (t-s)^{-(p+2)/p(\sum_{i=1}^d (1/\alpha_i))} \right.$$

$$\times \left\| \left[\int_{E_2} |h(s,y,z)|^2 \mu_2(dz) \right] \right.$$

$$\left. \times \left[1 + \mathbb{E}|u(s,\cdot)|^p \right]^{2/p} \right\|_{p/4} ds \right]^{p/2}$$

$$\leq C_p \left[\int_0^t (t-s)^{-(p+2)/p(\sum_{i=1}^d (1/\alpha_i))} \right.$$

$$\times \left\| \int_{E_2} |h(s,y,z)|^2 \mu_2(dz) \right\|_{p/2}$$

$$\left. \times \left\| (1 + \mathbb{E}|u(s,\cdot)|^p)^{2/p} \right\|_{p/2} ds \right]^{p/2}$$

$$\leq C_p \left[\sup_{0 \leq t \leq T} \left\| \int_{E_2} |h(s,y,z)|^2 \mu_2(dz) \right\|_{p/2}^{p/2} \right.$$

$$\cdot \sup_{0 \leq t \leq T} \left\| (1 + \mathbb{E}|u(s,\cdot)|^p)^{2/p} \right\|_{p/2}^{p/2}$$

$$\left. \cdot \left[\int_0^t (t-s)^{-(p+2)/(p-2)(\sum_{i=1}^d (1/\alpha_i))} \right]^{(p-2)/2} \right.$$

$$\leq C_{p,T} \left[\sup_{0 \leq t \leq T} \left\| \int_{E_2} |h(s,y,z)|^2 \mu_2(dz) \right\|_{p/2}^{p/2} \right.$$

$$\times \left(1 + \|u(\cdot)\|_{\mathbb{B}}^p \right) < \infty.$$

$$(50)$$

Thus we have proved that the operator $\mathcal{T}$ defined by (43) is an operator from $\mathbb{B}$ to itself. On the other hand, from the similar argument as in (45)–(50), let $\eta > 0$ sufficiently large; then $\mathcal{T} \in \mathbb{B}$. Thus we complete the proof of the proposition. $\square$

In what follows, we will prove that the operator $\mathcal{T} : \mathbb{B} \mapsto \mathbb{B}$ is a contract operator.

Proposition 9. *Suppose the assumptions (1), (2), and (3) of Theorem 6 are satisfied. Then, for each $p > 2(1 + \sum_{i=1}^d (1/\alpha_i))/(\sum_{i=1}^d (1/\alpha_i) - 1)$, the operator $\mathcal{T}_\alpha$ is a contraction on $\mathbb{B}$. In other words, there exists a constant $\varrho \in (0,1)$ such that*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{\mathbb{B}} \leq \varrho \|u - v\|_{\mathbb{B}}, \quad \text{for } u, v \in \mathbb{B}. \qquad (51)$$

Proof. Let u_0 and v_0 be initials of $(\mathcal{F}_t)_{t \geq 0}$-adapted random fields $u, v \in \mathbb{B}$ such that $u_0 = v_0$. Let us consider $\mathcal{T}_1(u)$

firstly. Note that, for $\rho = p/3$, together with (a) of Lemma 7 with $1/r = (1/p) - (1/\rho) + 1 = 1 - (2/p)$ and condition (1) of Theorem 6, one can get

$$\mathbb{E} \left[\left\| \mathcal{T}_2(u)(t,\cdot) - \mathcal{T}_2(v)(t,\cdot) \right\|_p^p \right]$$

$$\leq C\mathbb{E} \left[\int_0^t (t-s)^{\sum_{i=1}^d ((r-1)/\alpha_i)} \right.$$

$$\left. \times \left\| b(s,y,u(s,y)) - b(s,y,v(s,y)) \right\|_p ds \right]^p \qquad (52)$$

$$\leq C_p \left[\int_0^t (t-s)^{\sum_{i=1}^d ((r-1)/\alpha_i)} \mathbb{E}\|u(s,\cdot) - v(s,\cdot)\|_p ds \right]^p.$$

Then

$$\left\| \mathcal{T}_2(u)(t,\cdot) - \mathcal{T}_2(v)(t,\cdot) \right\|_{\mathbb{B}}^p$$

$$= \sup_{0 \leq t \leq T} e^{-\eta t} \mathbb{E} \left[\left\| \mathcal{T}_2(u)(t,\cdot) - \mathcal{T}_2(v)(t,\cdot) \right\|_p^p \right]$$

$$\leq C \sup_{0 \leq t \leq T} \mathbb{E} \left[\int_0^t e^{-(\eta/p)(t-s)}(t-s)^{\sum_{i=1}^d ((r-1)/\alpha_i)} e^{-(\eta/p)s} \right.$$

$$\left. \times \|u(s,\cdot) - v(s,\cdot)\|_p ds \right]^p$$

$$\leq C_p \sup_{0 \leq t \leq T} \mathbb{E} \left[\int_0^t e^{-\eta s} \|u(s,\cdot) - v(s,\cdot)\|_p^p ds \right] \qquad (53)$$

$$\cdot \left[\int_0^t \left(e^{-(\eta/p)(t-s)}(t-s)^{\sum_{i=1}^d ((r-1)/\alpha_i)} \right)^{p/(p-1)} ds \right]^{p-1}$$

$$\leq C_p \sup_{0 \leq t \leq T} \left[\int_0^t e^{-\eta s} \mathbb{E}\|u(s,\cdot) - v(s,\cdot)\|_p^p ds \right]$$

$$\cdot \left[\int_0^t \left(e^{-(\eta/p)(t-s)}(t-s)^{-(1/\alpha)(1-r)} \right)^{p/(p-1)} ds \right]^{p-1}$$

$$\leq \varrho \|u - v\|_{\mathbb{B}}^p,$$

with $\varrho \in (0,1)$ by choosing $\eta > 0$ large enough. Now we are going to consider the term $\mathcal{T}_5(u)(t,x)$. From a similar argument as in (50), thanks to the generalized B-D-G inequality, we derive from the conditions of Theorem 6 that

$$\left\| \mathcal{T}_5(u)(t,\cdot) - \mathcal{T}_5(v)(t,\cdot) \right\|_{\mathbb{B}}^p$$

$$= \sup_{0 \leq t \leq T} e^{-\eta t} \mathbb{E} \left[\left\| \mathcal{T}_5(u)(t,\cdot) - \mathcal{T}_5(v)(t,\cdot) \right\|_p^p \right]$$

$$
\begin{aligned}
&= \sup_{0 \le t \le T} e^{-\eta t} \int_{\mathbb{R}^d} \mathbb{E} \Bigg| \int_0^{t+} \int_{\mathbb{R}^d} \int_{E_2} \mathbb{G}_{\alpha,\delta}\left(t-s, x-y\right) h\left(s, y, z\right) \\
&\qquad\qquad\qquad \cdot \left[\sigma\left(s, y, u\left(s, y\right)\right)\right. \\
&\qquad\qquad\qquad\qquad \left. -\sigma\left(s, y, v\left(s, y\right)\right)\right] \\
&\qquad\qquad\qquad\qquad\qquad \times M\left(dy, dz, ds\right) \Bigg|^p dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C_p \sup_{0 \le t \le T} e^{-\eta t} \\
&\quad \times \int_{\mathbb{R}^d} \Bigg(\int_0^t \int_{\mathbb{R}^d} \int_{E_2} \left|\mathbb{G}_{\alpha,\delta}\left(t-s, x-y\right) h\left(s, y, z\right)\right|^2 \\
&\qquad\qquad\qquad \cdot \left(\mathbb{E}\left|u\left(s, y\right) - v\left(s, y\right)\right|^p\right)^{2/p} \\
&\qquad\qquad\qquad\qquad \times \mu_2\left(dz\right) dy\, ds \Bigg)^{p/2} dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C_p \sup_{0 \le t \le T} e^{-\eta t} \\
&\quad \times \int_{\mathbb{R}^d} \Bigg(\int_0^t \int_{\mathbb{R}^d} \int_{E_2} \left|\mathbb{G}_{\alpha,\delta}\left(t-s, x-y\right) h\left(s, y, z\right)\right|^2 e^{2\eta s/p} \\
&\qquad\qquad\qquad \cdot \left(e^{-\eta s}\mathbb{E}\left|u\left(s, y\right) - v\left(s, y\right)\right|^p\right)^{2/p} \\
&\qquad\qquad\qquad\qquad \times \mu_2\left(dz\right) dy\, ds \Bigg)^{p/2} dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C_p \sup_{0 \le t \le T} \\
&\quad \times \int_{\mathbb{R}^d} \Bigg(\int_0^t \int_{\mathbb{R}^d} \int_{E_2} e^{-\eta s}\mathbb{E}\left|u\left(s, y\right) - v\left(s, y\right)\right|^p \mu_2\left(dz\right) dy\, ds \Bigg) \\
&\quad \cdot \Bigg(\int_0^t \int_{\mathbb{R}^d} \int_{E_2} \left|\mathbb{G}_{\alpha,\delta}\left(t-s, x-y\right) h\left(s, y, z\right)\right|^{2p/(p-2)} \\
&\qquad\qquad\qquad \times e^{-(2\eta(t-s)/(p-2))} \mu_2\left(dz\right) dy\, ds \Bigg)^{(p-2)/2} dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C_{p,T} \mu_2\left(E_2\right) \|u - v\|_{\mathbb{B}}^p \\
&\quad \cdot \sup_{0 \le t \le T} \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \int_{E_2} \left|\mathbb{G}_{\alpha,\delta}\left(t-s, x-y\right) h\left(s, y, z\right)\right|^p \\
&\qquad\qquad\qquad\qquad \times e^{-\eta(t-s)} \mu_2\left(dz\right) dy\, ds\, dx \\
&\le \varrho \|u - v\|_{\mathbb{B}}^p.
\end{aligned}
$$

$$\text{(54)}$$

Then $\mathscr{T}_5(u)$ is a contraction on $\mathbb{B}$.

A similar procedure as the above arguments yields that $\mathscr{T}_4(u)$ is a contraction on $\mathbb{B}$ by letting $\eta > 0$ large enough. Therefore, it follows from (49) that $\mathscr{T}(\cdot)$ is a contraction on $\mathbb{B}$ if $\eta > 0$ large enough. Thus the proof of **Proposition 9** is complete.

$\square$

Based on Propositions 8 and 9 and fixed point principal on the set $\{u \in \mathbb{B} : u(0) = u_0\}$, we conclude that (3) admits a unique solution $u \in \mathbb{B}$. Thus the conclusion of Theorem 6 follows.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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