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Research Article

Oscillation for a Nonlinear Dynamic System with a Forced Term on Time Scales

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We consider a class of two-dimensional nonlinear dynamic system with a forced term on a time scale $\mathbb{T}$ and obtain sufficient conditions for all solutions of the system to be oscillatory. Our results not only unify the oscillation of two-dimensional differential systems and difference systems but also improve the oscillation results that have been established by Saker, 2005, since our results are not restricted to the case where $b(t) \neq 0$ for all $t \in \mathbb{T}$ and $g(u) = u$. Some examples are given to illustrate the results.

1. Introduction

Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$, which is unbounded above. This paper is concerned with the two-dimensional dynamic system

$$x^{\Delta}(t) = b(t)\, g(y(t)),$$
$$y^{\Delta}(t) = -a(t)\, f(x^{\sigma}(t)) + r(t), \quad (1)$$

on $\mathbb{T}$. We assume that $t_0 \in \mathbb{T}$ and it is convenient to let $t_0 > 0$ and define the time scale interval $t \in [t_0, \infty)_{\mathbb{T}}$. For system (1), we assume the following.

(H_1) $a(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $b(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}}, [0,\infty))$, and $\int_{t_0}^{\infty} b(\tau)\Delta\tau = \infty$.

(H_2) $f, g \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing functions with sign property $uf(u) > 0$ and $ug(u) > 0$, for all $u \neq 0$.

(H_3) $\int_{t_0}^{\infty} |r(s)|\Delta s < \infty$.

The problem of oscillation and nonoscillation of second-order dynamic equations on time scales has become an important research field due to its tremendous potential for various applications. We refer the reader to the recent papers [1–3] and the references therein. It is an interesting problem to extend oscillation criteria for second-order dynamic equations to the case of two-dimensional dynamic systems.

The system (1) includes two-dimensional linear and nonlinear differential and difference systems, which were investigated in the literature; see, for example, [4, 5] and the references therein. As a special case of (1), when $r(t) = 0$, system (1) can be reduced to

$$x^{\Delta}(t) = b(t)\, g(y(t)),$$
$$y^{\Delta}(t) = -a(t)\, f(x^{\sigma}(t)), \quad (2)$$

whose oscillation and nonoscillation results have been obtained by some authors; see, for example, [6–8] and the references therein. When $b(t) \neq 0$, for all $t \in \mathbb{T}$ and $g(u) = u$, system (1) can be reduced to a single dynamic equation

$$\left(\frac{1}{b(t)} x^{\Delta}(t)\right)^{\Delta} + a(t)\, f(x^{\sigma}(t)) = r(t), \quad (3)$$

whose oscillatory behavior has been investigated; see, for example, [9, 10] and the references cited therein.

However, to the best of our knowledge, there are few results dealing with the oscillation of the solutions of forced dynamic systems on time scales up to now. Motivated by [4, 5, 11], we will consider the oscillation property of system (1) and establish some oscillation criteria for system (1) in this paper. Our results not only unify the oscillation of two-dimensional differential systems and difference systems but also improve the oscillation results that had been established by Saker [9], since our results are not restricted to the case where $b(t) \neq 0$, for all $t \in \mathbb{T}$ and $g(u) = u$.

The remainder of this paper is organized as follows. Section 2 contains some basic definitions and the necessary results about time scales. In Section 3, we present some useful lemmas. In Section 4, we present and prove the main results. Examples are given to illustrate the applicability of the obtained results.

2. Preliminary

For completeness, we recall the following concepts and results concerning time scales that will be used in the sequel. More details can be found in [12–14].

The forward and backward jump operators are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{4}$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and, for any function $f(t) : \mathbb{T} \to \mathbb{R}$, the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$.

Let

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{otherwise.} \end{cases} \tag{5}$$

Lemma 1. *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and $f(t)f^{\sigma}(t) \neq 0$. Then, g/f is differentiable at t and*

$$\left(\frac{g}{f}\right)^{\Delta}(t) = \frac{f(t)\, g^{\Delta}(t) - f^{\Delta}(t)\, g^{\sigma}(t)}{f(t)\, f^{\sigma}(t)}. \tag{6}$$

Lemma 2. *If $f, g \in C_{\mathrm{rd}}$ and $a, b \in \mathbb{T}$, then*

$$\int_a^b f(t)\, g^{\Delta}(t)\, \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g^{\sigma}(t)\, \Delta t. \tag{7}$$

Lemma 3 (chain rule). *Assume that $g : \mathbb{T} \to \mathbb{R}$ is continuously differentiable and $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable; then $g \circ f : \mathbb{T} \to \mathbb{R}$ is differentiable and*

$$(g \circ f)^{\Delta}(t) = \int_0^1 g'\left(f(t) + h\mu(t) f^{\Delta}(t)\right) dh\, f^{\Delta}(t). \tag{8}$$

3. Some Basic Lemmas

A solution $(x(t), y(t))$ of (1) is said to be continuable if it exists on the entire interval $[t_0, \infty)_{\mathbb{T}}$. A continuable nontrivial solution is said to be oscillatory if $x(t)$, $y(t)$ are both oscillatory. A component $x(t)$ (or $y(t)$) of a solution $(x(t), y(t))$ is said to be oscillatory if and only if $x(t)$ (or $y(t)$) is neither eventually positive nor eventually negative. Notice that if $b(t) \geq 0$, the oscillation of y follows from that of x. Furthermore, we observe that the substitutions $u = -x$, $v = -y$ transform (1) into the system

$$\begin{gathered} u^{\Delta}(t) = b(t)\, g_1(v(t)), \\ v^{\Delta}(t) = -a(t)\, f_1(u^{\sigma}(t)) + r(t), \end{gathered} \tag{9}$$

where $f_1(u(t)) = -f(-u(t))$, $u \in \mathbb{R}$, and $g_1(v(t)) = -g(-v(t))$, $v \in \mathbb{R}$. The functions f_1 and g_1 are subject to the conditions imposed on f and g. Therefore, we restrict our discussion only to the case where $x(t)$ is positive. In order to prove our results, we need the following lemmas.

Lemma 4. *Suppose that* (H_1) *and* (H_2) *hold. If* $(x(t), y(t))$ *is a nonoscillatory solution of system* (1), *then the component* $x(t)$ *is also nonoscillatory.*

Proof. Assume that $(x(t), y(t))$ is a solution of (1) and $x(t)$ is oscillatory, but $y(t)$ is nonoscillatory. Without loss of generality, we let $y(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. In view of the first equation of system (1) and (H_1) and (H_2), we have $x^{\Delta}(t) \geq 0$ on $[t_0, \infty)_{\mathbb{T}}$. Thus, $x(t) > 0$ or $x(t) < 0$ for all large t on $[t_0, \infty)_{\mathbb{T}}$, which leads to a contradiction. □

Lemma 5. *Suppose that conditions* (H_1) *and* (H_2) *hold, and let* $(x(t), y(t))$ *denote a nonoscillatory solution of the system* (1) *on interval* $[\tau, \infty)_{\mathbb{T}}$, $\tau \geq t_0$, *with* $x(t) > 0$ *for all* $t \geq \tau$; *moreover, let* $\tau^* \geq \tau$. *If there exists a positive constant L such that*

$$G(t) \geq L, \quad t \geq \tau^*, \tag{10}$$

where the function $G(t) = G(x(t), y(t))$ is defined as

$$\begin{aligned} G(t) := {} & -\frac{y(\tau)}{f(x(\tau))} + \int_{\tau}^{t} \left[a(s) - \frac{r(s)}{f(x^{\sigma}(s))}\right] \Delta s \\ & + \int_{\tau}^{\tau^*} \frac{y(s)\, x^{\Delta}(s) \int_0^1 f'\left[x(s) + h\mu(s) x^{\Delta}(s)\right] dh}{f(x(s))\, f(x^{\sigma}(s))} \Delta s, \end{aligned} \tag{11}$$

then

$$y(t) \leq -Lf(x(\tau^*)), \quad t \geq \tau^* \in \mathbb{T}. \tag{12}$$

Proof. From the second equation of (1) and Lemma 3, we obtain

$$\begin{aligned}\int_{\tau}^{t} a(s)\,\Delta s &= -\int_{\tau}^{t} \frac{y^{\Delta}(s) - r(s)}{f(x^{\sigma}(s))}\Delta s \\ &= -\int_{\tau}^{t} \frac{y^{\Delta}(s)}{f(x^{\sigma}(s))}\Delta s + \int_{\tau}^{t} \frac{r(s)}{f(x^{\sigma}(s))}\Delta s \\ &= \frac{y(\tau)}{f(x(\tau))} - \frac{y(t)}{f(x(t))} + \int_{\tau}^{t} \frac{r(s)}{f(x^{\sigma}(s))}\Delta s \\ &\quad - \int_{\tau}^{t} \left(\left(y^{\sigma}(s)\, x^{\Delta}(s) \int_{0}^{1} f'\left[x(s) + h\mu(s)\, x^{\Delta}(s)\right] dh\right) \right. \\ &\quad \left. \times \left(f(x(s))\, f(x^{\sigma}(s))\right)^{-1}\right) \Delta s.\end{aligned} \tag{13}$$

By (10) and (11), we have

$$\begin{aligned}-\frac{y(t)}{f(x(t))} &= G(t) + \int_{\tau^*}^{t} \left(\left(y^{\sigma}(s)\, x^{\Delta}(s) \int_{0}^{1} f'\left[x(s) + h\mu(s)\, x^{\Delta}(s)\right] dh\right) \right. \\ &\quad \left. \times \left(f(x(s))\, f(x^{\sigma}(s))\right)^{-1}\right) \Delta s \\ &\geq L + \int_{\tau^*}^{t} \left(\left(y^{\sigma}(s)\, x^{\Delta}(s) \int_{0}^{1} f'\left[x(s) + h\mu(s)\, x^{\Delta}(s)\right] dh\right) \right. \\ &\quad \left. \times \left(f(x(s))\, f(x^{\sigma}(s))\right)^{-1}\right) \Delta s, \quad t \geq \tau^*.\end{aligned} \tag{14}$$

Since

$$y(s)\, x^{\Delta}(s) = b(s)\, y(s)\, g(y(s)) \geq 0, \tag{15}$$

it follows from (H_2) that $y(t) \leq 0$ and $x^{\Delta}(t) \leq 0$, for all $t \geq \tau^*$.

Putting

$$\begin{aligned}-\frac{v(t)}{f(x(t))} &= L + \int_{\tau^*}^{t} \left(\left(y^{\sigma}(s)\, x^{\Delta}(s) \int_{0}^{1} f'\left[x(s) + h\mu(s)\, x^{\Delta}(s)\right] dh\right) \right. \\ &\quad \left. \times \left(f(x(s))\, f(x^{\sigma}(s))\right)^{-1}\right) \Delta s,\end{aligned} \tag{16}$$

then

$$\begin{aligned}\left(-\frac{v(t)}{f(x(t))}\right)^{\Delta}(t) &= \frac{y^{\sigma}(t)\, x^{\Delta}(t) \int_{0}^{1} f'\left[x(t) + h\mu(t)\, x^{\Delta}(t)\right] dh}{f(x(t))\, f(x^{\sigma}(t))} \\ &\geq 0.\end{aligned} \tag{17}$$

In view of $f(x(t)) \geq 0$, we have

$$y(t) \leq v(t) < 0, \tag{18}$$

which implies that

$$\begin{aligned}\left(-\frac{v(t)}{f(x(t))}\right)^{\Delta}(t) \geq \frac{v^{\sigma}(t)\, x^{\Delta}(t) \int_{0}^{1} f'\left[x(t) + h\mu(t)\, x^{\Delta}(t)\right] dh}{f(x(t))\, f(x^{\sigma}(t))} \geq 0,\end{aligned} \tag{19}$$

since

$$-\frac{v(\tau^*)}{f(x(\tau^*))} = L = \frac{w(\tau^*)}{f(x(\tau^*))}, \tag{20}$$

where $w(t)$ satisfies

$$\begin{aligned}\frac{w(t)}{f(x(t))} &= L - \int_{\tau^*}^{t} \left(\left(w^{\sigma}(s)\, x^{\Delta}(s) \int_{0}^{1} f'\left[x(s) + h\mu(s)\, x^{\Delta}(s)\right] dh\right) \right. \\ &\quad \left. \times \left(f(x(s))\, f(x^{\sigma}(s))\right)^{-1}\right) \Delta s.\end{aligned} \tag{21}$$

Using nonlinear version of comparison theorem on time scales [13, Corollary 6.12], we have

$$v(t) \leq -w(t), \quad t \geq \tau^*. \tag{22}$$

Therefore,

$$y(t) \leq -w(t), \quad t \geq \tau^*. \tag{23}$$

By Lemmas 1 and 3, we obtain

$$\begin{aligned}\left(\frac{w(t)}{f(x(t))}\right)^{\Delta}(t) &= \frac{w^{\Delta}(t)\, f(x(t)) - f^{\Delta}(x(t))\, w^{\sigma}(t)}{f(x(t))\, f(x^{\sigma}(t))} = \frac{w^{\Delta}(t)}{f(x^{\sigma}(t))} \\ &\quad - \frac{w^{\sigma}(t)\, x^{\Delta}(t) \int_{0}^{1} f'\left[x(t) + h\mu(t)\, x^{\Delta}(t)\right] dh}{f(x(t))\, f(x^{\sigma}(t))} \\ &= -\frac{w^{\sigma}(t)\, x^{\Delta}(t) \int_{0}^{1} f'\left[x(t) + h\mu(t)\, x^{\Delta}(t)\right] dh}{f(x(t))\, f(x^{\sigma}(t))}.\end{aligned} \tag{24}$$

Then, we get $w^{\Delta} = 0$, $w(t) = w(\tau^*) = Lf(x(\tau^*))$. Hence,

$$y(t) \leq -Lf(x(\tau^*)), \quad t \geq \tau^*. \tag{25}$$

The proof is completed. □

4. Main Results

For simplicity, we list the conditions used in the main results as

$$f \in C^1(\mathbb{R}, \mathbb{R}), \tag{26}$$

$$\int_{t_0}^{\infty} a(s)\,\Delta s = \infty, \tag{27}$$

$$-\infty < \int_{t_0}^{\infty} a(s)\,\Delta s < \infty. \tag{28}$$

For every $v > 0$ and sufficiently small u,

$$g(u)\,g(v) \leq g(uv) \leq g(u)(-g(-v)). \tag{29}$$

Theorem 6. *Suppose that* (H_1)–(H_3), (26), *and* (27) *hold. Then, every solution* $(x(t), y(t))$ *of system* (1) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose that system (1) has a nonoscillatory solution $(x(t), y(t))$ on $[t_0, \infty)_{\mathbb{T}}$. By Lemma 4, we know that $x(t)$ is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $x(t) > 0$, for all $t \in [t_0, \infty)_{\mathbb{T}}$. In view of (H_2) and (26), there exist $\tau \geq t_0$ and $C_1 > 0$, such that $|f(x(t))| \geq C_1$, for $t \geq \tau$. By (H_3), we have

$$\left|\int_{\tau}^{t} \frac{r(s)}{f(x(s))}\Delta s\right| \leq \int_{\tau}^{t} \left|\frac{r(s)}{f(x(s))}\right| \Delta s \leq \frac{1}{C_1}\int_{\tau}^{t} |r(s)|\,\Delta s \leq C_2, \tag{30}$$

where C_2 is a finite positive constant. In view of (27) and (30), there exists a $\tau^* \geq \tau$ sufficiently large, such that (10) is satisfied for all $t \geq \tau^*$. Applying Lemma 5, we obtain

$$y(t) \leq -Lf(x(\tau^*)) < 0, \quad t \geq \tau^*. \tag{31}$$

Since $g(t)$ is nondecreasing, we have

$$x^{\Delta}(t) = b(t)\,g(y(t)) \leq b(t)\,g(-Lf(x(\tau^*))) < 0, \quad t \geq \tau^*. \tag{32}$$

Integrating the above inequality from τ to t, we get $x(t) \to -\infty$ as $t \to \infty$, which is a contradiction. The proof is complete. □

Example 7. Consider the system

$$\begin{aligned} x^{\Delta}(t) &= (2t+3)\,y(t), \\ y^{\Delta}(t) &= -\frac{1}{t+6}x^{\sigma}(t) + (-1)^t \frac{1}{t(t+3)}, \end{aligned} \tag{33}$$

where $\mathbb{T} = 3\mathbb{N} = \{3n \mid n \in \mathbb{N}\}$.

Let $b(t) = 2t+3$, $a(t) = 1/(t+6)$, $f(x) = g(x) = x$, and $r(t) = (-1)^t/t(t+3)$. Since

$$\begin{aligned} \int_{3}^{\infty} a(s)\,\Delta s &= \sum_{i=1}^{\infty} \frac{1}{3i+6} = \infty, \\ \int_{3}^{\infty} |r(s)|\,\Delta s &= \sum_{i=1}^{\infty} \frac{1}{3i(3i+3)} < \infty. \end{aligned} \tag{34}$$

The system is oscillatory by Theorem 6. In fact,

$$x(t) = \frac{(-1)^t}{t}, \qquad y(t) = \frac{(-1)^{t+1}}{t(t+3)} \tag{35}$$

is such an oscillatory solution.

Theorem 8. *Suppose that* (H_1)–(H_3), (26), (28), *and* (29) *hold. Suppose further that*

$$0 < \int_{\varepsilon}^{\infty} \frac{du}{g(f(u))}, \qquad \int_{-\varepsilon}^{-\infty} \frac{du}{g(f(u))} < \infty, \tag{36}$$

for every $\varepsilon > 0$. *Then, system* (1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$, *if*

$$\int_{t_0}^{\infty} b(t)\,g\left(\int_{t}^{\infty} a(s)\,\Delta s - l\int_{t}^{\infty} |r(s)|\,\Delta s\right)\Delta t = \infty, \tag{37}$$

for some $l > 0$.

Proof. Suppose that system (1) has a nonoscillatory solution $(x(t), y(t))$ on $[t_0, \infty)_{\mathbb{T}}$. By Lemma 4, we know that $x(t)$ is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. In view of (H_2) and (26), there exist $\tau \geq t_0$ and $C_1 > 0$ such that $f(x(t)) \geq C_1$, for $t \geq \tau$.

As seen in the proof of Lemma 5, we have

$$\begin{aligned} \frac{y(t)}{f(x(t))} &= \frac{y(\tau)}{f(x(\tau))} - \int_{\tau}^{t}\left(a(s) - \frac{r(s)}{f(x^{\sigma}(s))}\right)\Delta s \\ &\quad - \int_{\tau}^{t} \frac{y(s)\,x^{\Delta}(s)\int_0^1 f'\left[x(s) + h\mu(s)\,x^{\Delta}(s)\right]dh}{f(x(s))\,f(x^{\sigma}(s))}\Delta s. \end{aligned} \tag{38}$$

Note that

$$\int_{\tau}^{\infty} \frac{y(s)\,x^{\Delta}(s)\int_0^1 f'\left[x(s) + h\mu(s)\,x^{\Delta}(s)\right]dh}{f(x(s))\,f(x^{\sigma}(s))}\Delta s < \infty. \tag{39}$$

Otherwise, (11) is valid for some positive number $\tau^* \in \mathbb{T}$. Then, by Lemma 5, we have $y(t) \leq -Lf(x(\tau^*))$, for all $t \geq \tau^*$. Hence,

$$x^{\Delta}(t) = b(t)\,g[y(t)] \leq b(t)\,g[-Lf(x(\tau^*))] \tag{40}$$

holds, and its subsequent contradiction holds as before. It now follows

$$\begin{aligned}\frac{y(t)}{f(x(t))} &= \frac{y(\tau)}{f(x(\tau))} - \int_\tau^t \left(a(s) - \frac{r(s)}{f(x^\sigma(s))}\right)\Delta s \\ &\quad - \int_\tau^t \frac{y(s)x^\Delta(s)\int_0^1 f'\left[x(s)+h\mu(s)x^\Delta(s)\right]dh}{f(x(s))f(x^\sigma(s))}\Delta s \\ &= \beta + \int_t^\infty \left(a(s) - \frac{r(s)}{f(x^\sigma(s))}\right)\Delta s \\ &\quad + \int_t^\infty \frac{y(s)x^\Delta(s)\int_0^1 f'\left[x(s)+h\mu(s)x^\Delta(s)\right]dh}{f(x(s))f(x^\sigma(s))}\Delta s,\end{aligned} \tag{41}$$

where

$$\begin{aligned}\beta &= \frac{y(\tau)}{f(x(\tau))} - \int_\tau^\infty \left(a(s) - \frac{r(s)}{f(x^\sigma(s))}\right)\Delta s \\ &\quad - \int_\tau^\infty \frac{y(s)x^\Delta(s)\int_0^1 f'\left[x(s)+h\mu(s)x^\Delta(s)\right]dh}{f(x(s))f(x^\sigma(s))}\Delta s.\end{aligned} \tag{42}$$

We now show that $\beta \geq 0$. Indeed, if $\beta < 0$, then (28), (30), and (39), respectively, imply that

$$\left|\int_t^\infty a(s)\Delta s\right| \leq -\frac{\beta}{4}, \quad t \geq \tau^*, \tag{43}$$

$$\left|\int_t^\infty \frac{r(s)}{f(x^\sigma(s))}\Delta s\right| \leq -\frac{\beta}{4}, \quad t \geq \tau^*, \tag{44}$$

$$\int_{\tau^*}^\infty \frac{y(s)x^\Delta(s)\int_0^1 f'\left[x(s)+h\mu(s)x^\Delta(s)\right]dh}{f(x(s))f(x^\sigma(s))}\Delta s \leq -\frac{\beta}{4}. \tag{45}$$

By (43), (44), and (45), we have

$$\begin{aligned}G(t) &= -\frac{y(\tau)}{f(x(\tau))} + \int_\tau^t \left[a(s) - \frac{r(s)}{f(x^\sigma(s))}\right]\Delta s \\ &\quad + \int_\tau^{\tau^*} \frac{y(s)x^\Delta(s)\int_0^1 f'\left[x(s)+h\mu(s)x^\Delta(s)\right]dh}{f(x(s))f(x^\sigma(s))}\Delta s \\ &= -\beta - \int_t^\infty \left[a(s) - \frac{r(s)}{f(x^\sigma(s))}\right]\Delta s \\ &\quad - \int_{\tau^*}^\infty \frac{y(s)x^\Delta(s)\int_0^1 f'\left[x(s)+h\mu(s)x^\Delta(s)\right]dh}{f(x(s))f(x^\sigma(s))}\Delta s \\ &\geq -\beta + \frac{\beta}{4} + \frac{\beta}{4} + \frac{\beta}{4} = -\frac{3\beta}{4} > 0.\end{aligned} \tag{46}$$

Then, by Lemma 5, let $L = -(3\beta/4)$; we have $y(t) \leq -Lf(x(\tau^*))$, for all $t \geq \tau^*$. Hence,

$$x^\Delta(t) = b(t)g(y(t)) \leq b(t)g(-Lf(x(\tau^*))) \tag{47}$$

holds, and its subsequent contradiction holds as before. In view of (41) and $\beta \geq 0$, we have

$$\begin{aligned}y(t) &\geq f(x(t))\int_t^\infty \left[a(s) - \frac{r(s)}{f(x^\sigma(s))}\right]\Delta s \\ &\geq f(x(t))\left[\int_t^\infty a(s)\Delta s - \int_t^\infty \left|\frac{r(s)}{f(x(s))}\right|\Delta s\right] \\ &\geq f(x(t))\left[\int_t^\infty a(s)\Delta s - \frac{1}{C_1}\int_t^\infty |r(s)|\Delta s\right] \\ &= f(x(t))\left[\int_t^\infty a(s)\Delta s - l\int_t^\infty |r(s)|\Delta s\right],\end{aligned} \tag{48}$$

for all large t, where $l = 1/C_1$. For the sake of convenience, let

$$A(t) = \int_t^\infty a(s)\Delta s - l\int_t^\infty |r(s)|\Delta s \tag{49}$$

for all large t; then $\lim_{t\to\infty} A(t) = 0$ and, in view of (29),

$$\begin{aligned}x^\Delta(t) &= b(t)g(y(t)) \geq b(t)g(f(x(t))A(t)) \\ &\geq b(t)g(f(x(t)))g(A(t)).\end{aligned} \tag{50}$$

Thus, by (36), we have

$$\int_{T_1}^t b(s)g(A(s))\Delta s \leq \int_{T_1}^t \frac{x^\Delta(s)}{g(f(x(s)))}\Delta s; \tag{51}$$

however,

$$\int_{T_1}^t \frac{x^\Delta(s)}{g(f(x(s)))}\Delta s \leq \int_{x(T_1)}^\infty \frac{du}{g(f(u))} < \infty, \tag{52}$$

which is contrary to (37). The proof is completed. □

Remark 9. Theorems 6 and 8 extend and improve some results of [2–5, 9].

Example 10. Consider the system

$$\begin{aligned}x^\Delta(t) &= y(t), \\ y^\Delta(t) &= -\frac{1}{t\sigma(t)}x^\sigma(t)\left[1 + (x^\sigma(t))^2\right] + \frac{1}{t\sigma(t)},\end{aligned} \tag{53}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

Here, $b(t) = 1$, $a(t) = 1/t\sigma(t)$, $f(u) = u(1+u^2)$, $g(u) = u$, and $r(t) = 1/t\sigma(t)$. It is easy to see that $f(u)$, $g(u)$ satisfy the conditions of Theorem 8, and

$$\int_{t_0}^{\infty} b(s)\Delta s = \infty,$$
$$\int_{t_0}^{\infty} a(s)\Delta s = \int_{t_0}^{\infty} r(s)\Delta s = \int_{t_0}^{\infty} \frac{1}{s\sigma(s)}\Delta s = \frac{1}{t_0} < \infty,$$
$$0 < \int_{\varepsilon}^{\infty} \frac{du}{g(f(u))} = \int_{-\varepsilon}^{-\infty} \frac{du}{g(f(u))} < \infty,$$
$$\int_{t_0}^{\infty} b(t) g\left(\int_t^{\infty} a(s)\Delta s - l\int_t^{\infty} |r(s)|\Delta s\right)\Delta t = \int_{t_0}^{\infty} (1-l)\left[\int_t^{\infty} \frac{1}{s\sigma(s)}\Delta s\right]\Delta t = \infty. \tag{54}$$

Hence, it follows from Theorem 8 that system (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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