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Research Article

Global and Blow-Up Solutions for Nonlinear Hyperbolic Equations with Initial-Boundary Conditions

Ülkü Dinlemez[1] and Esra Aktaş[2]

[1] *Department of Mathematics, Faculty of Science, Gazi University, Teknikokullar, Ankara, Turkey*
[2] *Incirli Mahallesi, Karaelmas Sokak, Yunusemre Caddesi 51/18, İncirli, Ankara, Turkey*

Correspondence should be addressed to Ülkü Dinlemez; ulku@gazi.edu.tr

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We consider an initial-boundary value problem to a nonlinear string equations with linear damping term. It is proved that under suitable conditions the solution is global in time and the solution with a negative initial energy blows up in finite time.

1. Introduction

We study the damped nonlinear string equation with source term $|u|^\alpha u$:

$$u_{tt} + u_t = \left(\sigma\left(\left|u_x\right|^2\right)u_x\right)_x + |u|^\alpha u, \quad (x,t) \in (0,1) \times [0,T], \tag{1}$$

where $1 < \alpha$, $\sigma(s)$ is a smooth function for $0 \leq s$ with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in [0,1], \tag{2}$$

and boundary conditions

$$\begin{gathered} u(1,t) = 0, \quad t \in (0,T), \\ \sigma\left(\left|u_x(0,t)\right|^2 u_x(0,t)\right) - u_t(0,t) = 2\phi(t), \quad t \in [0,T], \\ \sigma\left(\left|u_x(0,t)\right|^2 u_x(0,t)\right) + u_t(0,t) = 2\psi(t), \quad t \in [0,T]. \end{gathered} \tag{3}$$

The problem (1)–(3) can be regarded as modelling a nonlinear string with vertical displacement function $u(x,t)$ in $\mathbb{R}$. And this problem has nonlinear mechanical damping of the form $|u|^\alpha u$. The right end of the string makes it steady. The input $\phi(t)$ function and the output $\psi(t)$ function are applied on the left.

Wu and Li [1] studied the motion for a nonlinear beam model with nonlinear damping $a|\phi_t|^{m-1}\phi_t$ and external forcing $b|\phi|^{p-1}\phi$ terms. They showed that this model has a unique global solution and blow-up solution under the same conditions. Levine et al. [2] and Levine and Serrin [3] studied abstract version. Georgiev and Todorova [4] studied nonlinear wave equations involving the nonlinear damping term $|u_t|^{m-1}u_t$ and source term of type $|u_t|^{p-1}u_t$. They proved global existence theorem with large initial data for $1 < p \leq m$. Hao and Li [5] studied the global solutions for a nonlinear string with boundary input and output. Dinlemez [6] proved the global existence and uniqueness of weak solutions for the initial-boundary value problem for a nonlinear wave equation with strong structural damping and nonlinear source terms in $\mathbb{R}$. A lot of papers in connection with blow-up, global solutions and existence of weak solutions were studied in [7–15].

In this paper we first find energy equation for the problem (1)–(3). Then we prove the solutions of the problem (1)–(3) are global in time under some conditions on the function $\sigma(s)$, input $\phi(t)$, and the output $\psi(t)$. Finally we establish a blow-up result for solutions with a negative initial energy. Our approach is similar to the one in [5].

2. Main Results

Now we give the following lemma for energy equation for the problem (1)–(3).

Lemma 1. *Let* $1 < \alpha$ *and* $u(x,t)$ *be a solution of the problem* (1)–(3)*. Then the energy equation of the problem* (1)–(3) *is*

$$E(t) = \frac{1}{2}\|u_t\|_2^2 - \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} + \frac{1}{2}\int_0^1\int_0^{|u_x|^2}\sigma(\xi)\,d\xi\,dx, \tag{4}$$

$$\frac{d}{dt}E(t) = \phi^2(t) - \psi^2(t) - \|u_t\|_2^2. \tag{5}$$

Proof. Multiplying (1) with u_t and integrating over $(0,1)$, then we get

$$\begin{aligned}\frac{d}{dt}&\left\{\frac{1}{2}\|u_t\|_2^2 - \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}\right\}\\ &= \int_0^1\left(\sigma\left(|u_x|^2\right)u_x\right)_x u_t\,dx - \|u_t\|_2^2.\end{aligned} \tag{6}$$

Applying integration by parts in the right hand side of (6), we find

$$\begin{aligned}\int_0^1\left(\sigma\left(|u_x|^2\right)u_x\right)_x u_t\,dx &= -\sigma\left(|u_x(0,t)|^2\right)u_x(0,t)\,u_t(0,t)\\ &\quad - \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^{|u_x|^2}\sigma(\xi)\,d\xi\,dx.\end{aligned} \tag{7}$$

And using boundary conditions in equality (7), we obtain

$$\begin{aligned}\frac{d}{dt}&\left\{\frac{1}{2}\|u_t\|_2^2 - \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} + \frac{1}{2}\int_0^1\int_0^{|u_x|^2}\sigma(\xi)\,d\xi\,dx\right\}\\ &= \phi^2(t) - \psi^2(t) - \|u_t\|_2^2.\end{aligned} \tag{8}$$

Hence the proof is completed. □

Next we give the following theorem for global solutions in time.

Theorem 2. *Assume that* $u(x,t)$ *is a solution of the problem* (1)–(3) *with* $1 < \alpha$ *and*

(i) $\sigma(s)$ *satisfies the following condition:*

$$|s|^\alpha \le \sigma(s), \quad \text{for } s \in \mathbb{R}^+ \cup \{0\}, \tag{9}$$

(ii) *the input and the output functions satisfy*

$$\phi^2(t) \le \psi^2(t). \tag{10}$$

Then the solution $u(x,t)$ *is global in time.*

Proof. Let

$$\begin{aligned}G(t) &:= E(t) + \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}\\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\int_0^1\int_0^{|u_x|^2}\sigma(\xi)\,d\xi\,dx + \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}.\end{aligned} \tag{11}$$

Differentiating $G(t)$ with respect to t and using (5), we get

$$\frac{d}{dt}G(t) = \phi^2(t) - \psi^2(t) - \|u_t\|_2^2 + 2\int_0^1 |u|^\alpha u u_t\,dx. \tag{12}$$

Using the Cauchy-Schwarz inequality in the last term of (12), we obtain

$$\begin{aligned}2\int_0^1 |u|^\alpha u u_t\,dx &\le 2\int_0^1 |u|^{\alpha+1}|u_t|\,dx\\ &\le \|u\|_{2(\alpha+1)}^{2(\alpha+1)} + \|u_t\|_2^2,\end{aligned} \tag{13}$$

and it follows from (12), (13), and (10) that we have

$$\frac{d}{dt}G(t) \le \|u\|_{2(\alpha+1)}^{2(\alpha+1)} + \|u_t\|_2^2. \tag{14}$$

By assumption (9) and integrating over $(0,|u_x|^2)$ and $(0,1)$, respectively, we yield

$$\frac{1}{\alpha+1}\|u_x\|_{2(\alpha+1)}^{2(\alpha+1)} \le \int_0^1\int_0^{|u_x|^2}\sigma(\xi)\,d\xi\,dx. \tag{15}$$

Furthermore, we have

$$\begin{aligned}|u(x,t)|^{2(\alpha+1)} &= \left|\int_x^1 u_\xi(\xi,t)\,d\xi\right|^{2(\alpha+1)}\\ &\le \int_x^1 \left|u_\xi(\xi,t)\right|^{2(\alpha+1)}d\xi\\ &\le \int_0^1 |u_x(x,t)|^{2(\alpha+1)}dx = \|u_x(x,t)\|_{2(\alpha+1)}^{2(\alpha+1)},\end{aligned} \tag{16}$$

and then

$$\|u(x,t)\|_{2(\alpha+1)}^{2(\alpha+1)} \le \|u_x(x,t)\|_{2(\alpha+1)}^{2(\alpha+1)}. \tag{17}$$

Combining (11), (14), (15), and (17), we get

$$\frac{dG(t)}{dt} \le \frac{1}{\xi_1}G(t), \tag{18}$$

where $\xi_1 = \min\{1/2, 1/(\alpha+1)\}$. Using Gronwall's inequality, we have

$$G(t) \le G(0)\,e^{(1/\xi_1)t}. \tag{19}$$

Therefore together with the continuation principle and the definition of $G(t)$ we complete the proof of Theorem 2. □

Then we give the following theorem for the blow-up solutions of the problem (1)–(3).

Theorem 3. *Let* $u(x,t)$ *be a solution of the problem* (1)–(3) *with* $1 < \alpha$*. Assume that*

(i) *there exists* $1 < \varepsilon < (\alpha+2)/2$ *such that the function* $\sigma(s)$ *satisfies*

$$\sigma(s)\,s \le \frac{\varepsilon}{2}\int_0^s \sigma(\zeta)\,d\zeta \quad \text{for } s \in \mathbb{R}^+ \cup \{0\}, \tag{20}$$

(ii) *the initial values satisfy*

$$E(0) \leq 0, \quad 0 < \int_0^1 u_0(x) u_1(x)\, dx, \tag{21}$$

(iii) *the input and output functions satisfy*

$$\begin{aligned} &\phi^2(t) \leq \psi^2(t), \\ &(\psi(t) + \phi(t)) \left(\int_0^t (\psi(s) - \phi(s))\, ds + u_0(0) \right) \leq 0, \end{aligned} \tag{22}$$

(iv) $u(x,t)$ *satisfies* $1 \leq \|u\|$.

Then the solution $u(x,t)$ *blows up in finite time* $T_{\max}$, *and*

$$T_{\max} \leq \left(\frac{\alpha+4}{\alpha\eta} \right) N^{-\alpha/(\alpha+4)}(0), \tag{23}$$

where η *is some positive constant independent of the initial value* α *and* $N(t)$ *are given by* (25).

Proof. We define

$$M(t) := -E(t), \qquad \gamma := \frac{\alpha}{2(\alpha+2)}, \tag{24}$$

$$N(t) := M^{1-\gamma}(t) + \int_0^1 u(x,t) u_t(x,t)\, dx. \tag{25}$$

By virtue of (5), (21), (22), and (24), we get

$$\frac{dM(t)}{dt} = \|u_t\|_2^2 + \psi^2(t) - \phi^2(t) \geq 0, \tag{26}$$

$$0 \leq M(0) \leq M(t), \quad \text{for } 0 \leq t. \tag{27}$$

Taking a derivative of (25) and using (26), we have

$$\begin{aligned} \frac{dN(t)}{dt} &= (1-\gamma) M^{-\gamma}(t) M'(t) + \int_0^1 u_t^2 dx + \int_0^1 u u_{tt} dx \\ &= (1-\gamma) M^{-\gamma}(t) \left(\|u_t\|_2^2 + \psi^2(t) - \phi^2(t) \right) + \|u_t\|_2^2 \\ &\quad + \int_0^1 u u_{tt} dx. \end{aligned} \tag{28}$$

Multiplying (1) by u and integrating over the interval $[0,1]$ and then using boundary conditions (3), we obtain

$$\begin{aligned} \int_0^1 u u_{tt} dx &= \|u\|_{\alpha+2}^{\alpha+2} - \int_0^1 u u_t dx - (\psi(t) + \phi(t)) \\ &\quad \times \left(\int_0^t (\psi(s) - \phi(s))\, ds + u_0(0) \right) \\ &\quad - \int_0^1 \sigma(|u_x|^2) u_x^2 dx. \end{aligned} \tag{29}$$

From the definition of $M(t)$ we yield

$$\begin{aligned} 0 = \varepsilon M(t) + \frac{\varepsilon}{2}\|u_t\|^2 + \frac{\varepsilon}{2} \int_0^1 \int_0^{|u_x|^2} \sigma(\xi)\, d\xi\, dx \\ - \frac{\varepsilon}{\alpha+2} \|u\|_{\alpha+2}^{\alpha+2}. \end{aligned} \tag{30}$$

Combining (29) and (30) in (28), we get

$$\begin{aligned} \frac{dN(t)}{dt} &= (1-\gamma) M^{-\gamma}(t) \left(\|u_t\|_2^2 + \psi^2(t) - \phi^2(t) \right) + \|u_t\|_2^2 \\ &\quad + \|u\|_{\alpha+2}^{\alpha+2} - \int_0^1 u u_t dx - (\psi(t) + \phi(t)) \\ &\quad \times \left(\int_0^t (\psi(s) - \phi(s))\, ds + u_0(0) \right) \\ &\quad - \int_0^1 \sigma(|u_x|^2) u_x^2 dx + \varepsilon M(t) + \frac{\varepsilon}{2}\|u_t\|_2^2 \\ &\quad + \frac{\varepsilon}{2} \int_0^1 \int_0^{|u_x|^2} \sigma(\xi)\, d\xi\, dx \\ &\quad - \frac{\varepsilon}{\alpha+2} \|u\|_{\alpha+2}^{\alpha+2}. \end{aligned} \tag{31}$$

Using (22) in (31), we obtain

$$\begin{aligned} &\left(1 + \frac{\varepsilon}{2}\right) \|u_t\|_2^2 + \left(\frac{1}{2} - \frac{\varepsilon}{\alpha+2} \right) \|u\|_{\alpha+2}^{\alpha+2} + \frac{1}{2}\|u\|_{\alpha+2}^{\alpha+2} \\ &\quad + \int_0^1 \left(\frac{\varepsilon}{2} \int_0^{|u_x|^2} \sigma(\xi)\, d\xi - \sigma(|u_x|^2) u_x^2 \right) dx \\ &\quad + \varepsilon M(t) - \int_0^1 u u_t dx \leq \frac{dN(t)}{dt}. \end{aligned} \tag{32}$$

Thanks to Young's inequality,

$$AB \leq \frac{\delta^p}{p} A^p + \frac{\delta^{-q}}{q} B^q, \quad 0 \leq A, B\ \forall 0 < \delta,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{33}$$

for $\int_0^1 u u_t dx$ with $p = q = 2$ and $\gamma = 2$, and then we get

$$\int_0^1 u u_t dx \leq \int_0^1 |u u_t|\, dx \leq \|u_t\|^2 + \frac{1}{4}\|u\|^2. \tag{34}$$

From embedding for $L^p(0,1)$ and using (iv), we have $\|u\|_2^2 \leq \|u\|_{\alpha+2}^{\alpha+2}$ and putting (34) in (32) we have

$$\begin{aligned} &\left(1 + \frac{\varepsilon}{2}\right) \|u_t\|_2^2 + \left(\frac{1}{2} - \frac{\varepsilon}{\alpha+2} \right) \|u\|_{\alpha+2}^{\alpha+2} + \frac{1}{2}\|u\|_2^2 \\ &\quad + \int_0^1 \left(\frac{\varepsilon}{2} \int_0^{|u_x|^2} \sigma(\xi)\, d\xi - \sigma(|u_x|^2) u_x^2 \right) dx + \varepsilon M(t) \\ &\quad - \|u_t\|_2^2 - \frac{1}{4}\|u\|_2^2 \leq \frac{dN(t)}{dt}. \end{aligned} \tag{35}$$

From (20), we get

$$\varepsilon M(t) + \frac{\varepsilon}{2}\|u_t\|_2^2 + \left(\frac{1}{2} - \frac{\varepsilon}{\alpha+2}\right)\|u\|_{\alpha+2}^{\alpha+2} + \frac{1}{4}\|u\|_2^2 \le \frac{dN(t)}{dt}. \tag{36}$$

Choosing ε and $\kappa = \min\{\varepsilon/2, (1/2-\varepsilon/(\alpha+2)), 1/4\}$, we obtain

$$\kappa\left\{M(t) + \|u_t\|_2^2 + \|u\|_{\alpha+2}^{\alpha+2} + \|u\|_2^2\right\} \le \frac{dN(t)}{dt}. \tag{37}$$

Thanks to (21) and (27), we yield

$$0 < N(0) \le N(t), \quad \forall 0 < t. \tag{38}$$

Now we estimate $[N(t)]^{1/(1-\gamma)}$. From Holder's inequality,

$$\left|\int_0^1 uu_t dx\right| \le \|u\|_2\|u_t\|_2 \le \|u\|_{\alpha+2}\|u_t\|_2; \tag{39}$$

then using Young's inequality again we get

$$\left|\int_0^1 uu_t dx\right| \le \frac{\delta^{2(1-\gamma)}}{2(1-\gamma)}\|u_t\|_2^{2(1-\gamma)} + \frac{1-2\gamma}{2(1-\gamma)}\delta^{-2(1-\gamma)/(1-2\gamma)}\|u\|_{\alpha+2}^{2(1-\gamma)/(1-2\gamma)}, \tag{40}$$

where $0 < \delta$ and $1/p + 1/q = 1$ with $p = 2(1-\gamma)$. And so we have

$$\left|\int_0^1 uu_t dx\right|^{1/(1-\gamma)} \le 2^{1/(1-\gamma)}\left(\frac{\delta^2}{(2(1-\gamma))^{1/(1-\gamma)}}\|u_t\|_2^2 + \left(\frac{1-2\gamma}{2(1-\gamma)}\right)^{1/(1-\gamma)}\delta^{-2/(1-2\gamma)}\|u\|_{\alpha+2}^{2/(1-2\gamma)}\right). \tag{41}$$

Choosing $\beta = \max\{\delta^2/(1-\gamma)^{1/(1-\gamma)}, ((1-2\gamma)/(1-\gamma))^{1/(1-\gamma)}\delta^{-2/(1-2\gamma)}\}$, we obtain

$$\left|\int_0^1 uu_t dx\right|^{1/(1-\gamma)} \le \beta\left(\|u_t\|_2^2 + \|u\|_{\alpha+2}^{\alpha+2}\right). \tag{42}$$

Therefore we yield

$$\begin{aligned}(N(t))^{1/(1-\gamma)} &= \left(M^{1-\gamma}(t) + \int_0^1 u(x,t)u_t(x,t)\,dx\right)^{1/(1-\gamma)} \\ &\le 2^{1/(1-\gamma)}\left(M(t) + \left|\int_0^1 u(x,t)u_t(x,t)dx\right|^{1/(1-\gamma)}\right) \\ &\le C\left(M(t) + \|u_t\|_2^2 + \|u\|_{\alpha+2}^{\alpha+2} + \|u\|_2^2\right),\end{aligned} \tag{43}$$

where C depends on δ and α. From (37) and (43), we have

$$\eta(N(t))^{1/(1-\gamma)} \le \frac{dN(t)}{dt}, \tag{44}$$

where $\eta = \kappa/C$. Integrating (44) over $(0,t)$, then we get

$$\frac{1}{(N(0))^{-\alpha/(\alpha+4)} - (\alpha/(\alpha+4))\eta t} \le (N(t))^{\alpha/(\alpha+4)}. \tag{45}$$

Hence $N(t)$ blows up in finite time $T_{\max}$. $T_{\max}$ is given by the inequality as below:

$$T_{\max} \le \frac{\alpha+4}{\alpha\eta}(N(0))^{-\alpha/(\alpha+4)}. \tag{46}$$

Consequently the solution blows up in finite time. And the proof of Theorem 3 is now finished. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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