Research Article

Global Behavior of the Difference Equation $x_{n+1} = x_{n-1}g(x_n)$

Hongjian Xi,1,2 Taixiang Sun,1 Bin Qin,2 and Hui Wu1

1 College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China
2 College of Information and Statistics, Guangxi University of Finance and Economics, Nanning, Guangxi 530003, China

Correspondence should be addressed to Taixiang Sun; stx1963@163.com

Received 17 November 2013; Accepted 23 December 2013; Published 3 February 2014

1. Introduction

Recently there have been published quite a lot of works concerning global behavior of the difference equations [1–8]. These results are not only valuable in their own right, but they can provide insight into their differential counterparts.

In [9], Kulenović and Ladas considered the positive solutions for difference equation

$$x_{n+1} = x_{n-1} + Ax_n^{1+a}, \quad n = 0, 1, \ldots$$  \hspace{1cm} (1)

with $A > 0$. They gave some partial results on the convergence of this equation.

Kalikow et al. [10] studied the following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{1 + f(x_n)}, \quad n = 0, 1, 2, \ldots$$ \hspace{1cm} (E1)

where initial values $x_{-1}, x_0 \in [0, +\infty)$ and $f$ is in a certain class of increasing continuous functions. They showed that the set of initial conditions $(x_{-1}, x_0)$ of (E1) in the first quadrant that converge to any given boundary point of the first quadrant forms a unique strictly increasing continuous function.

Motivated by the above studies, in this paper, we consider the following difference equation:

$$x_{n+1} = x_{n-1}g(x_n), \quad n = 0, 1, \ldots$$ \hspace{1cm} (2)

where initial values $x_{-1}, x_0 \in [0, +\infty)$ and $g : [0, +\infty) \to (0, 1]$ is a strictly decreasing continuous surjective function.

Our main result is the following theorem.

**Theorem 1.**

(1) Every positive solution of (2) converges to

$$a, 0, a, 0, \ldots \quad \text{or} \quad 0, a, 0, a, \ldots$$ \hspace{1cm} (3)

for some $a \in [0, +\infty)$.

(2) Assume $a \in (0, +\infty)$. Then the set of initial conditions $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solutions of (2) converge to

$$a, 0, a, 0, \ldots \quad \text{or} \quad 0, a, 0, a, \ldots$$ \hspace{1cm} (4)

is a unique strictly increasing continuous function or an empty set.

2. The Main Result

**Proof of Theorem 1(1).** Let $\{x_n\}_{n=1}^{\infty}$ be a positive solution of (2). Then $x_{2n}$ and $x_{2n+1}$ are decreasing sequences since $g(x) \leq 1$. Let $\lim_{n \to \infty} x_{2n} = p$ and $\lim_{n \to \infty} x_{2n-1} = q$. Then we have

$$p = pg(q),$$ \hspace{1cm} (5)

which implies $p = 0$ or $g(q) = 1$. If $g(q) = 1$, then $q = 0$ since $g : [0, +\infty) \to (0, 1]$ is a strictly decreasing continuous surjective function with $g(0) = 1$. This completes proof of Theorem 1(1).
Write \( D = [0, +\infty) \times [0, +\infty) \) and define \( f : D \to D \) by
\[
 f (x, y) = (y, xg(y)),
\]
for all \((x, y) \in D\). It is easy to see that if \( x_n, h_{n-1} \) is a solution of (2), then \( f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \) for any \( n \geq 0 \). In the following, let
\[
 L_0 = [a] \times [0, +\infty), \quad L_1 = [0, +\infty) \times \{a\},
 R_0 = [a, +\infty) \times [0], \quad R_1 = [0] \times [a, +\infty)
\]
for some \( a \in (0, +\infty) \).

**Lemma 2.** The following statements are true:

(i) \( f \) is a homeomorphism;

(ii) \( f(L_1) = L_0 \);

(iii) \( f(R_0) = R_1 \) and \( f(R_1) = R_0 \).

**Proof.** (i) Since \( f(x_1, y_1) \neq f(x_2, y_2) \) for any \((x_1, y_1), (x_2, y_2) \) in \( D \) with \((x_1, y_1) \neq (x_2, y_2) \) and \( f^{-1}(u, v) = (v/g(u), u) \) is continuous for any \((u, v) \in D\), \( f \) is a homeomorphism.

(ii) Let \((x, y) \in L_1 \) and \((u, v) = f(x, y) = (y, xg(y))\). Then \( y = a, x \geq 0 \), and
\[
 u = y = a, \quad v = xg(y) = xg(a) \geq 0,
\]
which implies \( f(L_1) \subseteq L_0 \).

Let \((u, v) \in L_0 \). Then \( u = a \) and \( v \geq 0 \). Choose \((x, y) = (v/g(a), a) \in L_1 \). Then \( f(x, y) = (u, v) \). Thus \( f(L_1) = L_0 \).

The proof of (iii) is similar to that of (ii). This completes the proof of Lemma 2.

In order to show Theorem 1(2), we will construct two families of strictly increasing functions \( y = h_{2n}(x) \) and \( x = g_{2n+1}(y) \) \((n \geq 1)\) as follows. Set
\[
x = g_2(y) = \frac{a}{g(y)} \quad (y \geq 0).
\]
Then \( y = h_2(x) = g_2^{-1}(x) = g^{-1}(a/x) \) is a strictly increasing function which maps \([a, +\infty)\) onto \([0, +\infty)\). Set
\[
x = g_3(y) = \frac{h_2(y)}{g(y)} \quad (y \geq a).
\]
Then \( x = g_3(y) \) is a strictly increasing function which maps \([a, +\infty)\) onto \([0, +\infty)\).

Assume that, for some positive integer \( n \), we already define strictly increasing functions \( y = h_{2n}(x) \) and \( x = g_{2n+1}(y) \) such that both \( h_{2n} \) and \( g_{2n+1} \) map \([a, +\infty)\) onto \([0, +\infty)\). Set
\[
x = g_{2n+2}(y) = \frac{g_{2n+1}(y)}{g(y)} \quad (y \geq 0).
\]
Then both \( y = h_{2n+2}(x) = g_{2n+2}^{-1}(x) \) and \( x = g_{2n+3}(y) = h_{2n+2}(y)/g(y) \) are strictly increasing functions which map \([a, +\infty)\) onto \([0, +\infty)\). In such a way, we construct two families of strictly increasing functions \( y = h_{2n}(x) \) and \( x = g_{2n+1}(y) \) \((n \geq 1)\).

Set \( P_0 = [a] \times [0, +\infty) \) and \( Q_0 = [0, +\infty) \times \{a\} \). For any \( n \geq 1 \), write
\[
P_n = f^{-2}(P_{n-1}), \quad Q_n = f^{-2}(Q_{n-1}),
\]
\[
L_n = f^{-1}(L_{n-1}).
\]
Let \((x, y) \in L_2 \). Since \( f(L_2) = L_1 \) and \((u, v) = f(x, y) = (y, xg(y)) \) in \( L_1 \), it follows that
\[
xg(y) = v = a, \quad y = u \geq 0.
\]
Thus \( x = g_2(y) = a/g(y) \) and \( L_2 = \{(x, y) : y = h_2(x), x \geq a\} \).

Let \((x, y) \in L_3 \). Since \( f(L_3) = L_2 \) and \((u, v) = f(x, y) = (y, xg(y)) \) in \( L_2 \), it follows that
\[
xg(y) = v = h_2(u) = h_2(x), \quad y = u \geq a.
\]
Thus \( x = g_3(y) = h_2(y)/g(y) \) \((y \geq a)\) and \( L_3 = \{(x, y) : x = g_3(y), y \geq a\} \). Using induction, one can easily show that, for any \( n \geq 1 \),
\[
L_{2n} = \{(x, y) : y = h_{2n}(x), x \geq a\},
\]
\[
L_{2n+1} = \{(x, y) : x = g_{2n+1}(y), y \geq a\}.
\]
Since \( f \) is a homeomorphism and \( P_n = f^{-2}(P_{n-1}) \) with \( L_{2n} \cup R_0 \) is the boundary of \( P_n \), we have that, for any \( n \geq 1 \),
\[
P_n = \{(x, y) : 0 \leq y \leq h_{2n}(x), x \geq a\}.
\]
In a similar fashion, we may show that
\[
Q_n = \{(x, y) : 0 \leq x \leq g_{2n+1}(y), y \geq a\}.
\]
Since \( L_2 \subseteq P_0, L_3 \subseteq Q_0 \), and \( f \) is a homeomorphism, we have that \( P_1 \subseteq P_0 \) and \( Q_1 \subseteq Q_0 \), which implies that, for any \( n \geq 1 \),
\[
L_{2n} \subseteq P_{n-1}, \quad L_{2n+1} \subseteq Q_{n-1},
\]
\[
P_n \subseteq P_{n-1}, \quad Q_n \subseteq Q_{n-1}.
\]
It follows from (12) and (18) that, for \( x \geq a \),
\[
0 \leq \cdots \leq h_4(x) \leq h_2(x)
\]
and for \( y \geq a \),
\[
0 \leq \cdots \leq g_3(y) \leq g_5(y).
\]
Noting (19) and (20), we may assume that, for every \( x \geq a \),
\[
H(x) = \lim_{n \to \infty} h_{2n}(x)
\]
and for every \( y \geq a \),
\[
G(y) = \lim_{n \to \infty} g_{2n+1}(y).
\]
Set
\[
L = \{(x, y) : y = H(x), x \geq a\},
\]
\[
M = \{(x, y) : x = G(y), y \geq a\}.
\]
Lemma 3. The following statements are true:

(i) \( f(L) = M \) and \( f(M) = L \);

(ii) both \( y = H(x) \) and \( x = G(y) \) are increasing continuous functions which map \([a, +\infty)\) onto \([0, +\infty)\).

Proof. (i) Let \((x_0, y_0) \in L\). Then we have \( y_0 = \lim_{n \to \infty} h_{2n}(x_0) \), which follows that

\[
 f(x_0, y_0) = f \left( x_0, \lim_{n \to \infty} h_{2n}(x_0) \right) = \lim_{n \to \infty} f \left( x_0, h_{2n}(x_0) \right). \tag{24}
\]

Since \( f(L_{2n}) = L_{2n-1} \), we have

\[
 f \left( x_0, h_{2n}(x_0) \right) = (h_{2n}(x_0), x_0 g \left( h_{2n}(x_0) \right)) \tag{25}
\]

Let \( y_n = x_0 g \left( h_{2n}(x_0) \right) \). It follows from (24) and (25) that

\[
 f(x_0, y_0) = \lim_{n \to \infty} (g_{2n-1}(y_n), y_n) = (y_0, x_0 g \left( y_0 \right)), \tag{26}
\]

so we have

\[
 \lim_{n \to \infty} y_n = x_0 g \left( y_0 \right), \quad \lim_{n \to \infty} g_{2n-1}(y_n) = G \left( x_0 g \left( y_0 \right) \right). \tag{27}
\]

It follows from (25) and (27) that

\[
 f(x_0, y_0) = \left( G \left( x_0 g \left( y_0 \right) \right), x_0 g \left( y_0 \right) \right) \in M. \tag{28}
\]

Thus we have \( f(L) \subset M \).

Let \((x_0, y_0) \in M\). Then we have \( x_0 = \lim_{n \to \infty} g_{2n+1}(y_0) \), which follows that

\[
 f^{-1}(x_0, y_0) = f^{-1} \left( \lim_{n \to \infty} g_{2n+1}(y_0) \right), \tag{29}
\]

Since \( f^{-1}(L_{2n+1}) = L_{2n+2} \), we have

\[
 f^{-1} \left( g_{2n+1}(y_0) \right) = \left( \frac{y_0}{g \left( g_{2n+1}(y_0) \right)}, g_{2n+1}(y_0) \right) \tag{30}
\]

Let \( z_n = y_0 / g \left( g_{2n+1}(y_0) \right) \). It follows from (29) and (30) that

\[
 f^{-1}(x_0, y_0) = \lim_{n \to \infty} \left( z_n, h_{2n+2}(z_n) \right) = \left( \frac{y_0}{g \left( x_0 \right)}, x_0 \right), \tag{31}
\]

so we have

\[
 \lim_{n \to \infty} z_n = \frac{y_0}{g \left( x_0 \right)}, \quad \lim_{n \to \infty} h_{2n+2}(z_n) = H \left( \frac{y_0}{g \left( x_0 \right)} \right). \tag{32}
\]

It follows from (31) and (32) that

\[
 f^{-1}(x_0, y_0) = \left( \frac{y_0}{g \left( x_0 \right)}, H \left( \frac{y_0}{g \left( x_0 \right)} \right) \right) \in L. \tag{33}
\]

Thus we have \( f(L) = M \). In a similar fashion, we can show that \( f(M) = L \).

(ii) Since \( y = h_{2n}(x) \) \((n \geq 1)\) are strictly increasing functions, we have that \( y = H(x) \) is an increasing function. For any \( x_0 > a \), let

\[
 \lim_{x \to x^*_0} H(x) = y_0^+, \quad \lim_{x \to x^*_0} H(x) = y_0^-; \tag{34}
\]

then \( y_0^+ \geq H(x_0) \geq y_0^- \).

Now we claim that \( y_0^+ = y_0^- \). Indeed, if \( y_0^+ > y_0^- \), then it follows from (6) that

\[
 f^2 \left( x_1, y_1 \right) = (x_1 g \left( y_1 \right), y_1 g \left[ x_1 g \left( y_1 \right) \right]), \tag{35}
\]

So we have that

\[
 x_0 g \left( y_0^+ \right) < x_0 g \left( y_0^- \right), \quad y_0^+ g \left[ x_0 g \left( y_0^- \right) \right] > y_0^- g \left[ x_0 g \left( y_0^+ \right) \right]. \tag{36}
\]

It follows from (34) and (36) that there exist \((x_1, y_1), (x_2, y_2) \in L\) such that

\[
 f^2 \left( x_1, y_1 \right) = (x_1 g \left( y_1 \right), y_1 g \left[ x_1 g \left( y_1 \right) \right]), \tag{37}
\]

\[
 f^2 \left( x_2, y_2 \right) = (x_2 g \left( y_2 \right), y_2 g \left[ x_2 g \left( y_2 \right) \right]), \tag{38}
\]

It follows from Lemma 3(i) and (37) that

\[
 (x_1 g \left( y_1 \right), y_1 g \left[ x_1 g \left( y_1 \right) \right]), (x_2 g \left( y_2 \right), y_2 g \left[ x_2 g \left( y_2 \right) \right]) \in L, \tag{39}
\]

and this is a contradiction. The claim is proven.

In a similar fashion, we may show that \( \lim_{x \to a} H(x) = H(a) = 0 \). Thus \( y = H(x) \) \((x \geq a)\) is an increasing continuous function. In a similar fashion, we may show that \( x = G(y) \) \((y \geq a)\) is an increasing continuous function. Lemma 3 is proven. \(\square\)
where $a \in (0, +\infty)$ and $a \leq b$. It follows from Lemma 2(ii) and Lemma 3(ii) that

\[
M^1 = \{(x, y) : x = G(y) = 0, y \in [a, b]\}, \\
M^2 = \{(x, y) : x = G(y) > 0, y \in (b, +\infty)\},
\]

\[
f(M^1) = L^1, \\
f(M^2) = L^2.
\]  

(41)

Proof of Theorem 1(2). Noting (40), we consider the following two cases.

Case 1 ($a = b$). It follows from (40) that

\[
L = L^2 \cup \{(a, 0)\}.
\]  

(42)

Let $(x_{-1}, x_0) \in L^2$ and $\{x_{n, i}\}_{i \geq 1}$ be a solution of (2) with initial value $(x_{-1}, x_0)$; it follows from Lemma 3(i) that

\[
(x_{2n-1}, x_{2n}) = f^n(x_{-1}, x_0) \in L,
\]  

(43)

which implies that $\lim_{n \to \infty} x_{2n-1}, x_{2n} \in L$. It follows from (42) and Theorem I(1) that

\[
\lim_{n \to \infty} (x_{2n-1}, x_{2n}) = (a, 0).
\]  

(44)

Next we claim that $y = H(x)$ is a strictly increasing function. Indeed, if there exists $(x_{-1}, x_0), (y_{-1}, y_0) \in L$ such that $y_{-1} > x_{-1}$ and $x_0 = y_0$, then there exists $r \in (1, +\infty)$ such that $y_{-1} = r x_{-1}$. Set

\[
f^n(x_{-1}, x_0) = (x_{n-1}, x_n), \\
f^n(y_{-1}, y_0) = (y_{n-1}, y_n),
\]

\[
n = 1, 2, \ldots.
\]  

(45)

Then we have

\[
y_1 = y_{-1} g(\gamma_0) \geq r x_{-1} g(y_0) = r x_1, \\
y_2 = y_0 g(y_1) \leq x_0 g(x_1) = x_2.
\]  

(46)

Using induction, one can show that, for any $n \geq 0$,

\[
y_{2n-1} \geq r x_{2n-1}, \\
y_{2n} \leq x_{2n}.
\]  

(47)

It follows from (44) and (47) that

\[
(a, 0) = \lim_{n \to \infty} (y_{2n-1}, y_{2n}) \neq \lim_{n \to \infty} (x_{2n-1}, x_{2n}) = (a, 0).
\]  

(48)

from which it follows that

\[
f^{2n}(x_{-1}, x_0) = (x_{2n-1}, x_{2n}) \in P_0 - P_1.
\]  

(50)

Then we have $x_{2n-1} < a$, which implies $\lim_{n \to \infty} x_{2n-1}, x_{2n} \notin (a, 0).$

If $x_{-1} \geq a$ and $x_0 < H(x_{-1})$, then let $y_{-1} = x_{-1}$ and $y_0 = H(x_{-1})$, and there exists $r \in (1, +\infty)$ such that $y_0 = r x_0$. We can show that, for any $n \geq 1$,

\[
y_{2n} \geq r x_{2n}, \\
x_{2n+1} = x_{2n-1} g(y_{2n}) \geq \cdots \geq x_{1} y_1,
\]  

(51)

which implies

\[
\lim_{n \to \infty} (x_{2n}, x_{2n}) \neq \lim_{n \to \infty} (y_{2n-1}, y_{2n}) = (a, 0).
\]  

(52)

From all abovementioned, the set of initial conditions $(x_{-1}, x_0)$ such that the positive solutions of (2) converge to

\[
0, a, 0, a, \ldots
\]  

(53)

is $y = H(x)$ ($x > a$).

In a similar fashion, we also may show that the set of initial conditions $(x_{-1}, x_0)$ such that the positive solutions of (2) converge to $0, a, 0, a, \ldots$ is an empty set. This completes the proof of Theorem I(2). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This project was supported by NNSF of China (11261005), NSF of Guangxi (2012GXNSFDA276040), and SF of ED of Guangxi (2013ZD061).

References


