Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 705496, 8 pages
http://dx.doi.org/10.1155/2014/705496

Research Article

Global Exponential Robust Stability of High-Order Hopfield Neural Networks with S-Type Distributed Time Delays

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Received 24 March 2014; Accepted 1 June 2014; Published 26 June 2014

Academic Editor: Reinaldo Martinez Palhares

By employing differential inequality technique and Lyapunov functional method, some criteria of global exponential robust stability for the high-order neural networks with S-type distributed time delays are established, which are easy to be verified with a wider adaptive scope.

1. Introduction

Neural networks and their various generalizations have been successfully employed in many areas such as pattern recognition, cognitive modeling, adaptive control, and combinatorial optimization [1–7]. Hopfield neural networks (HNNs), as some forms of recurrent artificial neural networks, have been widely studied in recent years [8–12]. The earlier HNNs model proposed by Hopfield [13, 14] was based on the theory of analog circuit consisting of capacitors, resistors, and amplifiers and can be formulated as a system of ordinary equations. Time delays are inevitable in the interactions of neurons in biological and artificial neural networks. The existence of delays is frequently a source of instability for neural networks [9, 10, 15–19].

Over the past few decades, the stability of HNNs with time delays has attracted considerable attention in the literature [20–23]. One of the most investigated problems in the study of HNNs is global exponential stability of the equilibrium point. An equilibrium point of HNNs is *globally exponentially stable*, if the domain of attraction of the equilibrium point is the whole space and the convergence is in real time.

It is worth noting that although the signal propagation is sometimes instantaneous and can be modeled with discrete delays, it may also be distributed during a certain time period so that the distributed delays should be incorporated in the model [24]. Discrete delays and distributed delays attract the attention of many scholars and have been widely studied [17, 19, 25]. To the best of our knowledge, the stability problem for system with both discrete and distributed delays has been a challenging issue, mainly due to the mathematical difficulties in dealing with discrete and distributed delays simultaneously. In 2002, Wang and Xu [26] presented a new neural network model with S-type distributed time delays and demonstrated that S-type distributed time delays include discrete or continuously distributed time delays, but it is not true in the opposite way. In the following years, S-type distributed time delayed neural network models have raised great interest [12, 27–29].

Compared with traditional Hopfield neural networks, the high-order Hopfield type neural networks (HOHNNs) [11, 12, 30–34] have the advantages of stronger approximation properties, faster convergence rate, greater storage capacity, and higher fault tolerance. Therefore, it is of considerable interest to explore the theoretical foundations and practical applications of HOHNNs.

Motivated by the aforementioned discussion, we studied the problem of global exponential robust stability of HOHNNs with S-type distributed time delays. By employing differential inequality technique and a new Lyapunov functional method, some criteria for the global exponential robust stability of the high-order neural networks with S-type distributed time delays have been established, which are easy to be verified with a wider adaptive scope. Meanwhile, the systems in [10, 12, 26, 31] are some special cases of the HOHNNs with S-type distributed time delays.

2. Model Description and Preliminaries

We consider the following HOHNNs with S-type distributed time delays:

$$\begin{aligned}
\frac{dx_i(t)}{dt} = & -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) \\
& + \sum_{k=1}^{r}\sum_{j=1}^{n} w_{ij}^{(k)} g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t+s)\right) \\
& + \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n} w_{ijl}^{(k)} g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t+s)\right) g_l \\
& \times \left(\int_{-\infty}^{0} d_s\eta_{lk}(s)\, x_l(t+s)\right) + I_i \\
& 0 < \underline{a}_i \le a_i, \quad \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, \\
& \underline{w}_{ij}^{(k)} \le w_{ij}^{(k)} \le \overline{w}_{ij}^{(k)}, \quad \underline{w}_{ijl}^{(k)} \le w_{ijl}^{(k)} \le \overline{w}_{ijl}^{(k)}, \\
& \underline{I}_i \le I_i \le \overline{I}_i, \\
& i, j, l = 1, \ldots, n; \quad k = 1, \ldots, r,
\end{aligned} \tag{1}$$

where $x_i(t)$, a_i, b_{ij}, f_j, g_j, and I_i have the same meanings as those in [28], $w_{ij}^{(k)}$, $w_{ijl}^{(k)}$ are the first- and second-order synaptic weights of the system (1) (see [12]), and $\int_{-\infty}^{0} d_s\eta_{lk}(s)x_l(t+s)$ are Lebesgue-Stieltjes integrable, where $\eta_{jk}(s)$ and $\eta_{lk}(s)$ are nondecreasing bounded variation functions which satisfy

$$0 < \int_{-\infty}^{0} d_s\eta_{jk}(s) = \xi_{jk}. \tag{2}$$

In this paper the superscript "T" presents the transpose and $C((-\infty, 0], R^n)$ denotes a set of continuous bounded functions.

For system (1), the initial condition is

$$\begin{aligned}
& x_i(s) = \varphi_i(s), \quad s \in (-\infty, 0], \\
& \varphi_i(s) \in C\left((-\infty, 0], R^n\right), \\
& i = 1, \ldots, n.
\end{aligned} \tag{3}$$

If there is an equilibrium point $x^* = [x_1^*, \ldots, x_n^*]^T$ of system (1) with conditions (3), we can rewrite system (1) as the following equivalent form:

$$\begin{aligned}
& \frac{d\left(x_i(t) - x_i^*\right)}{dt} \\
& = -a_i\left(x_i(t) - x_i^*\right) + \sum_{j=1}^{n} b_{ij}\left(f_j\left(x_j(t)\right) - f_j\left(x_j^*\right)\right) \\
& + \sum_{k=1}^{r}\sum_{j=1}^{n} w_{ij}^{(k)}\left(\int_{-\infty}^{0} g_j\left(d_s\eta_{jk}(s)\, x_j(t+s)\right)\right. \\
& \left. - g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j^*\right)\right) \\
& + \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n} w_{ijl}^{(k)}\left(g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t+s)\right) g_l\right. \\
& \times\left(\int_{-\infty}^{0} d_s\eta_{lk}(s)\, x_l(t+s)\right) \\
& - g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j^*\right) \\
& \left. \times g_l\left(\int_{-\infty}^{0} d_s\eta_{lk}(s)\, x_l^*\right)\right), \\
& 0 < \underline{a}_i \le a_i, \quad \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, \\
& \underline{w}_{ij}^{(k)} \le w_{ij}^{(k)} \le \overline{w}_{ij}^{(k)}, \quad \underline{w}_{ijl}^{(k)} \le w_{ijl}^{(k)} \le \overline{w}_{ijl}^{(k)}, \\
& \underline{I}_i \le I_i \le \overline{I}_i, \\
& i, j, l = 1, \ldots, n; \quad k = 1, \ldots, r.
\end{aligned} \tag{4}$$

From [12], we know that the following system (5) is equivalent to system (4):

$$\begin{aligned}
& \frac{d\left(x_i(t) - x_i^*\right)}{dt} \\
& = -a_i\left(x_i(t) - x_i^*\right) + \sum_{j=1}^{n} b_{ij}\left(f_j\left(x_j(t)\right) - f_j\left(x_j^*\right)\right) \\
& + \sum_{k=1}^{r}\sum_{j=1}^{n}\left(w_{ij}^{(k)} + \sum_{l=1}^{n}\left(w_{ijl}^{(k)} + w_{ilj}^{(k)}\right)\zeta_{lk}\right) \\
& \times\left(g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t+s)\right)\right. \\
& \left. - g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j^*\right)\right), \\
& 0 < \underline{a}_i \le a_i, \quad \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, \\
& \underline{w}_{ij}^{(k)} \le w_{ij}^{(k)} \le \overline{w}_{ij}^{(k)}, \quad \underline{w}_{ijl}^{(k)} \le w_{ijl}^{(k)} \le \overline{w}_{ijl}^{(k)}, \\
& \underline{I}_i \le I_i \le \overline{I}_i, \\
& i, j, l = 1, \ldots, n; \quad k = 1, \ldots, r,
\end{aligned} \tag{5}$$

where $\zeta_{lk} = (1/2)((g_l(\int_{-\infty}^{0} d_s\eta_{lk}(s)x_l(t+s)) + g_l(\int_{-\infty}^{0} d_s\eta_{lk}(s)x_l^*)))$.

Definition 1. The equilibrium point $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ of system (1) is called globally exponentially robustly stable, if, for any $\varphi_i(s) \in C((-\infty, 0], R)$, there exist scalars $\alpha > 0$ and $\Pi > 0$ such that the solution $x(t) = [x_1(t), \ldots, x_n(t)]^T$ to system (1) with initial condition $x_i(s) = \varphi_i(s) \in C((-\infty, 0], R)$, $i = 1, \ldots, n$, satisfies

$$\left|x_i(t) - x_i^*\right| \le \Pi e^{-\alpha t}, \quad t \ge 0, \ i = 1, \ldots, n. \tag{6}$$

Let $A = \operatorname{diag}(\underline{a}_1, \ldots, \underline{a}_n)$, $\sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, $\omega = \operatorname{diag}(\omega_1, \ldots, \omega_n)$, $M = [M_1, M_2, \ldots, M_n]^T$, $\Gamma = \operatorname{diag}(M, M, \ldots, M)$, $B^+ = (\widetilde{b}_{ij})_{n\times n} = (\max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\})_{n\times n}$, $W_k = (w_{ij}^{(k)})_{n\times n}$, $W_k^+ = (\widetilde{w}_{ij}^{(k)})_{n\times n} = (\max\{|\underline{w}_{ij}^{(k)}|, |\overline{w}_{ij}^{(k)}|\})_{n\times n}$, $W_{ik} = (w_{ijl}^{(k)})_{n\times n}$, $(W_{ik})^T = (w_{ilj}^{(k)})_{n\times n}$, $W_{ik}^+ = (\widetilde{w}_{ijl}^{(k)})_{n\times n} = (\max\{|\underline{w}_{ijl}^{(k)}|, |\overline{w}_{ijl}^{(k)}|\})_{n\times n}$, $W_{Hk}^+ = (W_{1k}^+ + (W_{1k}^+)^T, W_{2k}^+ + (W_{2k}^+)^T, \ldots, W_{nk}^+ + (W_{nk}^+)^T)^T_{n^2\times n}$, $\widetilde{I}_i = \max\{|\underline{I}_i|, |\overline{I}_i|\}$, $X \circ Y = (x_{ij}y_{ij})_{n\times n}$ be Hadamard product of two matrices,

$$\xi_k = \begin{pmatrix} \xi_{1k} & \cdots & \xi_{nk} \\ \vdots & \ddots & \vdots \\ \xi_{1k} & \cdots & \xi_{nk} \end{pmatrix}_{n\times n} \quad (k = 1, \ldots, r). \tag{7}$$

We assume throughout that the neuron activation functions $f_j(u_j)$, $g_j(u_j)$, $j = 1, \ldots, n$, satisfy the following conditions.

(H_1) Consider

$$\begin{gathered} \left|g_j(u_j)\right| \le M_j, \qquad 0 \le \frac{\left|f_j(u_j) - f_j(v_j)\right|}{\left|u_j - v_j\right|} \le \sigma_j, \\ 0 \le \frac{\left|g_j(u_j) - g_j(v_j)\right|}{\left|u_j - v_j\right|} \le \omega_j, \\ \text{for } u_j \ne v_j, u_j, v_j, \sigma_j, \omega_j \in R. \end{gathered} \tag{8}$$

(H_2) Consider

$$C = A - B^+\sigma - \sum_{k=1}^{r}\left(\left(W_k^+ + \Gamma^T W_{Hk}^+\right) \circ \xi_k\right)\omega \tag{9}$$

is an M-matrix.

(H_3) Consider

$$0 < \int_{-\infty}^{0} d_s\eta_{jk}(s) \le 1. \tag{10}$$

3. Main Results

Theorem 2. *The equilibrium of system* (1) *is globally exponentially robustly stable, if system* (1) *satisfies* (H_1), (H_2), *and* (H_3).

Proof. Part 1: Existence of the Equilibrium Point. Let

$$\begin{aligned} h_i(x_i, I_i) = a_i x_i &- \sum_{j=1}^{n} b_{ij} f_j(x_j) \\ &- \sum_{k=1}^{r}\sum_{j=1}^{n} w_{ij}^{(k)} g_j(\xi_{jk}x_j) \\ &- \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n} w_{ijl}^{(k)} g_j(\xi_{jk}x_j)\, g_l(\xi_{lk}x_k) - I_i = 0, \\ & i = 1, \ldots, n. \end{aligned} \tag{11}$$

It is obvious that the solutions to (11) are the equilibrium points of system (1).

Let us define homotopic mapping as follows:

$$H(x, \lambda) = (H_1(x_1, \lambda), \ldots, H_n(x_n, \lambda))^T, \tag{12}$$

where

$$H_i(x_i, \lambda) = \lambda h_i(x_i, I_i) + (1 - \lambda)x_i, \quad \lambda \in [0, 1]. \tag{13}$$

By homotopy invariance theorem (see [35]), topological degree theory (see [36]), (H_2), and the proof, which is similar to Theorem 1 in [28], we can conclude that (13) has at least one solution.

That is, system (1) has at least an equilibrium point.

Part 2: Global Existence of the Solutions to System (1). Since $C = A - B^+\sigma - \sum_{k=1}^{r}((W_k^+ + \Gamma^T W_{HK}^+) \circ \xi_k)\omega$ is an M-matrix, there exists a constant vector $L \in R^n$ such that $CL > 0$ (see [15]); that is,

$$\begin{aligned} J_i &= \underline{a}_i L_i \\ &- \sum_{j=1}^{n}\left(\widetilde{b}_{ij}\sigma_j + \sum_{k=1}^{r}\left(\widetilde{w}_{ij}^{(k)} + \sum_{l=1}^{n} M_l\left(\widetilde{w}_{ijl}^{(k)} + \widetilde{w}_{ilj}^{(k)}\right)\right)\right. \\ &\qquad \left.\times \xi_{jk}\omega_j\right) L_j > 0. \end{aligned} \tag{14}$$

Suppose $x(t) = [x_1(t), \ldots, x_n(t)]^T$ is a solution to system (1) and also satisfies the initial condition $x_i(s) = \varphi_i(s) \in C((-\infty, 0], R^n)$, $s \in (-\infty, 0]$, $i = 1, \ldots, n$.

Let us choose a positive constant N such that

$$\begin{aligned} N > \max\Bigg(&\frac{1}{L_i}\sup_{-\infty < s \le 0}\left|\varphi_i(s)\right|, \\ &J_i^{-1}\Bigg(\sum_{j=1}^{n}\widetilde{b}_{ij}\left|f_j(0)\right| + \sum_{k=1}^{r}\sum_{j=1}^{n}\widetilde{w}_{ij}^{(k)}\left|g_j(0)\right| \\ &+ \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n}\widetilde{w}_{ijl}^{(k)}\left|g_j(0)\right|\left|g_l(0)\right| + \widetilde{I}_i\Bigg)\Bigg). \end{aligned} \tag{15}$$

From (15) we know that

$$\left|x_i(t)\right| < NL_i, \quad i = 1, \ldots, n, \ t \in (-\infty, 0]. \tag{16}$$

Then, we will show that

$$\left|x_i(t)\right| < NL_i, \quad i = 1, \ldots, n, \ t \in (-\infty, +\infty). \tag{17}$$

If (17) is not true, there must be some positive integer i_0 and $t_0 > 0$, such that

$$\left|x_{i_0}(t)\right| < NL_{i_0}, \quad t \in (-\infty, t_0), \quad \left|x_{i_0}(t_0)\right| = NL_{i_0}. \tag{18}$$

From (H_1), (15), and (18), we know

$$\begin{aligned}
&\left.\frac{d\left|x_{i_0}(t)\right|}{dt}\right|_{t=t_0} \\
&\quad \le -\underline{a}_{i_0}\left|x_{i_0}(t_0)\right| + \sum_{j=1}^{n}\widetilde{b}_{ij}\left|f_j(x_j)\right| \\
&\qquad + \sum_{k=1}^{r}\sum_{j=1}^{n}\widetilde{w}_{ij}^{(k)}\left|g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t_0+s)\right)\right| \\
&\qquad + \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n}\widetilde{w}_{ijl}^{(k)} \\
&\qquad\qquad \times \left|g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t_0+s)\right) g_l\right. \\
&\qquad\qquad\quad \left.\times\left(\int_{-\infty}^{0} d_s\eta_{lk}(s)\, x_l(t_0+s)\right)\right| + \widetilde{I}_i \\
&\quad \le -\underline{a}_{i_0}\left|x_{i_0}(t_0)\right| + \sum_{j=1}^{n}\widetilde{b}_{ij}\left(\sigma_j\left|x_j(t_0)\right| + \left|f_j(0)\right|\right) \\
&\qquad + \sum_{k=1}^{r}\sum_{j=1}^{n}\widetilde{w}_{ij}^{(k)}\left(\omega_j\left|\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t_0+s)\right|\right. \\
&\qquad\qquad \left. + \left|g_j(0)\right|\right) \\
&\qquad + \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n}\widetilde{w}_{ijl}^{(k)}\left(M_l\omega_j\left|\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t_0+s)\right|\right. \\
&\qquad\qquad + M_j\omega_l\left|\int_{-\infty}^{0} d_s\eta_{lk}(s)\, x_l(t_0+s)\right| \\
&\qquad\qquad \left. + \left|g_j(0)\right|\left|g_l(0)\right|\right) + \widetilde{I}_i \\
&\quad \le -\underline{a}_{i_0}\left|x_{i_0}(t_0)\right| \\
&\qquad + \sum_{j=1}^{n}\widetilde{b}_{ij}\left(\sigma_j\left|x_j(t_0)\right| + \left|f_j(0)\right|\right) \\
&\qquad + \sum_{k=1}^{r}\sum_{j=1}^{n}\widetilde{w}_{ij}^{(k)}\left(\omega_j\left|\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t_0+s)\right| + \left|g_j(0)\right|\right) \\
&\qquad + \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(\widetilde{w}_{ijl}^{(k)} + \widetilde{w}_{ilj}^{(k)}\right) \\
&\qquad\qquad \times\left(M_l\omega_j\left|\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t_0+s)\right|\right. \\
&\qquad\qquad\quad \left. + \left|g_j(0)\right|\left|g_l(0)\right|\right) + \widetilde{I}_i \\
&\quad \le \Bigg[-\underline{a}_{i_0}L_i \\
&\qquad + \left(\sum_{j=1}^{n}\widetilde{b}_{ij}\sigma_j + \sum_{k=1}^{r}\sum_{j=1}^{n}\left(\widetilde{w}_{ij}^{(k)} + \sum_{l=1}^{n}\left(\widetilde{w}_{ijl}^{(k)} + \widetilde{w}_{ilj}^{(k)}\right)\right)\right. \\
&\qquad\qquad \left.\times\,\xi_{jk}\omega_j\right)L_j\Bigg]N \\
&\qquad + \left(\sum_{j=1}^{n}\widetilde{b}_{ij}\left|f_j(0)\right| + \sum_{k=1}^{r}\sum_{j=1}^{n}\widetilde{w}_{ij}^{(k)}\left|g_j(0)\right|\right. \\
&\qquad\qquad \left. + \sum_{k=1}^{r}\sum_{j=1}^{n}\sum_{l=1}^{n}\widetilde{w}_{ijl}^{(k)}\left|g_j(0)\right|\left|g_l(0)\right| + \widetilde{I}_i\right) < 0.
\end{aligned} \tag{19}$$

So

$$\left|x_{i_0}(t_0)\right| < \left|x_{i_0}(t)\right| < NL_{i_0}, \quad t \in (-\infty, t_0], \tag{20}$$

which leads by contradiction to (18).

Hence, (17) holds. That is, the solutions to system (1) are bounded. So the solutions to system (1) are of global existence.

Part 3: Global Exponential Stability of System (1). From (H_2), we know that there exists constant $p_i > 0$, such that

$$\begin{aligned}
&p_i\underline{a}_i - \sum_{j=1}^{n}p_j\left(\widetilde{b}_{ij}\sigma_j\right. \\
&\qquad \left. + \sum_{k=1}^{r}\left(\widetilde{w}_{ij}^{(k)} + \sum_{l=1}^{n}M_l\left(\widetilde{w}_{ijl}^{(k)} + \widetilde{w}_{ilj}^{(k)}\right)\right)\xi_{jk}\omega_j\right) \\
&> 0.
\end{aligned} \tag{21}$$

So, from (H_3), we can choose a constant $\alpha > 0$ sufficiently small, such that

$$p_i\left(\underline{a}_i - \alpha\right) - \sum_{k=1}^{r}\sum_{j=1}^{n} p_j\left(\widetilde{b}_{ij}\sigma_j + \left(\widetilde{w}_{ij}^{(k)} + \sum_{l=1}^{n} M_l\left(\widetilde{w}_{ijl}^{(k)} + \widetilde{w}_{ilj}^{(k)}\right)\right)\xi_{jk}\omega_j\right) \times \left|\int_{-\infty}^{0} e^{-\alpha s} d_s\eta_{jk}(s)\right| > 0. \tag{22}$$

Let $\varphi = [\varphi_1, \ldots, \varphi_n]^T \in C((-\infty, 0], R^n)$, and

$$\|\varphi - x^*\| = \max_{i=1,\ldots,n} \sup_{-\infty < s \le 0} p_i^{-1}\left|\varphi_i(t) - x_i^*\right| > 0. \tag{23}$$

Then, we will show that there exists $\Pi > 0$ such that

$$\left|x_i - x_i^*\right| < \Pi e^{-\alpha t}, \quad t \ge 0,\ i = 1, \ldots, n. \tag{24}$$

Define a Lyapunov functional by

$$V_i(t) = \left|x_i(t) - x_i^*\right| e^{\alpha t}, \quad i = 1, \ldots, n. \tag{25}$$

Its *Dini* derivative reads

$$\begin{aligned}
D^+(V_i(t)) &\le \alpha\left|x_i(t) - x_i^*\right| e^{\alpha t} - a_i\left|x_i(t) - x_i^*\right| e^{\alpha t} \\
&\quad + \Bigg[\sum_{j=1}^{n} b_{ij}\left(\left|f_j\left(x_j(t)\right) - f_j\left(x_j^*\right)\right|\right) \\
&\quad + \sum_{k=1}^{r}\sum_{j=1}^{n}\left(w_{ij}^{(k)} + \sum_{l=1}^{n}\left(w_{ijl}^{(k)}(t) + w_{ilj}^{(k)}(t)\right)\zeta_l\right) \\
&\quad \times \left|g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j(t+s)\right) - g_j\left(\int_{-\infty}^{0} d_s\eta_{jk}(s)\, x_j^*\right)\right|\Bigg] e^{\alpha t} \\
&\le p_i\left(-\underline{a}_i + \alpha\right) p_i^{-1}\left|x_i(t) - x_i^*\right| e^{\alpha t} \\
&\quad + \sum_{j=1}^{n} p_j\sigma_j\widetilde{b}_{ij} p_j^{-1}\left|x_j(t) - x_j^*\right| e^{\alpha t} \\
&\quad + \sum_{k=1}^{r}\sum_{j=1}^{n} p_j\left(\widetilde{w}_{ij}^{(k)} + \sum_{l=1}^{n}\left(\widetilde{w}_{ijl}^{(k)} + \widetilde{w}_{ilj}^{(k)}\right) M_l\right)\omega_j p_j^{-1} \\
&\quad \times \int_{-\infty}^{0}\left|x_j(t+s) - x_j^*\right| e^{\alpha(t+s)} d_s\eta_{jk}(s) \\
&\quad \times \left|\int_{-\infty}^{0} e^{-\alpha s} d_s\eta_{jk}(s)\right|.
\end{aligned} \tag{26}$$

If $T > 1$, we have

$$p_i^{-1}\left|x_i(t) - x_i^*\right| e^{\alpha t} < T\|\varphi - x^*\|, \quad t \le 0, \quad i = 1, \ldots, n. \tag{27}$$

Then, we will prove that

$$p_i^{-1} V_i(t) = p_j^{-1}\left|x(t) - x_i^*\right| e^{\alpha t} < T\|\varphi - x^*\|, \quad t > 0, \quad i = 1, \ldots, n. \tag{28}$$

If (28) is not true, there exists $i_0 \in \{1, 2, \ldots, n\}$ and $t_0 > 0$ such that

$$\begin{aligned}
&p_{i_0}^{-1} V_{i_0}(t_0) = T\|\varphi - x^*\|, \\
&p_{i_0}^{-1} V_{i_0}(t) < T\|\varphi - x^*\|, \quad t < t_0, \\
&p_j^{-1} V_j(t) < T\|\varphi - x^*\|, \quad \forall t \le t_0, \\
&j = 1, 2, \ldots, n, \quad j \ne i_0.
\end{aligned} \tag{29}$$

From (29), we have

$$\begin{aligned}
0 &\le \left.\frac{dV_{i_0}(t)}{dt}\right|_{t=t_0} \\
&\le p_{i_0}\left(-\underline{a}_{i_0} + \alpha\right) p_{i_0}^{-1}\left|x_{i_0}(t_0) - x_{i_0}^*\right| e^{\alpha t_0} \\
&\quad + \sum_{j=1}^{n} p_j\sigma_j\left|b_{i_0 j}\right| p_j^{-1}\left|x_j(t_0) - x_j^*\right| e^{\alpha t_0} \\
&\quad + \sum_{k=1}^{r}\sum_{j=1}^{n} p_j\Bigg(\left|w_{i_0 j}^{(k)}\right| + \sum_{l=1}^{n}\left(\left|w_{i_0 jl}^{(k)}\right| + \left|w_{i_0 lj}^{(k)}\right|\right) M_l\Bigg)\omega_j p_j^{-1} \\
&\quad \times \left|\int_{-\infty}^{0} e^{-\alpha s} d_s\eta_{jk}(s)\right| \\
&\quad \times \int_{-\infty}^{0}\left|x_j(t_0+s) - x_j^*\right| \times e^{\alpha(t_0+s)} d_s\eta_{jk}(s).
\end{aligned} \tag{30}$$

Because $p_{i_0}^{-1} V_{i_0}(t_0) = T\|\varphi - x^*\|$ and $p_j^{-1} V_j(t) \le T\|\varphi - x^*\|$, for all $t \le t_0$, $j = 1, \ldots, n$, we can obtain

$$\begin{aligned}
\left.\frac{dV_{i_0}(t)}{dt}\right|_{t=t_0} &\le \Bigg\{p_{i_0}\left(-\underline{a}_{i_0} + \alpha\right) + \sum_{j=1}^{n} p_j\Bigg(\widetilde{b}_{i_0 j}\sigma_j \\
&\quad + \sum_{k=1}^{r}\left(\widetilde{w}_{i_0 j}^{(k)} + \sum_{l=1}^{n} M_l\left(\widetilde{w}_{i_0 jl}^{(k)} + \widetilde{w}_{i_0 lj}^{(k)}\right)\right)\xi_{jk}\omega_j\Bigg) \\
&\quad \times \left|\int_{-\infty}^{0} e^{-\alpha s} d_s\eta_{jk}(s)\right|\Bigg\} T\|\varphi - x^*\| < 0.
\end{aligned} \tag{31}$$

It is obvious that (31) is in contradiction to (30). Hence, (28) holds. That is,

$$p_i^{-1}\left|x_i(t)-x^*\right| < T\left\|\varphi - x^*\right\| e^{-\alpha t}, \quad t>0,\ i=1,\dots,n. \tag{32}$$

So

$$\left|x_i(t)-x^*\right| < \Pi e^{-\alpha t}, \quad t>0,\ i=1,\dots,n, \tag{33}$$

where $\Pi = \max_{1\le i\le n}\{p_i\}T\|\varphi - x^*\| > 0$.

If there exists another equilibrium x^{**} of system (1), we have $|x_i^* - x_i^{**}| \le |x_i(t)-x_i^*| + |x_i(t)-x_i^{**}| \to 0, t \to \infty$, $i=1,\dots,n$.

From above proof, the system (1) has a unique equilibrium point x^*, which is globally exponentially robustly stable.

The proof of Theorem 2 is completed. □

Remark 3. The system (1) includes system with discrete time delays and with continuously distributed delays. Conversely, it is not true.

When $w_{ijl}^k = 0$ and

$$\eta_{jk}(s) = \begin{cases} \displaystyle\sum_{k=1}^{r}\eta_{jk}, & s=\tau_1=0, \\ \displaystyle\sum_{k=2}^{r}\eta_{jk}, & \tau_2\le s<0, \\ \displaystyle\sum_{k=3}^{r}\eta_{jk}, & \tau_3\le s<\tau_2, \\ & j=1,\dots,n, \\ \dots & \\ \eta_{jr}, & \tau_r\le s<\tau_{r-1}, \\ 0, & -\infty<s<\tau_r, \end{cases} \tag{34}$$

system (1) becomes a HNNs model with discrete time delays

$$\begin{gathered} \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) + \sum_{k=1}^{r}\sum_{j=1}^{n} w_{ij}^{(k)} g_j\left(\eta_{jk} x_j\left(t-\tau_k\right)\right) + I_i, \\ 0<\underline{a}_i\le a_i, \quad \underline{b}_{ij}\le b_{ij}\le \overline{b}_{ij}, \\ \underline{w}_{ij}^{(k)} \le w_{ij}^{(k)} \le \overline{w}_{ij}^{(k)}, \quad \underline{I}_i \le I_i \le \overline{I}_i, \\ i,j,l=1,\dots,n;\ k=1,\dots,r. \end{gathered} \tag{35}$$

When $w_{ijl}^k = 0$ and $\eta_{jk}(s) \in C^1(-\infty,0]$, $j=1,\dots,n$, $k=1,\dots,r$, the value of the synaptic connectivity from neuron j to i is a continuous function on $(\infty,0]$, which means that time delays influence the network continuously, and system (1) belongs to a HNNs model with continuous time delays:

$$\begin{gathered} \frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) + \sum_{k=1}^{r}\sum_{j=1}^{n} w_{ij}^{(k)} g_j\left(\int_{-\infty}^{0} \eta_{jk}'(s)\, x_j(t+s)\, ds\right) + I_i, \\ 0<\underline{a}_i\le a_i, \quad \underline{b}_{ij}\le b_{ij}\le \overline{b}_{ij}, \\ \underline{w}_{ij}^{(k)} \le w_{ij}^{(k)} \le \overline{w}_{ij}^{(k)}, \quad \underline{I}_i \le I_i \le \overline{I}_i, \\ i,j,l=1,\dots,n;\ k=1,\dots,r. \end{gathered} \tag{36}$$

So system (1) is widely representative.

Remark 4. When $a_i = \underline{a}_i > 0$, $\underline{b}_{ij} = b_{ij} = \overline{b}_{ij}$, $\underline{w}_{ij} = w_{ij} = \overline{w}_{ij}$, $\underline{w}_{ijl}^{(k)} = w_{ijl}^{(k)} = \overline{w}_{ijl}^{(k)}$, and $\underline{I}_i = I_i = \overline{I}_i$, system (1) becomes the system (1) in [12]. So the systems in [10, 12, 26, 31] are also the special cases of system (1) (see [12]).

4. Example

For the sake of simplicity, we consider given one-dimension HOHNNs with S-type distributed time delays as follows:

$$\begin{gathered} \frac{dx_1(t)}{dt} = -a_1 x_1(t) + b_{11} f_1\left(x_1(t)\right) + \sum_{k=1}^{3} w_{11}^{(k)} g_1\left(\int_{-\infty}^{0} d_s\eta_{1k}(s)\, x_1(t+s)\right) \\ + \sum_{k=1}^{3} w_{111}^{(k)} g_1\left(\int_{-\infty}^{0} d_s\eta_{1k}(s)\, x_1(t+s)\right) \times g_1\left(\int_{-\infty}^{0} d_s\eta_{1k}(s)\, x_1(t+s)\right) + I_1 \\ i,j,l=1; \quad k=1,\dots,3. \end{gathered} \tag{37}$$

In system (37), $f_1(z) = |z|$ and $g_1(z) = \tanh(z)$, $z \in R$, which satisfy (H_1), $\Gamma^T = M = M_1 = 1$, and $\sigma_1 = \omega_1 = 1$.

Let $\eta_{1k}(t) = k^{-1}e^t$; then $0 < \int_{-\infty}^{0} d_s\eta_{1k}(s) = \xi_{1k} = k^{-1} \le 1$, $k=1,\dots,3$, which satisfies (H_3).

The parameters of the system (37) are given as follows:

$$\begin{gathered}
\underline{a}_1 = 3.3 \leq a_1, \qquad -1 = \underline{b}_{11} \leq b_{11} \leq \overline{b}_{11} = 1, \\
0.2 = \underline{w}^{(1)}_{11} \leq w^{(1)}_{11} \leq \overline{w}^{(1)}_{11} = 0.4, \\
-0.4 = \underline{w}^{(2)}_{11} \leq w^{(2)}_{11} \leq \overline{w}^{(2)}_{11} = 0.4, \\
-0.1 = \underline{w}^{(3)}_{11} \leq w^{(3)}_{11} \leq \overline{w}^{(3)}_{11} = 0.2, \\
0 = \underline{w}^{(1)}_{111} \leq w^{(1)}_{111} \leq \overline{w}^{(1)}_{111} = 0.2, \\
-0.6 = \underline{w}^{(2)}_{111} \leq w^{(2)}_{111} \leq \overline{w}^{(3)}_{111} = 0.8, \\
-0.5 = \underline{w}^{(3)}_{111} \leq w^{(3)}_{111} \leq \overline{w}^{(3)}_{111} = 0.5, \\
0 = \underline{I}_1 \leq I_1 \leq \overline{I}_1 = 1.
\end{gathered} \tag{38}$$

From (H_2) and the above parameters, we can easily obtain that $C = 3.3 - 1 - [(0.4 + 0.4) \times 1 + (0.4 + 1.6) \times 2^{-1} + (0.2 + 1) \times 3^{-1}] = 0.1 > 0$ is an M-matrix.

Therefore, it follows from Theorem 2 that the null solution to system (37) is globally exponentially robustly stable.

5. Conclusion

We have investigated the global exponential robust stability of high-order Hopfield neural networks with S-type distributed time delays, which is of theoretical as well as practical importance for the development of neural networks with time delays. The system (1) considered here is more general compared to the systems in literatures [10, 12, 26, 31]. By employing differential inequality technique and Lyapunov functional method, some criteria of global exponential robust stability for the high-order neural networks with S-type distributed time delays are established, which are easily verifiable and have a wider applicable range. The linear matrix inequality (LMI) approach is also widely used to establish the desired sufficient conditions for stability analysis of delayed neural networks [11, 37]. Wen et al. [17] have done some great work in control and filtering problems for neural systems. In future extension, we will do some research in stability of high-order Hopfield neural networks with S-type distributed time delays using LMI method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Haiyong Zheng was partially supported by the National Natural Science Foundation of China (nos. 61271406 and 61301240). Linshan Wang was partially supported by the National Natural Science Foundation of China (no. 11171374) and the Natural Science Foundation of Shandong (no. ZR2011AZ001). Yangfan Wang was partially supported by the National Natural Science Foundation of China (no. 31302182) and Fundamental Research Funds for Central Universities (nos. 201362034 and 201313003).

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