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Research Article

On a Class of q-Bernoulli, q-Euler, and q-Genocchi Polynomials

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The main purpose of this paper is to introduce and investigate a class of q-Bernoulli, q-Euler, and q-Genocchi polynomials. The q-analogues of well-known formulas are derived. In addition, the q-analogue of the Srivastava-Pintér theorem is obtained. Some new identities, involving q-polynomials, are proved.

1. Introduction

Throughout this paper, we always make use of the classical definition of quantum concepts as follows.

The q-shifted factorial is defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n \in \mathbb{N},$$
$$(a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad |q|<1,\ a\in\mathbb{C}. \tag{1}$$

It is known that

$$(a;q)_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)}(-1)^k a^k. \tag{2}$$

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad (q\neq 1,\ a\in\mathbb{C});$$
$$[0]_q! = 1, \qquad [n]_q! = [n]_q[n-1]_q!. \tag{3}$$

The q-polynomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, \quad (k\leqslant n, k,\ n\in\mathbb{N}). \tag{4}$$

In the standard approach to the q-calculus two exponential functions are used, these q-exponential functions and improved type q-exponential function (see [1]) are defined as follows:

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty}\frac{1}{\left(1-(1-q)q^k z\right)},$$
$$0<|q|<1,\ |z|<\frac{1}{|1-q|},$$
$$E_q(z) = e_{1/q}(z) = \sum_{n=0}^{\infty}\frac{q^{(1/2)n(n-1)}z^n}{[n]_q!}$$
$$= \prod_{k=0}^{\infty}\left(1+(1-q)q^k z\right), \quad 0<|q|<1,\ z\in\mathbb{C},$$
$$\mathscr{E}_q(z) = e_q\left(\frac{z}{2}\right)E_q\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty}\frac{(-1,q)_n}{2^n}\frac{z^n}{[n]_q!}$$
$$= \prod_{k=0}^{\infty}\frac{\left(1+(1-q)q^k(z/2)\right)}{\left(1-(1-q)q^k(z/2)\right)}, \quad 0<q<1,\ |z|<\frac{2}{1-q}. \tag{5}$$

The form of improved type of q-exponential function $\mathscr{E}_q(z)$ motivated us to define a new q-addition and q-subtraction as

$$\left(x\oplus_q y\right)^n := \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{(-1,q)_k(-1,q)_{n-k}}{2^n}x^k y^{n-k}, \quad n=0,1,2,\ldots,$$
$$\left(x\ominus_q y\right)^n := \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{(-1,q)_k(-1,q)_{n-k}}{2^n}x^k (-y)^{n-k}, \quad n=0,1,2,\ldots. \tag{6}$$

It follows that

$$\mathscr{E}_q(tx)\,\mathscr{E}_q(ty) = \sum_{n=0}^{\infty}\left(x\oplus_q y\right)^n\frac{t^n}{[n]_q!}. \tag{7}$$

The Bernoulli numbers $\{B_m\}_{m\geq 0}$ are rational numbers in a sequence defined by the binomial recursion formula:

$$\sum_{k=0}^{m}\binom{m}{k}B_k - B_m = \begin{cases}1, & m=1,\\ 0, & m>1,\end{cases} \tag{8}$$

or equivalently, the generating function

$$\sum_{k=0}^{\infty}B_k\frac{t^k}{k!} = \frac{t}{e^t-1}. \tag{9}$$

q-Analogues of the Bernoulli numbers were first studied by Carlitz [2] in the middle of the last century when he introduced a new sequence $\{\beta_m\}_{m\geqslant 0}$:

$$\sum_{k=0}^{m}\binom{m}{k}\beta_k q^{k+1} - \beta_m = \begin{cases}1, & m=1,\\ 0, & m>1.\end{cases} \tag{10}$$

Here and in the remainder of the paper, for the parameter q we make the assumption that $|q|<1$. Clearly we recover (8) if we let $q\to 1$ in (10). The q-binomial formula is known as

$$(1-a)_q^n = (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{(1/2)k(k-1)}(-1)^k a^k. \tag{11}$$

The above q-standard notation can be found in [3].

Carlitz has introduced the q-Bernoulli numbers and polynomials in [2]. Srivastava and Pintér proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [4]. They also gave some generalizations of these polynomials. In [4–16], the authors investigated some properties of the q-Euler polynomials and q-Genocchi polynomials. They gave some recurrence relations. In [17], Cenkci et al. gave the q-extension of Genocchi numbers in a different manner. In [18], Kim gave a new concept for the q-Genocchi numbers and polynomials. In [19], Simsek et al. investigated the q-Genocchi zeta function and l-function by using generating functions and Mellin transformation. There are numerous recent studies on this subject by, among many other authors, Cigler [20], Cenkci et al. [17, 21], Choi et al. [22], Cheon [23], Luo and Srivastava [8–10], Srivastava et al. [4, 24], Nalci and Pashaev [25] Gaboury and Kurt, [26], Kim et al. [27], and Kurt [28].

We first give the definitions of the q-numbers and q-polynomials. It should be mentioned that the definition of q-Bernoulli numbers in Definition 1 can be found in [25].

Definition 1. Let $q\in\mathbb{C}, 0<|q|<1$. The q-Bernoulli numbers $\mathfrak{b}_{n,q}$ and polynomials $\mathfrak{B}_{n,q}(x,y)$ are defined by means of the generating functions:

$$\widehat{\mathfrak{B}}(t) := \frac{te_q(-t/2)}{e_q(t/2)-e_q(-t/2)} = \frac{t}{\mathscr{E}_q(t)-1} = \sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!}, \quad |t|<2\pi,$$
$$\frac{t}{\mathscr{E}_q(t)-1}\mathscr{E}_q(tx)\,\mathscr{E}_q(ty) = \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \quad |t|<2\pi. \tag{12}$$

Definition 2. Let $q\in\mathbb{C}, 0<|q|<1$. The q-Euler numbers $\mathfrak{e}_{n,q}$ and polynomials $\mathfrak{E}_{n,q}(x,y)$ are defined by means of the generating functions:

$$\widehat{\mathfrak{E}}(t) := \frac{2e_q(-t/2)}{e_q(t/2)+e_q(-t/2)} = \frac{2}{\mathscr{E}_q(t)+1} = \sum_{n=0}^{\infty}\mathfrak{e}_{n,q}\frac{t^n}{[n]_q!}, \quad |t|<\pi,$$
$$\frac{2}{\mathscr{E}_q(t)+1}\mathscr{E}_q(tx)\,\mathscr{E}_q(ty) = \sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \quad |t|<\pi. \tag{13}$$

Definition 3. Let $q\in\mathbb{C}, 0<|q|<1$. The q-Genocchi numbers $\mathfrak{g}_{n,q}$ and polynomials $\mathfrak{G}_{n,q}(x,y)$ are defined by means of the generating functions:

$$\widehat{\mathfrak{G}}(t) := \frac{2te_q(-t/2)}{e_q(t/2)+e_q(-t/2)} = \frac{2t}{\mathscr{E}_q(t)+1} = \sum_{n=0}^{\infty}\mathfrak{g}_{n,q}\frac{t^n}{[n]_q!}, \quad |t|<\pi,$$
$$\frac{2t}{\mathscr{E}_q(t)+1}\mathscr{E}_q(tx)\,\mathscr{E}_q(ty) = \sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \quad |t|<\pi. \tag{14}$$

Note that Cigler [20] defined q-Genocchi numbers as

$$t\frac{e_q(t)+e_q(-t)}{e_q(t)+e_q(-t)}=\sum_{n=0}^{\infty}g_{2n,q}\frac{(-1)^{n-1}(-q;q)_{2n-1}t^{2n}}{[2n]_q!}. \quad (15)$$

Then comparing $\mathfrak{g}_{n,q}$ with $g_{n,q}$, we see that

$$(-1)^{n-1}2^{2n+1}\mathfrak{g}_{2n+2,q}=(-q;q)_{2n+1}g_{2n+2,q}. \quad (16)$$

Definition 4. Let $q\in\mathbb{C}, 0<|q|<1$. The q-tangent numbers $\mathfrak{T}_{n,q}$ are defined by means of the generating functions:

$$\begin{aligned}\tanh_q t &= -i\tan_q(it)=\frac{e_q(t)-e_q(-t)}{e_q(t)+e_q(-t)}=\frac{\mathscr{E}_q(2t)-1}{\mathscr{E}_q(2t)+1}\\ &=\sum_{n=1}^{\infty}\mathfrak{T}_{2n+1,q}\frac{(-1)^k t^{2n+1}}{[2n+1]_q!}.\end{aligned} \quad (17)$$

It is obvious that, by letting q tend to 1 from the left side, we lead to the classic definition of these polynomials:

$$\begin{aligned}&\mathfrak{b}_{n,q}:=\mathfrak{B}_{n,q}(0), \qquad \lim_{q\to 1^-}\mathfrak{B}_{n,q}(x)=B_n(x),\\ &\lim_{q\to 1^-}\mathfrak{B}_{n,q}(x,y)=B_n(x+y), \qquad \lim_{q\to 1^-}\mathfrak{b}_{n,q}=B_n,\\ &\mathfrak{e}_{n,q}:=\mathfrak{E}_{n,q}(0), \qquad \lim_{q\to 1^-}\mathfrak{E}_{n,q}(x)=E_n(x),\\ &\lim_{q\to 1^-}\mathfrak{E}_{n,q}(x,y)=E_n(x+y), \qquad \lim_{q\to 1^-}\mathfrak{e}_{n,q}=E_n,\\ &\mathfrak{g}_{n,q}:=\mathfrak{G}_{n,q}(0), \qquad \lim_{q\to 1^-}\mathfrak{G}_{n,q}(x)=G_n(x),\\ &\lim_{q\to 1^-}\mathfrak{G}_{n,q}(x,y)=G_n(x+y) \qquad \lim_{q\to 1^-}\mathfrak{g}_{n,q}=G_n.\end{aligned} \quad (18)$$

Here $B_n(x)$, $E_n(x)$, and $G_n(x)$ denote the classical Bernoulli, Euler, and Genocchi polynomials, respectively, which are defined by

$$\frac{t}{e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \qquad \frac{2}{e^t+1}e^{tx}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},$$
$$\frac{2t}{e^t+1}e^{tx}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}. \quad (19)$$

The aim of the present paper is to obtain some results for the above newly defined q-polynomials. It should be mentioned that q-Bernoulli and q-Euler polynomials in our definitions are polynomials of x and y and when $y=0$, they are polynomials of x. First advantage of this approach is that for $q\to 1^-$, $\mathfrak{B}_{n,q}(x,y)$ ($\mathfrak{E}_{n,q}(x,y)$, $\mathfrak{G}_{n,q}(x,y)$) becomes the classical Bernoulli $\mathfrak{B}_n(x+y)$ (Euler $\mathfrak{E}_n(x+y)$, Genocchi $\mathfrak{G}_{n,q}(x,y)$) polynomial and we may obtain the q-analogues of well-known results, for example, Srivastava and Pintér [11], Cheon [23], and so forth. Second advantage is that, similar to the classical case, odd numbered terms of the Bernoulli numbers $\mathfrak{b}_{k,q}$ and the Genocchi numbers $\mathfrak{g}_{k,q}$ are zero, and even numbered terms of the Euler numbers $\mathfrak{e}_{n,q}$ are zero.

2. Preliminary Results

In this section we will provide some basic formulae for the q-Bernoulli, q-Euler, and q-Genocchi numbers and polynomials in order to obtain the main results of this paper in the next section.

Lemma 5. *The q-Bernoulli numbers $\mathfrak{b}_{n,q}$ satisfy the following q-binomial recurrence:*

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{b}_{k,q}-\mathfrak{b}_{n,q}=\begin{cases}1, & n=1,\\ 0, & n>1.\end{cases} \quad (20)$$

Proof. By a simple multiplication of (8) we see that

$$\widehat{\mathfrak{B}}(t)\,\mathscr{E}_q(t)=t+\widehat{\mathfrak{B}}(t). \quad (21)$$

So

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{b}_{k,q}\frac{t^n}{[n]_q!}=t+\sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!}. \quad (22)$$

The statement follows by comparing t^m coefficients. □

We use this formula to calculate the first few $\mathfrak{b}_{k,q}$:

$$\begin{aligned}\mathfrak{b}_{0,q}&=1,\\ \mathfrak{b}_{1,q}&=-\frac{1}{2},\\ \mathfrak{b}_{2,q}&=\frac{1}{4}\frac{q(q+1)}{q^2+q+1}=\frac{q[2]_q}{4[3]_q},\\ \mathfrak{b}_{3,q}&=0.\end{aligned} \quad (23)$$

The similar property can be proved for q-Euler numbers

$$\sum_{k=0}^{m}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{e}_{k,q}+\mathfrak{e}_{m,q}=\begin{cases}2, & m=0,\\ 0, & m>0.\end{cases} \quad (24)$$

and q-Genocchi numbers

$$\sum_{k=0}^{m}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{g}_{k,q}+\mathfrak{g}_{m,q}=\begin{cases}2, & m=1,\\ 0, & m>1.\end{cases} \quad (25)$$

Using the above recurrence formulae we calculate the first few $\mathfrak{e}_{n,q}$ and $\mathfrak{g}_{n,q}$ terms as well:

$$\begin{aligned}&\mathfrak{e}_{0,q}=1, \qquad \mathfrak{g}_{0,q}=0,\\ &\mathfrak{e}_{1,q}=-\frac{1}{2}, \qquad \mathfrak{g}_{1,q}=1,\\ &\mathfrak{e}_{2,q}=0, \qquad \mathfrak{g}_{2,q}=-\frac{[2]_q}{2}=-\frac{q+1}{2},\\ &\mathfrak{e}_{3,q}=\frac{[3]_q[2]_q-[4]_q}{8}=\frac{q(1+q)}{8}, \qquad \mathfrak{g}_{3,q}=0.\end{aligned} \quad (26)$$

Remark 6. The first advantage of the new q-numbers $\mathfrak{b}_{k,q}$, $\mathfrak{e}_{k,q}$, and $\mathfrak{g}_{k,q}$ is that similar to classical case odd numbered terms of the Bernoulli numbers $\mathfrak{b}_{k,q}$ and the Genocchi numbers $\mathfrak{g}_{k,q}$ are zero, and even numbered terms of the Euler numbers $\mathfrak{e}_{n,q}$ are zero.

Next lemma gives the relationship between q-Genocchi numbers and q-Tangent numbers.

Lemma 7. *For any $n \in \mathbb{N}$, we have*

$$\mathfrak{T}_{2n+1,q} = \mathfrak{g}_{2n+2,q}\frac{(-1)^{k-1}2^{2n+1}}{[2n+2]_q}. \tag{27}$$

Proof. First we recall the definition of q-trigonometric functions:

$$\cos_q t = \frac{e_q(it)+e_q(-it)}{2}, \qquad \sin_q t = \frac{e_q(it)-e_q(-it)}{2i},$$
$$i\tan_q t = \frac{e_q(it)-e_q(-it)}{e_q(it)+e_q(-it)}, \qquad \cot_q t = i\frac{e_q(it)+e_q(-it)}{e_q(it)-e_q(-it)}. \tag{28}$$

Now by choosing $z = 2it$ in $\widehat{\mathfrak{B}}(z)$, we get

$$\widehat{\mathfrak{B}}(2it) = \frac{2it}{\mathscr{E}_q(2it)-1} = \frac{te_q(-it)}{\sin_q t} = \sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{(2it)^n}{[n]_q!}. \tag{29}$$

It follows that

$$\begin{aligned}\widehat{\mathfrak{B}}(2it) &= \frac{te_q(-it)}{\sin_q t} = \frac{t}{\sin_q t}\left(\cos_q t - i\sin_q t\right) = t\cot_q t - it\\ &= \mathfrak{b}_{0,q} + 2it\mathfrak{b}_{1,q} + \sum_{n=2}^{\infty}\mathfrak{b}_{n,q}\frac{(2it)^n}{[n]_q!}\\ &= 1 - it + \sum_{n=2}^{\infty}\mathfrak{b}_{n,q}\frac{(2it)^n}{[n]_q!}.\end{aligned} \tag{30}$$

Since the function $t\cot_q t$ is even in the above sum odd coefficients $\mathfrak{b}_{2k+1,q}$, $k = 1, 2, \ldots$, are zero, and we get

$$t\cot_q t = 1 + \sum_{n=2}^{\infty}\mathfrak{b}_{n,q}\frac{(2it)^n}{[n]_q!} = 1 + \sum_{n=1}^{\infty}\mathfrak{b}_{n,q}\frac{(2it)^{2n}}{[2n]_q!}. \tag{31}$$

By choosing $z = 2it$ in $\widehat{\mathfrak{G}}(z)$, we get

$$\widehat{\mathfrak{G}}(2it) = \frac{4it}{\mathscr{E}_q(2it)+1} = \frac{2ite_q(-it)}{\cos_q t} = \sum_{n=0}^{\infty}\mathfrak{g}_{n,q}\frac{(2it)^n}{[n]_q!},$$

$$\begin{aligned}\widehat{\mathfrak{G}}(2it) &= \frac{4it}{\mathscr{E}_q(2it)+1} = \frac{2ite_q(-it)}{\cos_q t}\\ &= \frac{2it}{\cos_q t}\left(\cos_q t - i\sin_q t\right)\\ &= 2it + 2t\tan_q t = \mathfrak{g}_{0,q} + 2it\mathfrak{g}_{1,q} + \sum_{n=2}^{\infty}\mathfrak{g}_{n,q}\frac{(2it)^n}{[n]_q!}\\ &= 2it + \sum_{n=2}^{\infty}\mathfrak{g}_{n,q}\frac{(2it)^n}{[n]_q!}.\end{aligned} \tag{32}$$

It follows that

$$\begin{aligned}2t\tan_q t &= \sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(2it)^{2n}}{[2n]_q!},\\ \tan_q t &= \sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(-1)^n(2t)^{2n-1}}{[2n]_q!},\\ \tanh_q t &= -i\tan_q(it) = -i\sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(-1)^n(2it)^{2n-1}}{[2n]_q!}\\ &= -\sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(2t)^{2n-1}}{[2n]_q!} = -\sum_{n=1}^{\infty}\mathfrak{g}_{2n+2,q}\frac{(2t)^{2n+1}}{[2n+2]_q!},\end{aligned} \tag{33}$$

Thus

$$\begin{aligned}\tanh_q t &= -i\tan_q(it) = \frac{e_q(t)-e_q(-t)}{e_q(t)+e_q(-t)} = \frac{\mathscr{E}_q(2t)-1}{\mathscr{E}_q(2t)+1}\\ &= \sum_{n=1}^{\infty}\mathfrak{T}_{2n+1,q}\frac{(-1)^k t^{2n+1}}{[2n+1]_q!},\\ \mathfrak{T}_{2n+1,q} &= \mathfrak{g}_{2n+2,q}\frac{(-1)^{k-1}2^{2n+1}}{[2n+2]_q}.\end{aligned} \tag{34}$$

□

The following result is a q-analogue of the addition theorem, for the classical Bernoulli, Euler, and Genocchi polynomials.

Lemma 8 (addition theorems). *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathfrak{b}_{k,q}\left(x\oplus_q y\right)^{n-k},$$

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x)\,y^{n-k},$$

$$\mathfrak{E}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{e}_{k,q}\left(x\oplus_q y\right)^{n-k},$$
$$\mathfrak{E}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{E}_{k,q}(x)\,y^{n-k},$$
$$\mathfrak{G}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{g}_{k,q}\left(x\oplus_q y\right)^{n-k},$$
$$\mathfrak{G}_{n,q}(x,y)=\sum_{k=0}^{n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{G}_{k,q}(x)\,y^{n-k}. \tag{35}$$

Proof. We prove only the first formula. It is a consequence of the following identity:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\frac{t}{\mathscr{E}_q(t)-1}\mathscr{E}_q(tx)\,\mathscr{E}_q(ty)\\&=\sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\left(x\oplus_q y\right)^n\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{b}_{k,q}\left(x\oplus_q y\right)^{n-k}\frac{t^n}{[n]_q!}.\end{aligned} \tag{36}$$

□

In particular, setting $y=0$ in (35), we get the following formulae for q-Bernoulli, q-Euler and q-Genocchi polynomials, respectively:

$$\mathfrak{B}_{n,q}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{b}_{k,q}x^{n-k},$$
$$\mathfrak{E}_{n,q}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{e}_{k,q}x^{n-k}, \tag{37}$$
$$\mathfrak{G}_{n,q}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{g}_{k,q}x^{n-k}. \tag{38}$$

Setting $y=1$ in (35), we get

$$\mathfrak{B}_{n,q}(x,1)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x),$$
$$\mathfrak{E}_{n,q}(x,1)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{E}_{k,q}(x), \tag{39}$$
$$\mathfrak{G}_{n,q}(x,1)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{G}_{k,q}(x).$$

Clearly (39) is q-analogues of

$$B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x),$$
$$E_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}E_k(x),$$
$$G_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}G_k(x), \tag{40}$$

respectively.

Lemma 9. *The odd coefficients of the q-Bernoulli numbers, except the first one, are zero. That means* $\mathfrak{b}_{n,q}=0$ *where* $n=2r+1$ $(r\in\mathbb{N})$.

Proof. It follows from the fact that the function

$$\begin{aligned}f(t)&=\sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!}-\mathfrak{b}_{1,q}t\\&=\frac{t}{\mathscr{E}_q(t)-1}+\frac{t}{2}=\frac{t}{2}\left(\frac{\mathscr{E}_q(t)+1}{\mathscr{E}_q(t)-1}\right),\end{aligned} \tag{41}$$

□

By using q-derivative we obtain the next lemma.

Lemma 10. *One has*

$$\begin{aligned}D_{q,x}\mathfrak{B}_{n,q}(x)&=[n]_q\frac{\mathfrak{B}_{n-1,q}(x)+\mathfrak{B}_{n-1,q}(qx)}{2},\\D_{q,x}\mathfrak{E}_{n,q}(x)&=[n]_q\frac{\mathfrak{E}_{n-1,q}(x)+\mathfrak{E}_{n-1,q}(qx)}{2},\\D_{q,x}\mathfrak{G}_{n,q}(x)&=[n]_q\frac{\mathfrak{G}_{n-1,q}(x)+\mathfrak{G}_{n-1,q}(qx)}{2}.\end{aligned} \tag{42}$$

Lemma 11 (difference equations). *One has*

$$\mathfrak{B}_{n,q}(x,1)-\mathfrak{B}_{n,q}(x)=\frac{(-1;q)_{n-1}}{2^{n-1}}[n]_q x^{n-1},\quad n\geq 1, \tag{43}$$

$$\mathfrak{E}_{n,q}(x,1)+\mathfrak{E}_{n,q}(x)=2\frac{(-1;q)_{n}}{2^{n}}x^{n},\quad n\geq 0, \tag{44}$$

$$\mathfrak{G}_{n,q}(x,1)+\mathfrak{G}_{n,q}(x)=2\frac{(-1;q)_{n-1}}{2^{n-1}}[n]_q x^{n-1},\quad n\geq 1. \tag{45}$$

Proof. We prove the identity for the q-Bernoulli polynomials. From the identity

$$\frac{t\mathscr{E}_q(t)}{\mathscr{E}_q(t)-1}\mathscr{E}_q(tx)=t\mathscr{E}_q(tx)+\frac{t}{\mathscr{E}_q(t)-1}\mathscr{E}_q(tx), \tag{46}$$

it follows that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x)\frac{t^n}{[n]_q!}\\&\quad=\sum_{n=0}^{\infty}\frac{(-1,q)_n}{2^n}x^n\frac{t^{n+1}}{[n]_q!}+\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}.\end{aligned} \tag{47}$$

□

From (43) and (37), (44) and (38), we obtain the following formulae.

Lemma 12. *One has*

$$x^n = \frac{2^n}{(-1;q)_n [n]_q} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{(-1;q)_{n+1-k}}{2^{n+1-k}} \mathfrak{B}_{k,q}(x),$$

$$x^n = \frac{2^{n-1}}{(-1;q)_n} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1;q)_{n-k}}{2^{n-k}} \mathfrak{E}_{k,q}(x) + \mathfrak{E}_{n,q}(x) \right),$$

$$x^n = \frac{2^{n-1}}{(-1;q)_n [n+1]_q} \times \left(\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{(-1;q)_{n+1-k}}{2^{n+1-k}} \mathfrak{G}_{k,q}(x) + \mathfrak{G}_{n+1,q}(x) \right). \tag{48}$$

The above formulae are q-analogues of the following familiar expansions:

$$x^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(x),$$

$$x^n = \frac{1}{2} \left[\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) \right], \tag{49}$$

$$x^n = \frac{1}{2(n+1)} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} E_k(x) + E_{n+1}(x) \right],$$

respectively.

Lemma 13. *The following identities hold true:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} \mathfrak{B}_{k,q}(x,y) - \mathfrak{B}_{n,q}(x,y) = [n]_q (x \oplus_q y)^{n-1},$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} \mathfrak{E}_{k,q}(x,y) + \mathfrak{E}_{n,q}(x,y) = 2(x \oplus_q y)^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} \mathfrak{G}_{k,q}(x,y) + \mathfrak{G}_{n,q}(x,y) = 2[n]_q (x \oplus_q y)^{n-1}. \tag{50}$$

Proof. We prove the identity for the q-Bernoulli polynomials. From the identity

$$\frac{t\mathscr{E}_q(t)}{\mathscr{E}_q(t)-1} \mathscr{E}_q(tx) \mathscr{E}_q(ty) = t\mathscr{E}_q(tx) \mathscr{E}_q(ty) + \frac{t}{\mathscr{E}_q(t)-1} \mathscr{E}_q(tx) \mathscr{E}_q(ty), \tag{51}$$

it follows that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} \mathfrak{B}_{k,q}(x,y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_k (-1,q)_{n-k}}{2^n} x^k y^{n-k} \frac{t^{n+1}}{[n]_q!} + \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!}. \tag{52}$$

□

3. Some New Formulae

The classical Cayley transformation $z \rightarrow \mathrm{Cay}(z,a) := (1+az)/(1-az)$ motivated us to derive the formula for $\mathscr{E}_q(qt)$. In addition, by substituting $\mathrm{Cay}(z,(q-1)/2)$ in the generating formula we have

$$\widehat{\mathfrak{B}}_q(qt) \widehat{\mathfrak{B}}_q(t) = \left(\widehat{\mathfrak{B}}_q(qt) - q\widehat{\mathfrak{B}}_q(t) \left(1 + (1-q)\frac{t}{2}\right)\right) \times \frac{1}{1-q} \times \frac{2}{\mathscr{E}_q(t)+1}. \tag{53}$$

The right hand side can be presented by q-Euler numbers. Now equating coefficients of t^n we get the following identity. In the case that $n = 0$, we find the first improved q-Euler number which is exactly 1.

Proposition 14. *For all* $n \geq 1$,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q} \mathfrak{B}_{n-k,q} q^k = -q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} \mathfrak{B}_{k,q} \mathfrak{E}_{n-k,q} [k-1]_q - \frac{q}{2} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k-1}}{2^{n-k-1}} \mathfrak{B}_{k,q} \mathfrak{E}_{n-k-1,q} [n]_q. \tag{54}$$

Let us take a q-derivative from the generating function, after simplifying the equation, by knowing the quotient rule for quantum derivative, and also using

$$\mathscr{E}_q(qt) = \frac{1-(1-q)(t/2)}{1+(1-q)(t/2)} \mathscr{E}_q(t),$$

$$D_q(\mathscr{E}_q(t)) = \frac{\mathscr{E}_q(qt) + \mathscr{E}_q(t)}{2}, \tag{55}$$

one has

$$\widehat{\mathfrak{B}}_q(qt) \widehat{\mathfrak{B}}_q(t) = \frac{2+(1-q)t}{2\mathscr{E}_q(t)(q-1)} \left(q\widehat{\mathfrak{B}}_q(t) - \widehat{\mathfrak{B}}_q(qt) \right). \tag{56}$$

It is clear that $\mathscr{E}_q^{-1}(t) = \mathscr{E}_q(-t)$. Now, by equating coefficients of t^n we obtain the following identity.

Proposition 15. *For all $n \geq 1$,*

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q} \mathfrak{B}_{2n-k,q} q^k = -q \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-k}}{2^{2n-k}} \mathfrak{B}_{k,q} [k-1]_q (-1)^k + \frac{q(1-q)}{2} \sum_{k=0}^{2n-1} \begin{bmatrix} 2n-1 \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-1-k}}{2^{2n-1-k}} \mathfrak{B}_{k,q} \times [k-1]_q (-1)^k,$$

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q} \mathfrak{B}_{2n-k+1,q} q^k = q \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \frac{(-1,q)_{2n+1-k}}{2^{2n+1-k}} \mathfrak{B}_{k,q} [k-1]_q (-1)^k - \frac{q(1-q)}{2} \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-k}}{2^{2n-k}} \mathfrak{B}_{k,q} [k-1]_q (-1)^k. \quad (57)$$

We may also derive a differential equation for $\widehat{\mathfrak{B}}_q(t)$. If we differentiate both sides of the generating function with respect to t, after a little calculation we find that

$$\frac{\partial}{\partial t} \widehat{\mathfrak{B}}_q(t) = \widehat{\mathfrak{B}}_q(t) \left(\frac{1}{t} - \frac{(1-q)\mathscr{E}_q(t)}{\mathscr{E}_q(t)-1} \left(\sum_{k=0}^{\infty} \frac{4q^k}{4-(1-q)^2 q^{2k}} \right) \right). \quad (58)$$

If we differentiate $\widehat{\mathfrak{B}}_q(t)$ with respect to q, we obtain, instead,

$$\frac{\partial}{\partial q} \widehat{\mathfrak{B}}_q(t) = -\widehat{\mathfrak{B}}_q^2(t) \mathscr{E}_q(t) \sum_{k=0}^{\infty} \frac{4t\left(kq^{k-1} - (k+1)q^k\right)}{4-(1-q)^2 q^{2k}}. \quad (59)$$

Again, using the generating function and combining this with the derivative we get the partial differential equation.

Proposition 16. *Consider the following:*

$$\frac{\partial}{\partial t} \widehat{\mathfrak{B}}_q(t) - \frac{\partial}{\partial q} \widehat{\mathfrak{B}}_q(t) = \frac{\widehat{\mathfrak{B}}_q(t)}{t} + \frac{\widehat{\mathfrak{B}}_q^2(t)\mathscr{E}_q(t)}{t} \times \sum_{k=0}^{\infty} \frac{4t\left(kq^{k-1} - (k+1)q^k\right) - q^k(1-q)}{4-(1-q)^2 q^{2k}}. \quad (60)$$

4. Explicit Relationship between the q-Bernoulli and q-Euler Polynomials

In this section, we give some explicit relationship between the q-Bernoulli and q-Euler polynomials. We also obtain new formulae and some special cases for them. These formulae are extensions of the formulae of Srivastava and Pintér, Cheon, and others.

We present natural q-extensions of the main results in the papers [9, 11]; see Theorems 17 and 19.

Theorem 17. *For $n \in \mathbb{N}_0$, the following relationships hold true:*

$$\mathfrak{B}_{n,q}(x,y) = \frac{1}{2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n} \left[\mathfrak{B}_{k,q}(x) + \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j} \mathfrak{B}_{j,q}(x)}{2^{k-j} m^{k-j}} \right] \times \mathfrak{E}_{n-k,q}(my) = \frac{1}{2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n} \left[\mathfrak{B}_{k,q}(x) + \mathfrak{B}_{k,q}\left(x, \frac{1}{m}\right) \right] \mathfrak{E}_{n-k,q}(my). \quad (61)$$

Proof. Using the following identity

$$\frac{t}{\mathscr{E}_q(t)-1} \mathscr{E}_q(tx) \mathscr{E}_q(ty) = \frac{t}{\mathscr{E}_q(t)-1} \mathscr{E}_q(tx) \cdot \frac{\mathscr{E}_q(t/m)+1}{2} \cdot \frac{2}{\mathscr{E}_q(t/m)+1} \mathscr{E}_q\left(\frac{t}{m} my\right) \quad (62)$$

we have

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{1}{2} \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \sum_{n=0}^{\infty} \frac{(-1,q)_n}{m^n 2^n} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x) \frac{t^n}{[n]_q!} + \frac{1}{2} \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x) \frac{t^n}{[n]_q!} =: I_1 + I_2. \quad (63)$$

It is clear that

$$I_2 = \frac{1}{2} \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q m^{k-n} \mathfrak{B}_{k,q}(x) \mathfrak{E}_{n-k,q}(my) \frac{t^n}{[n]_q!}. \quad (64)$$

On the other hand

$$I_1 = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \times \sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \mathfrak{B}_{j,q}(x)\frac{(-1,q)_{n-j}}{m^{n-j}2^{n-j}}\frac{t^n}{[n]_q!} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n-k,q}(my) \times \sum_{j=0}^{k}\begin{bmatrix} n \\ j \end{bmatrix}_q \frac{\mathfrak{B}_{j,q}(x)(-1,q)_{k-j}}{m^{n-k}m^{k-j}2^{k-j}}\frac{t^n}{[n]_q!}. \tag{65}$$

Therefore

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n} \times \left[\mathfrak{B}_{k,q}(x) + \sum_{j=0}^{k}\begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}\mathfrak{B}_{j,q}(x)}{2^{k-j}m^{k-j}}\right] \times \mathfrak{E}_{n-k,q}(my)\frac{t^n}{[n]_q!}. \tag{66}$$

It remains to equate the coefficients of t^n. □

Next we discuss some special cases of Theorem 17.

Corollary 18. *For $n \in \mathbb{N}_0$ the following relationship holds true:*

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\mathfrak{B}_{k,q}(x) + \frac{(-1;q)_{k-1}}{2^k}[k]_q x^{k-1}\right)\mathfrak{E}_{n-k,q}(y). \tag{67}$$

The formula (67) is a q-extension of the Cheon's main result [23].

Theorem 19. *For $n \in \mathbb{N}_0$, the following relationships*

$$\mathfrak{E}_{n,q}(x,y) = \frac{1}{[n+1]_q} \times \sum_{k=0}^{n+1}\frac{1}{m^{n+1-k}}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \times \left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}}\mathfrak{E}_{j,q}(y) - \mathfrak{E}_{k,q}(y)\right) \times \mathfrak{B}_{n+1-k,q}(mx) \tag{68}$$

hold true between the q-Bernoulli polynomials and q-Euler polynomials.

Proof. The proof is based on the following identity:

$$\frac{2}{\mathscr{E}_q(t)+1}\mathscr{E}_q(tx)\mathscr{E}_q(ty) = \frac{2}{\mathscr{E}_q(t)+1}\mathscr{E}_q(ty) \cdot \frac{\mathscr{E}_q(t/m)-1}{t} \cdot \frac{t}{\mathscr{E}_q(t/m)-1}\mathscr{E}_q\left(\frac{t}{m}mx\right). \tag{69}$$

Indeed

$$\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(x,y)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{(-1,q)_n}{m^n2^n}\frac{t^{n-1}}{[n]_q!} \times \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(mx)\frac{t^n}{m^n[n]_q!} - \sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(y)\frac{t^{n-1}}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(mx)\frac{t^n}{m^n[n]_q!} =: I_1 - I_2. \tag{70}$$

It follows that

$$I_2 = \frac{1}{t}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(mx)\frac{t^n}{m^n[n]_q!} = \frac{1}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^{n-k}}\mathfrak{E}_{k,q}(y)\mathfrak{B}_{n-k,q}(mx)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\frac{1}{[n+1]_q} \times \sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{1}{m^{n+1-k}}\mathfrak{E}_{k,q}(y)\mathfrak{B}_{n+1-k,q}(mx)\frac{t^n}{[n]_q!},$$

$$I_1 = \frac{1}{t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(mx)\frac{t^n}{m^n[n]_q!} \times \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{m^{n-k}2^{n-k}}\mathfrak{E}_{k,q}(y)\frac{t^n}{[n]_q!} = \frac{1}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^{n-k}}\mathfrak{B}_{n-k,q}(mx) \times \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}}\mathfrak{E}_{j,q}(y)\frac{t^{n-1}}{[n]_q!}. \tag{71}$$

□

Next we give an interesting relationship between the q-Genocchi polynomials and the q-Bernoulli polynomials.

Theorem 20. *For $n \in \mathbb{N}_0$, the following relationship*

$$\begin{aligned}
&\mathfrak{G}_{n,q}(x,y) \\
&= \frac{1}{[n+1]_q} \\
&\quad \times \sum_{k=0}^{n+1} \frac{1}{m^{n-k}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \\
&\quad\quad \times \left(\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j} 2^{k-j}} \mathfrak{G}_{j,q}(x) - \mathfrak{G}_{k,q}(x) \right) \\
&\quad\quad \times \mathfrak{B}_{n+1-k,q}(my), \\
&\mathfrak{B}_{n,q}(x,y) \\
&= \frac{1}{2[n+1]_q} \\
&\quad \times \sum_{k=0}^{n+1} \frac{1}{m^{n-k}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \\
&\quad\quad \times \left(\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j} 2^{k-j}} \mathfrak{B}_{j,q}(x) + \mathfrak{B}_{k,q}(x) \right) \\
&\quad\quad \times \mathfrak{G}_{n+1-k,q}(my)
\end{aligned} \tag{72}$$

holds true between the q-Genocchi and the q-Bernoulli polynomials.

Proof. Using the following identity

$$\begin{aligned}
&\frac{2t}{\mathscr{E}_q(t)+1} \mathscr{E}_q(tx) \mathscr{E}_q(ty) \\
&\qquad = \frac{2t}{\mathscr{E}_q(t)+1} \mathscr{E}_q(tx) \cdot \left(\mathscr{E}_q\left(\frac{t}{m}\right) - 1 \right) \frac{m}{t} \\
&\qquad\quad \cdot \frac{t/m}{\mathscr{E}_q(t/m) - 1} \cdot \mathscr{E}_q\left(\frac{t}{m} my\right)
\end{aligned} \tag{73}$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(x,y) \frac{t^n}{[n]_q!} \\
&= \frac{m}{t} \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(x,y) \frac{t^n}{[n]_q!} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(-1,q)_n}{m^n 2^n} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \\
&\quad - \frac{m}{t} \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(x,y) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \\
&= \frac{m}{t} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{m^{n-k} 2^{n-k}} \mathfrak{G}_{k,q}(x) - \mathfrak{G}_{n,q}(x) \right) \frac{t^n}{[n]_q!} \\
&\quad \times \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \\
&= \frac{m}{t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{m^{n-k}} \begin{bmatrix} n \\ k \end{bmatrix}_q \\
&\qquad \times \left(\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j} 2^{k-j}} \mathfrak{G}_{j,q}(x) - \mathfrak{G}_{k,q}(x) \right) \\
&\qquad \times \mathfrak{B}_{n-k,q}(my) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \frac{1}{m^{n-k}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \\
&\qquad \times \left(\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j} 2^{k-j}} \mathfrak{G}_{j,q}(x) - \mathfrak{G}_{k,q}(x) \right) \\
&\qquad \times \mathfrak{B}_{n+1-k,q}(my) \frac{t^n}{[n]_q!}.
\end{aligned} \tag{74}$$

The second identity can be proved in a like manner. □

Conflict of Interests

The authors declare that there is no conflict of interests with any commercial identities regarding the publication of this paper.

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