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Research Article

Invariant Means and Reversible Semigroup of Relatively Nonexpansive Mappings in Banach Spaces

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The purpose of this paper is to study modified Halpern type and Ishikawa type iteration for a semigroup of relatively nonexpansive mappings $\mathfrak{I} = \{T(s) : s \in S\}$ on a nonempty closed convex subset C of a Banach space with respect to a sequence of asymptotically left invariant means $\{\mu_n\}$ defined on an appropriate invariant subspace of $l^\infty(S)$, where S is a semigroup. We prove that, given some mild conditions, we can generate iterative sequences which converge strongly to a common element of the set of fixed points $F(\mathfrak{I})$, where $F(\mathfrak{I}) = \bigcap\{F(T(s)) : s \in S\}$.

1. Introduction

Let E be a real Banach space with the topological dual E^* and let C be a closed and convex subset of E. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$.

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Halpern [1] and is defined as follows:

$$x_0 = u \in C, \quad \text{chosen arbitrarily,}$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad \forall n \geq 1, \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. He pointed out that the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that if the iteration (1) converges to a fixed point of T, then these conditions must be satisfied. The second iteration process is known as Mann's iteration process [2] which is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \geq 1, \tag{2}$$

where the initial x_1 is taken in C arbitrary and the sequence $\{\alpha_n\}$ is in $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [3] which is defined as follows:

$$y_n = \beta_n x_n + (1 - \beta_n) T x_n,$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \quad \forall n \geq 1, \tag{3}$$

where the initial x_1 is taken in C arbitrary and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

In 2007, Lau et al. [4] proposed the following modification of Halpern's iteration (1) for amenable semigroups of nonexpansive mappings in a Banach space.

Theorem 1. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings from a compact convex subset C of a strictly convex and smooth Banach space E into C, let X be an amenable and $\mathfrak{I}$-stable subspace of $l^\infty(S)$, and let $\{\mu_n\}$ be a strongly left regular sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $x_1 = x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T(\mu_n) x_n, \quad n \geq 2. \tag{4}$$

Then $\{x_n\}$ converges strongly to Px, where P denotes the unique sunny nonexpansive retraction of C onto $F(\mathfrak{I})$.

Let C be a closed and convex subset of E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Point p in C is said to be an *asymptotic fixed point* of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called *relatively nonexpansive* [6–8], if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings was studied in [6, 7, 9].

Recently, Kim [10] proved a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the shrinking method.

Theorem 2. *Let S be a left reversible semigroup and let $\mathfrak{J} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into C with $F(\mathfrak{J}) \neq \emptyset$. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be a asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \alpha_n < 1$ and $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{gathered}
x_0 \in C, \quad \text{chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
y_n = J^{-1}\left(\alpha_n J x_1 + (1-\alpha_n) J T_{\mu_n} x_n\right), \\
C_{n+1} = \{z \in C_n : \\
\phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1.
\end{gathered}
\tag{5}
$$

Then $\{x_n\}$ converges strongly to $\Pi_{F(\mathfrak{J})}x_1$, where $\Pi_{F(\mathfrak{J})}$ is the generalized projection from C onto $F(\mathfrak{J})$.

Let S be a semigroup. The purpose of this paper is to study modified Halpern type and Ishikawa type iterations for a semigroup of relatively nonexpansive mappings $\mathfrak{J} = \{T(s) : s \in S\}$ on a nonempty closed convex subset C of a Banach space with respect to a sequence of asymptotically left invariant means $\{\mu_n\}$ defined on an appropriate invariant subspace of $l^\infty(S)$. We prove that, given some mild conditions, we can generate iterative sequences which converge strongly to a common element of the set of fixed points $F(\mathfrak{J})$, where $F(\mathfrak{J}) = \bigcap\{F(T(s)) : s \in S\}$.

2. Preliminaries

A real Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* if

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{6}$$

exists for each $x, y \in U$. It is said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. Denote by $\langle\cdot,\cdot\rangle$ the duality product. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\right\}, \tag{7}$$

for $x \in E$. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfies that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ and then $x_n \to x$, where $\rightharpoonup$ and $\to$ denote the weak convergence and the strong convergence, respectively.

We know the following:

(1) the duality mapping J is monotone, that is, $\langle x-y, x^*-y^*\rangle \ge 0$ whenever $x^* \in Jx$ and $y^* \in Jy$;

(2) if E is strictly convex, then J is one-to-one; that is, if $Jx \cap Jy$ is nonempty, then $x = y$;

(3) if E is strictly convex, then J is strictly monotone; that is, $x = y$ whenever $\langle x-y, x^*-y^*\rangle = 0$, $x^* \in Jx$ and $y^* \in Jy$;

(4) if E is uniformly convex, then E has the Kadec-Klee property;

(5) if E is uniformly convex, then E is reflexive and strictly convex;

(6) if E is smooth, then J is single-valued and norm-to-weak* continuous;

(7) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E;

(8) if E is reflexive, then J is onto;

(9) if E is smooth and reflexive, then J is norm-to-weak continuous; that is, $Jx_n \rightharpoonup Jx$ whenever $x_n \to x$;

(10) if E is smooth, strictly convex, and reflexive, then J is single-valued, one-to-one and onto; in this case, the inverse mapping J^{-1} coincides with the duality mapping on E;

(11) if E^* is strictly convex, then J is single-valued;

(12) the norm of E^* is Fréchet differentiable if and only if E is strictly convex and reflexive Banach space which has the Kadec-Klee property.

For more details, see [11].

As well known, if C is a nonempty, closed, and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive (see, the reference therein). This fact actually characterizes Hilbert spaces. Consequently, it is not true to more general Banach spaces. In this connection, Alber [12] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{8}$$

for $x, y \in E$. Observe that, in a Hilbert space H, (8) reduces to

$$\phi(x, y) = \|x - y\|^2, \tag{9}$$

for $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns an arbitrary point $x \in E$ to the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem:

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x). \tag{10}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [12, 13]). In a Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(ϕ_1) $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$ for all $x, y \in E$,

(ϕ_2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$,

(ϕ_3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$,

(ϕ_4) if E is a reflexive, strictly convex, and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \quad \text{iff } x = y. \tag{11}$$

For more details see [14].

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real-valued functionals on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s) f)(t) = f(st), \qquad (r(s) f)(t) = f(ts), \tag{12}$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let X be a subspace of $l^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. It is well known that μ is mean on X if and only if

$$\inf_{s \in S} f(s) \le \mu(f) \le \sup_{s \in S} f(s), \tag{13}$$

for each $f \in X$.

Let X be a translation invariant subspace of $l^\infty(S)$ (i.e., $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be *left invariant* (resp., *right invariant*) if

$$\mu(l(s) f) = \mu(f), \quad (\text{resp., } \mu(r(s) f) = \mu(f)) \tag{14}$$

for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [15–19]. X is said to be *left* (resp., *right*) *amenable* if X has a left (resp., right) invariant mean. X is amenable if X is left and right amenable. We call a semigroup S *amenable* if X is amenable. Further, amenable semigroups include all commutative semigroups and solvable groups. However, the free group or semigroup of two generators is not left or right amenable (see [20–22]).

A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically left* (resp., *right*) *invariant* if

$$\lim_\alpha (\mu_\alpha(l(s) f) - \mu_\alpha(f)) = 0, \\ \left(\text{resp., } \lim_\alpha (\mu_\alpha(r(s) f) - \mu_\alpha(f)) = 0\right), \tag{15}$$

for each $f \in X$ and $s \in S$, and it is said to be *left* (resp., *right*) *strongly asymptotically invariant* (or *strong regular*) if

$$\lim_\alpha \|l^*(s)\mu_\alpha - \mu_\alpha\| = 0, \\ \left(\text{resp., } \lim_\alpha \|r^*(s)\mu_\alpha - \mu_\alpha\| = 0\right), \tag{16}$$

for each $s \in S$, where $l^*(s)$ and $r^*(s)$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day in [20] where they were called *weak* invariant* and *norm invariant*, respectively.

It is easy to see that if a semigroup S is left (resp., right) amenable, then the semigroup $S' = S \cup \{e\}$, where $es' = s'e = s'$ for all $s' \in S$, is also left (resp., right) amenable and converse.

From now on S denotes a semigroup with an identity e. S is called *left reversible* if any two right ideals of S have nonvoid intersection; that is, $aS \cap bS \ne \emptyset$ for $a, b \in S$. In this case, $(S, \preceq)$ is a directed system when the binary relation "$\preceq$" on S is defined by $a \preceq b$ if and only if $\{a\} \cup aS \supseteq \{b\} \cup bS$ for $a, b \in S$. It is easy to see that $t \preceq ts$ for all $t, s \in S$. Further, if $t \preceq s$ then $pt \preceq ps$ for all $p \in S$. The class of left reversible semigroup includes all groups and commutative semigroups. If a semigroup S is left amenable, then S is left reversible. But the converse is not true [23–28].

Let S be a semigroup and let C be a closed and convex subset of E. Let $F(T)$ denote the fixed point set of T. Then $\mathfrak{T} = \{T(s) : s \in S\}$ is called a *representation of S as relatively nonexpansive mappings on* C if $T(s)$ is relatively nonexpansive with $T(e) = I$ and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathfrak{T})$ the set of common fixed points of $\{T(s) : s \in S\}$; that is,

$$F(\mathfrak{T}) = \bigcap_{s \in S} F(T(s)) = \bigcap_{s \in S} \{x \in C : T(s)x = x\}. \tag{17}$$

We know that if μ is a mean on X and if for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x^*\rangle = \langle x_0, x^*\rangle$ for each $x^* \in E^*$. We denote such a point x_0 by $T_\mu x$. Note that $T_\mu x$ is contained in the closure of the convex hull of $\{T(s)x : s \in S\}$ for each $x \in C$. Note that $T_\mu z = z$ for each $z \in F(\mathfrak{T})$; see [29–31].

3. Lemmas

We need the following lemmas for the proof of our main results.

Lemma 3 (see [9]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T*

be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.

Lemma 4 (see [12, 32]). *Let E be a reflexive, strictly convex, and smooth Banach space and let C be a nonempty, closed, and convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \tag{18}$$

for all $y \in C$.

Lemma 5 (see [32]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\lim_{n\to\infty}\phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 6 (see [4, 33]). *Let μ be a left invariant mean on X. Then $F(\mathfrak{T}) = F(T_\mu) \cap C_a$, where C_a denotes the set of almost periodic elements in C; that is, all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of E.*

Lemma 7 (cf. [4, 10]). *Let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. If $z \in C_a$ and $\liminf_{n\to\infty}\|T_{\mu_n}z - z\| = 0$, then z is a common fixed point of $\mathfrak{T}$.*

4. Strong Convergence Theorems

In this section, we will establish two strong convergence theorems of various iterative sequences for finding common fixed point of relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach spaces (cf. [34–36]).

We begin with a strong convergence theorem of modified Halpern's type.

Theorem 8. *Let S be a left reversible semigroup and let $\mathfrak{T} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered} x_0 \in C, \quad \text{chosen arbitrarily}, \\ x_{n+1} = \Pi_C J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J T_{\mu_n} x_n\right), \quad \forall n \ge 0. \end{gathered} \tag{19}$$

If the interior of $F(\mathfrak{T})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{T})$.

Proof. We show first that the sequence $\{x_n\}$ converges strongly in C.

From Lemma 3, we know $F(T)$ is closed and convex. So, we can define the generalized projection Π_C onto $F(\mathfrak{T})$. Most of all, we have

$$\begin{aligned} \left\|T_{\mu_n}x_n\right\| &= \sup\left\{\left|\left\langle T_{\mu_n}x_n, x^*\right\rangle\right| : x^* \in E^*, \|x^*\| = 1\right\} \\ &= \sup\left\{\left|(\mu_n)_s \left\langle T(s)x_n, x^*\right\rangle\right| : x^* \in E^*, \|x^*\| = 1\right\} \\ &\le \sup\left\{(\mu_n)_s\left(\|T(s)x_n\|\|x^*\|\right) : x^* \in E^*, \|x^*\| = 1\right\} \\ &= (\mu_n)_s\|T(s)x_n\|. \end{aligned} \tag{20}$$

Then, from the definition of relatively nonexpansive, we have

$$\begin{aligned} \phi\left(u, T_{\mu_n}x_n\right) &= \|u\|^2 - 2\left\langle u, JT_{\mu_n}x_n\right\rangle + \left\|T_{\mu_n}x_n\right\|^2 \\ &= \|u\|^2 - 2(\mu_n)_s\langle u, JT(s)x_n\rangle \\ &\quad + (\mu_n)_s\|T(s)x_n\|^2 \\ &= (\mu_n)_s\phi(u, T(s)x_n) \\ &\le (\mu_n)_s\phi(u, x_n) = \phi(u, x_n), \end{aligned} \tag{21}$$

for all $u \in F(\mathfrak{T})$. From the convexity of $\|\cdot\|^2$ and (21), we get

$$\begin{aligned} &\phi(u, x_{n+1}) \\ &\quad = \phi\left(u, \Pi_C J^{-1}\left(\alpha_n Jx_0 + (1-\alpha_n)JT_{\mu_n}x_n\right)\right) \\ &\quad \le \phi\left(u, J^{-1}\left(\alpha_n Jx_0 + (1-\alpha_n)JT_{\mu_n}x_n\right)\right) \\ &\quad = \|u\|^2 - 2\left\langle u, \alpha_n Jx_0 + (1-\alpha_n)JT_{\mu_n}x_n\right\rangle \\ &\qquad + \left\|\alpha_n Jx_0 + (1-\alpha_n)JT_{\mu_n}x_n\right\|^2 \\ &\quad \le \|u\|^2 - 2\alpha_n\langle u, Jx_0\rangle - 2(1-\alpha_n)\left\langle u, JT_{\mu_n}x_n\right\rangle \\ &\qquad + \alpha_n\|x_0\|^2 + (1-\alpha_n)\left\|T_{\mu_n}x_n\right\|^2 \\ &\quad = \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi\left(u, T_{\mu_n}x_n\right) \\ &\quad \le \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, x_n). \end{aligned} \tag{22}$$

So, we have

$$\begin{aligned} &(1-\alpha_n)\{\phi(u, x_{n+1}) - \phi(u, x_n)\} \\ &\quad \le \alpha_n\{\phi(u, x_0) - \phi(u, x_{n+1})\} \\ &\quad \le \alpha_n\phi(u, x_0). \end{aligned} \tag{23}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\lim_{n\to\infty}\{\phi(u, x_{n+1}) - \phi(u, x_n)\} \le 0. \tag{24}$$

Therefore $\{\phi(u, x_n)\}$ is bounded and $\lim_{n\to\infty}\phi(u, x_n)$ exists. Then $\{x_n\}$ is also bounded. This implies that $\{T_{\mu_n}x_n\}$ is bounded. Since the interior of $F(\mathfrak{T})$ is nonempty, there exist $p \in F(\mathfrak{T})$ and $r > 0$ such that

$$p + rq \in F(\mathfrak{T}), \tag{25}$$

whenever $\|q\| \le 1$. By (ϕ_2), we have

$$\begin{aligned} \phi(u, x_n) &= \phi(u, x_{n+1}) + \phi(x_{n+1}, x_n) \\ &\quad + 2\langle u - x_{n+1}, Jx_{n+1} - Jx_n\rangle, \end{aligned} \tag{26}$$

for any $u \in F(\mathfrak{T})$. This implies

$$\langle x_{n+1} - u, Jx_n - Jx_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}(\phi(u, x_n) - \phi(u, x_{n+1})). \quad (27)$$

Also, we have

$$\begin{aligned} &\langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle \\ &= \langle x_{n+1} - (p + rq) + rq, Jx_n - Jx_{n+1} \rangle \\ &= \langle x_{n+1} - (p + rq), Jx_n - Jx_{n+1} \rangle \\ &\quad + r\langle q, Jx_n - Jx_{n+1} \rangle. \end{aligned} \quad (28)$$

On the other hand, by (24) and (25), we have that

$$\phi(p + rq, x_{n+1}) \le \phi(p + rq, x_n). \quad (29)$$

From (27), we get

$$\begin{aligned} 0 &\le \frac{1}{2}(\phi(p + rq, x_n) - \phi(p + rq, x_{n+1})) \\ &= \langle x_{n+1} - (p + rq), Jx_n - Jx_{n+1} \rangle \\ &\quad + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle \\ &\quad - r\langle q, Jx_n - Jx_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n). \end{aligned} \quad (30)$$

Then, by (27), we have

$$\begin{aligned} &r\langle q, Jx_n - Jx_{n+1} \rangle \\ &\le \langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \frac{1}{2}(\phi(p, x_n) - \phi(p, x_{n+1})), \end{aligned} \quad (31)$$

for $p \in F(\mathfrak{T})$. Hence

$$\langle q, Jx_n - Jx_{n+1} \rangle \le \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})). \quad (32)$$

Since q with $\|q\| \le 1$ is arbitrary, by (24), we have

$$\|Jx_n - Jx_{n+1}\| \le \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})). \quad (33)$$

So, we have

$$\begin{aligned} &\|Jx_{n+m} - Jx_n\| \\ &= \|Jx_{n+m} - Jx_{n+m-1} + Jx_{n+m-1} \\ &\quad - \cdots - Jx_{n+1} + Jx_{n+1} - Jx_n\| \\ &\le \sum_{i=n}^{n+m-1} \|Jx_i - Jx_{i+1}\| \\ &\le \frac{1}{2r}\sum_{i=n}^{n+m-1}(\phi(p, x_i) - \phi(p, x_{i+1})) \\ &= \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})). \end{aligned} \quad (34)$$

We know that $\{\phi(p, x_n)\}$ converges. Hence, $\{Jx_n\}$ is a Cauchy sequence. Since E^* is complete, $\{Jx_n\}$ converges strongly to some point in E^*. Since E is uniformly convex, E^* has a Fréchet differentiable norm. Then J^{-1} is continuous on E^*. Hence $\{x_n\}$ converges strongly to some point v in C.

Now, we show that $v \in F(\mathfrak{T})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{T})}x_n$.

By (33) and the convergence of $\{\phi(p, x_n)\}$, we have

$$\lim_{n\to\infty}\|Jx_n - Jx_{n+1}\| = 0. \quad (35)$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, it follows that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \quad (36)$$

Let $z_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n}x_n)$. Then, we have

$$\begin{aligned} &\|Jz_n - JT_{\mu_n}x_n\| \\ &= \|\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n}x_n - JT_{\mu_n}x_n\| \\ &= \alpha_n\|Jx_0 - JT_{\mu_n}x_n\|. \end{aligned} \quad (37)$$

Since $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|Jz_n - JT_{\mu_n}x_n\| = 0. \quad (38)$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we get

$$\lim_{n\to\infty}\|z_n - T_{\mu_n}x_n\| = 0. \quad (39)$$

From $x_{n+1} = \Pi_C z_n$ and Lemma 4, we have

$$\begin{aligned} &\phi(T_{\mu_n}x_n, x_{n+1}) + \phi(x_{n+1}, z_n) \\ &= \phi(T_{\mu_n}x_n, \Pi_C z_n) + \phi(\Pi_C z_n, z_n) \\ &\le \phi(T_{\mu_n}x_n, z_n). \end{aligned} \quad (40)$$

Since

$$\begin{aligned}
\phi\left(T_{\mu_n}x_n, z_n\right) &= \phi\left(T_{\mu_n}x_n, J^{-1}\left(\alpha_n Jx_0 + (1-\alpha_n) JT_{\mu_n}x_n\right)\right) \\
&= \left\|T_{\mu_n}x_n\right\|^2 - 2\left\langle T_{\mu_n}x_n, \alpha_n Jx_0 + (1-\alpha_n) JT_{\mu_n}x_n\right\rangle \\
&\quad + \left\|\alpha_n Jx_0 + (1-\alpha_n) JT_{\mu_n}x_n\right\|^2 \\
&\leq \left\|T_{\mu_n}x_n\right\|^2 - 2\alpha_n\left\langle T_{\mu_n}x_n, Jx_0\right\rangle \\
&\quad - 2(1-\alpha_n)\left\langle T_{\mu_n}x_n, JT_{\mu_n}x_n\right\rangle \\
&\quad + \alpha_n\|x_0\|^2 + (1-\alpha_n)\left\|T_{\mu_n}x_n\right\|^2 \\
&= \alpha_n\phi\left(T_{\mu_n}x_n, x_0\right) + (1-\alpha_n)\phi\left(T_{\mu_n}x_n, T_{\mu_n}x_n\right) \\
&= \alpha_n\phi\left(T_{\mu_n}x_n, x_0\right)
\end{aligned} \tag{41}$$

and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\phi\left(T_{\mu_n}x_n, z_n\right) = 0. \tag{42}$$

From (40), we get

$$\operatorname{Lim}_{n\to\infty}\phi\left(T_{\mu_n}x_n, x_{n+1}\right) = \lim_{n\to\infty}\phi\left(x_{n+1}, z_n\right) = 0. \tag{43}$$

By Lemma 5, we obtain

$$\lim_{n\to\infty}\left\|T_{\mu_n}x_n - x_{n+1}\right\| = \lim_{n\to\infty}\left\|x_{n+1} - z_n\right\| = 0. \tag{44}$$

Since $\|x_n - T_{\mu_n}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| + \|z_n - T_{\mu_n}x_n\|$, from (36), (39), and (44), we have

$$\lim_{n\to\infty}\left\|x_n - T_{\mu_n}x_n\right\| = 0. \tag{45}$$

From Lemma 7, we have $x_n \in F(\mathfrak{I})$. Since $F(\mathfrak{I})$ is closed and $\lim_{n\to\infty}x_n = v$, we have $v \in F(\mathfrak{I})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{I})}x_n$. □

We now establish a convergence theorem of modified Ishikawa type.

Theorem 9. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\alpha_n, \beta_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered}
x_0 \in C, \quad \textit{chosen arbitrarily}, \\
y_n = J^{-1}\left(\beta_n Jx_n + (1-\beta_n) JT_{\mu_n}x_n\right), \\
x_{n+1} = \Pi_C J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right), \quad \forall n \geq 0.
\end{gathered} \tag{46}$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$.

Proof. Firstly, we show that $\{x_n\}$ converges strongly in C.

From Lemma 3, we know $F(T)$ is closed and convex. So, we can define the generalized projection Π_C onto $F(\mathfrak{I})$. Let $u \in F(\mathfrak{I})$. From the definition of relatively nonexpansive and the convexity of $\|\cdot\|^2$, from (21), we have

$$\begin{aligned}
\phi(u, y_n) &= \phi\left(u, J^{-1}\left(\beta_n Jx_n + (1-\beta_n) JT_{\mu_n}x_n\right)\right) \\
&\leq \beta_n\phi(u, x_n) + (1-\beta_n)\phi\left(u, T_{\mu_n}x_n\right) \\
&\leq \phi(u, x_n),
\end{aligned} \tag{47}$$

for all $u \in F(\mathfrak{I})$. From (47), we obtain

$$\begin{aligned}
\phi(u, x_{n+1}) &= \phi\left(u, \Pi_C J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right)\right) \\
&\leq \phi\left(u, J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right)\right) \\
&\leq \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi\left(u, T_{\mu_n}y_n\right) \\
&\leq \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, y_n) \\
&\leq \phi(u, x_n).
\end{aligned} \tag{48}$$

Hence, $\{\phi(u, x_n)\}$ is bounded and $\lim_{n\to\infty}\phi(u, x_n)$ exists. This implies that $\{x_n\}$, $\{T_{\mu_n}x_n\}$, and $\{y_n\}$ are bounded. Since the interior of $F(\mathfrak{I})$ is nonempty, similar to the proof of Theorem 8, we obtain that $\{x_n\}$ converges strongly to v in C.

Next, we show that $v \in F(\mathfrak{I})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{I})}x_n$. Let

$$z_n = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right). \tag{49}$$

From Lemma 4, we have

$$\begin{aligned}
\phi(x_n, x_{n+1}) + \phi(x_{n+1}, z_n) &= \phi(x_n, \Pi_C z_n) + \phi(\Pi_C z_n, z_n) \\
&\leq \phi(x_n, z_n).
\end{aligned} \tag{50}$$

Also,

$$\begin{aligned}
\phi(x_n, z_n) &= \phi\left(x_n, J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right)\right) \\
&\leq \alpha_n\phi(x_n, x_n) + (1-\alpha_n)\phi\left(x_n, T_{\mu_n}y_n\right) \\
&\leq \alpha_n\phi(x_n, x_n) + (1-\alpha_n)\phi(x_n, y_n) \\
&\leq \phi(x_n, y_n),
\end{aligned} \tag{51}$$

$$\begin{aligned}
\|Jx_n - Jy_n\| &= \left\|Jx_n - \left(\beta_n Jx_n - (1-\beta_n) JT_{\mu_n}x_n\right)\right\| \\
&= (1-\beta_n)\left\|Jx_n - JT_{\mu_n}x_n\right\|.
\end{aligned} \tag{52}$$

From $\lim_{n\to\infty}\beta_n = 1$ and (52), we have

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{53}$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{54}$$

Hence,

$$\begin{aligned}\phi(x_n, y_n) &= \|x_n\|^2 - 2\langle x_n, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, Jy_n - Jx_n\rangle \\ &\quad - 2\langle x_n, Jx_n\rangle + \|y_n\|^2 \\ &\le \|y_n\|^2 - \|x_n\|^2 \\ &\quad + 2\|x_n\| \|Jy_n - Jx_n\| \\ &\le \|y_n - x_n\| (\|y_n\| + \|x_n\|) \\ &\quad + 2\|x_n\| \|Jy_n - Jx_n\|.\end{aligned} \tag{55}$$

By (53) and (54), we have

$$\lim_{n\to\infty} \phi(x_n, y_n) = 0. \tag{56}$$

From (50) and (51), we obtain

$$\lim_{n\to\infty} \phi(x_n, x_{n+1}) = \lim_{n\to\infty} \phi(x_n, z_n) = 0. \tag{57}$$

From Lemma 5, we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{58}$$

Since

$$\begin{aligned}\left\|Jz_n - JT_{\mu_n}y_n\right\| &= \left\|\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n - JT_{\mu_n}y_n\right\| \\ &= \alpha_n \left\|Jx_n - JT_{\mu_n}x_n\right\|\end{aligned} \tag{59}$$

and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty} \left\|Jz_n - JT_{\mu_n}y_n\right\| = 0. \tag{60}$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \left\|z_n - T_{\mu_n}y_n\right\| = 0. \tag{61}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and J is uniformly norm-to-norm continuous,

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0. \tag{62}$$

By (46) and (49), we have

$$\begin{aligned}JT_{\mu_n}x_n &= \frac{1}{1-\beta_n}(Jy_n - \beta_n Jx_n), \\ JT_{\mu_n}y_n &= \frac{1}{1-\alpha_n}(Jz_n - \alpha_n Jx_n).\end{aligned} \tag{63}$$

From (63), we obtain

$$\begin{aligned}&\left\|JT_{\mu_n}x_n - JT_{\mu_n}y_n\right\| \\ &\quad = \left\|\frac{1}{1-\beta_n}(Jy_n - \beta_n Jx_n) - \frac{1}{1-\alpha_n}(Jz_n - \alpha_n Jx_n)\right\| \\ &\quad = \left\|Jy_n + \frac{\beta_n}{1-\beta_n}(Jy_n - Jx_n) - \left(Jz_n + \frac{\alpha_n}{1-\alpha_n}(Jz_n - Jx_n)\right)\right\| \\ &\quad \le \|Jy_n - Jx_n\| + \|Jx_n - Jz_n\| \\ &\qquad + \frac{\beta_n}{1-\beta_n}\|Jy_n - Jx_n\| + \frac{\alpha_n}{1-\alpha_n}\|Jz_n - Jx_n\| \\ &\quad = \frac{1}{1-\alpha_n}\|Jz_n - Jx_n\| + \frac{1}{1-\beta_n}\|Jy_n - Jx_n\|.\end{aligned} \tag{64}$$

Combining (53), (62), and (64), we get

$$\lim_{n\to\infty} \left\|JT_{\mu_n}x_n - JT_{\mu_n}y_n\right\| = 0. \tag{65}$$

Since J^{-1} is uniformly norm-to-norm continuous, we have

$$\lim_{n\to\infty} \left\|T_{\mu_n}x_n - T_{\mu_n}y_n\right\| = 0. \tag{66}$$

Since

$$\begin{aligned}\left\|x_n - T_{\mu_n}x_n\right\| &\le \|x_n - z_n\| + \left\|z_n - T_{\mu_n}y_n\right\| \\ &\quad + \left\|T_{\mu_n}y_n - T_{\mu_n}x_n\right\|,\end{aligned} \tag{67}$$

therefore, by (58), (61), (66), and (67), we obtain

$$\lim_{n\to\infty} \left\|x_n - T_{\mu_n}x_n\right\| = 0. \tag{68}$$

From Lemma 7, we have $x_n \in F(\mathfrak{J})$. Since $F(\mathfrak{J})$ is closed and $\lim_{n\to\infty}x_n = v$, we have $v \in F(\mathfrak{J})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{J})}x_n$. □

If we set $\beta_n = 1$, then the iteration (46) reduces modified Mann type. Hence we obtain the following corollary.

Corollary 10. *Let S be a left reversible semigroup and let $\mathfrak{J} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence of real number such that $\alpha_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned}&x_0 \in C, \quad \textit{chosen arbitrarily}, \\ &x_{n+1} = \Pi_C J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}x_n\right), \quad \forall n \ge 0.\end{aligned} \tag{69}$$

If the interior of $F(\mathfrak{J})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{J})$.

In a Hilbert space, J is the identity operator. Theorems 8 and 9 reduce to the following.

Corollary 11. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a Hilbert space H into itself. Let X be a subspace of $l^{\infty}(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered} x_0 \in C, \quad \text{chosen arbitrarily}, \\ x_{n+1} = P_C\left(\alpha_n x_0 + (1-\alpha_n) T_{\mu_n} x_n\right), \quad \forall n \geq 0. \end{gathered} \tag{70}$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$, where P_C is a metric projection.

Corollary 12. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a Hilbert space H into itself. Let X be a subspace of $l^{\infty}(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\alpha_n, \beta_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n) T_{\mu_n} x_n, \\ x_{n+1} = P_C\left(\alpha_n x_n + (1-\alpha_n) T_{\mu_n} y_n\right), \quad \forall n \geq 0. \end{gathered} \tag{71}$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$, where P_C is a metric projection.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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