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Research Article

Some Connections between Class $\mathcal{U}$- and α-Convex Functions

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The class $\mathcal{U}(\lambda,\mu)$ of normalized analytic functions that satisfy $|(z/f(z))^{1+\mu}\cdot f'(z)-1|<\lambda$ for all z in the open unit disk is studied and sufficient conditions for an α-convex function to be in $\mathcal{U}(\lambda,\mu)$ are given.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the unit disk $\mathbb{D}=\{z:|z|<1\}$ with the normalization $f(0)=0$ and $f'(0)=1$.

For a function $f(z)\in\mathcal{A}$, we say that f is starlike of order α, $0\le\alpha<1$, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad z\in\mathbb{D}. \tag{1}$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all such functions. Also, we denote by $\mathcal{K}(\alpha)$ the class of convex functions of order α, $0\le\alpha<1$, that is, the class of functions $f(z)\in\mathcal{A}$ for which

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad z\in\mathbb{D}. \tag{2}$$

For $\alpha=0$, we have the classes of $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions, respectively. All of the above classes are subclasses of the class of univalent functions in $\mathbb{D}$ and even more, $\mathcal{K}\subset\mathcal{S}^*$. For details see [1].

Further, for $f(z)\in\mathcal{A}$ and $\mu\in\mathbb{C}$, let us define the operator

$$U(f,\mu;z)=\left(\frac{z}{f(z)}\right)^{1+\mu}\cdot f'(z) \tag{3}$$

and the class

$$\mathcal{U}(\lambda,\mu)=\left\{f\in\mathcal{A}:\frac{z}{f(z)}\neq 0,\ \left|U(f,\mu;z)-1\right|<\lambda,\ z\in\mathbb{D}\right\}. \tag{4}$$

This class and its special cases $\mathcal{U}(\lambda)\equiv\mathcal{U}(\lambda,1)$ and $\mathcal{U}\equiv\mathcal{U}(1)=\mathcal{U}(1,1)$ are widely studied in the past decades ([2–12]). It is known [2, 12] that functions in $\mathcal{U}(\lambda)$ are univalent if $0<\lambda\le 1$, but not necessarily univalent if $\lambda>1$. Further, Fournier and Ponnusamy [3] proved that assuming Re $\mu<1$ the following equivalency holds:

$$\mathcal{U}(\lambda,\mu)\subset\mathcal{S}^*\Longleftrightarrow 0<\lambda\le\frac{|1-\mu|}{\sqrt{(1-\mu)^2+\mu^2}}; \tag{5}$$

that is, in general case, $\mathcal{U}(\lambda,\mu)$ is not a subset of $\mathcal{S}^*$. In particular,

$$\mathcal{U}(1,\mu)\subset\mathcal{S}^*\Longleftrightarrow\mu=0; \tag{6}$$

that is, $\mathcal{U}\not\subseteq\mathcal{S}^*$, which can be also verified by the function

$$f(z)=\frac{z}{1+(1/2)z+(1/2)z^3}\in\mathcal{U}\setminus\mathcal{S}^*. \tag{7}$$

Finally, let us consider the classes

$$\mathcal{M}(\alpha,\gamma)=\{f\in\mathcal{A}:\ \operatorname{Re}J(f,\alpha;z)>\gamma,\ z\in\mathbb{D}\},$$
$$\mathcal{M}'(\alpha,\beta)=\{f\in\mathcal{A}:|J(f,\alpha;z)-1|<\beta,\ z\in\mathbb{D}\},\tag{8}$$

where

$$J(f,\alpha;z)\equiv(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right),\tag{9}$$

$\alpha,\gamma\in\mathbb{R}$, and $\beta>0$. These classes make a "bridge" between the classes of starlike and convex functions (of some order). The class $\mathcal{M}(\alpha,\gamma)$ is in fact the class of α-convex functions of order γ and $\mathcal{M}'(\alpha,\beta)$, in the case when $0<\beta\le 1$ is a subclass of $\mathcal{M}(\alpha,\gamma)$. Further, α-convex functions of some order are also starlike ([13], page 10). Therefore, it gives rise to the question (which is studied in this paper) of finding the sufficient conditions for

$$f\in\mathcal{U}\left(\lambda,-\frac{1}{\alpha}\right)\cap\mathcal{M}(\alpha,\gamma),$$
$$f\in\mathcal{M}'(\alpha,\beta)\Longrightarrow f\in\mathcal{U}\left(\lambda,-\frac{1}{\alpha}\right).\tag{10}$$

Let $f(z)$ and $g(z)$ be analytic in the unit disk. We say that $f(z)$ is subordinate to $g(z)$, and we write $f(z)\prec g(z)$; if $g(z)$ is univalent in $\mathbb{D}$, then $f(0)=g(0)$ and $f(\mathbb{D})\subseteq g(\mathbb{D})$. Further, we use the method of differential subordination introduced by Miller and Mocanu [14]. In fact, if $\phi:\mathbb{C}^2\to\mathbb{C}$ ($\mathbb{C}$ is the complex plane) is analytic in domain $D\subset\mathbb{C}$, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z),zp'(z))\in D$, when $z\in\mathbb{D}$, then we say that $p(z)$ satisfies a first-order differential subordination if

$$\phi\left(p(z),zp'(z)\right)\prec h(z).\tag{11}$$

The univalent function $q(z)$ is called a dominant of the differential subordination (11) if $p(z)\prec q(z)$ for all $p(z)$ satisfying (11). If $\tilde{q}(z)$ is a dominant of (11) and $\tilde{q}(z)\prec q(z)$ for all dominants of (11), then we say that $\tilde{q}(z)$ is the best dominant of the differential subordination (11).

We will make use of the following lemma.

Lemma 1 (see [15]). *Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w)\neq 0$ when $w\in q(\mathbb{D})$. Set $Q(z)=zq'(z)\phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$ and suppose that*

(i) *Q is starlike in the unit disk $\mathbb{D}$;*

(ii) $\operatorname{Re}(zh'(z))/Q(z)=\operatorname{Re}[(\theta'(q(z))/\phi(q(z)))+(zQ'(z)/Q(z))]>0$, $z\in\mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0)=q(0)$, $p(\mathbb{D})\subseteq D$, and

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z))=h(z),\tag{12}$$

then $p(z)\prec q(z)$, and q is the best dominant of (12).

2. Main Results and Consequences

Now we will prove the following theorem that will further lead to connections between class $\mathcal{U}(\lambda,\mu)$ and classes $\mathcal{M}'(\alpha,\beta)$ and $\mathcal{M}(\alpha,\gamma)$.

Theorem 2. *Let $f\in\mathcal{A}$, $0<\lambda\le 1$ and $\alpha\neq 0$. If $f'(z)\neq 0$ for all $z\in\mathbb{D}$, and if*

$$J(f,\alpha;z)\prec 1+\frac{\alpha\lambda z}{1+\lambda z}\equiv h(z),\tag{13}$$

then

$$U\left(f,-\frac{1}{\alpha};z\right)\prec 1+\lambda z;\tag{14}$$

that is, $f\in\mathcal{U}(\lambda,-1/\alpha)$, and $1+\lambda z$ is the best dominant of (13).

Proof. Let $p(z)=U(f,-1/\alpha;z)=(z/f(z))^{1-1/\alpha}\cdot f'(z), q(z)=1+\lambda z,\theta(\omega)=1$, and $\phi(\omega)=\alpha/\omega$, where $\omega\in(\mathbb{D})$. Then $q(z)$ is univalent in $\mathbb{D},\theta(\omega)$ and $\phi(\omega)$ are analytic in domain $D=\mathbb{C}\setminus\{0\}$ which contains $q(\mathbb{D})=\{1+z:|z|<\lambda\}$ ($q(z)=1+\lambda z$ and $\mathbb{D}$ is the unit disk), and $\phi(\omega)\neq 0$ when $\omega\in q(\mathbb{D})$. On the other hand, let

$$Q(z)=zq'(z)\phi(q(z))=z\lambda\frac{\alpha}{q(z)}=\frac{\alpha\lambda z}{1+\lambda z}=h(z)-1.\tag{15}$$

Then

$$\frac{zQ'(z)}{Q(z)}=\frac{zh'(z)}{Q(z)}=\frac{1}{1+\lambda z},$$
$$\operatorname{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\frac{1}{q(z)}>\frac{1}{1+\lambda}>0,\quad z\in\mathbb{D}.\tag{16}$$

The last inequality holds since $\operatorname{Re}q(z)=1+\lambda\cdot\operatorname{Re}z<1+\lambda$ for all $z\in\mathbb{D}$ and $q(\mathbb{D})$ does not contain the origin. So, conditions (i) and (ii) from Lemma 1 are satisfied.

Further, p is analytic in $\mathbb{D}$ and $p(0)=q(0)=1$. Also, $p(z)\neq 0$ for all $z\in\mathbb{D}$; that is, $p(\mathbb{D})\subseteq D$, since $f'(z)\neq 0$ for all $z\in\mathbb{D}$ (condition of the theorem); $z/f(z)=1\neq 0$ for $z=0$ (because $f\in\mathcal{A}$) and $f(z)$ has no poles on $\mathbb{D}$. Hence from Lemma 1 and the fact that

$$J(f,\alpha;z)=1+\alpha\cdot\frac{zp'(z)}{p(z)}=\theta(p(z))+zp'(z)\phi(p(z))\prec\frac{\alpha\lambda z}{1+\lambda z}=\theta(q(z))+zq'(z)\phi(q(z)),\tag{17}$$

we receive that $p(z)\prec q(z)$, that is, relation (14), and we also receive that $q(z)$ is the best dominant of (13). □

Applying the definition of subordination to Theorem 2, we receive the following.

Corollary 3. *Let* $f \in \mathcal{A}$, $0 < \lambda \le 1$, *and* $\alpha \ne 0$. *Also, let* $f(z)/z \ne 0$ *and* $f'(z) \ne 0$ *for all* $z \in \mathbb{D}$. *If*

$$\left|J(f,\alpha;z) - 1\right| < \frac{\lambda|\alpha|}{1+\lambda}, \quad z \in \mathbb{D}, \tag{18}$$

then $f \in \mathcal{U}(\lambda, -1/\alpha)$; *that is,*

$$\left|U\left(f, -\frac{1}{\alpha}; z\right) - 1\right| < \lambda, \quad z \in \mathbb{D}. \tag{19}$$

This result is sharp; that is, the constant λ *in inequality* (19) *cannot be replaced by a smaller one such that the implication still holds.*

Proof. First, let us note that the function h defined by expression (13) is univalent in the unit disk such that

$$\min\{|h(z) - 1| : |z| = 1\} = \frac{\lambda|\alpha|}{1+\lambda}. \tag{20}$$

So, the disk $\{w : |w-1| < \lambda|\alpha|/(1+\lambda)\}$ is contained in $h(\mathbb{D})$ which, having in mind the definition of subordination, means that inequality (18) implies subordination (13). Further, from Theorem 2 follows subordination (14), which is equivalent to the inequality (19). Even more, Theorem 2 says that $q(z) = 1 + \lambda z$ is the best dominant of (13).

In order to prove the sharpness of the result let us assume the opposite; that is, there exists λ_*, $0 < \lambda_* < \lambda$, such that inequality (18) implies

$$\left|U\left(f, -\frac{1}{\alpha}; z\right) - 1\right| < \lambda_*, \quad z \in \mathbb{D}; \tag{21}$$

that is,

$$U\left(f, -\frac{1}{\alpha}; z\right) \prec 1 + \lambda_* z \equiv q_*(z). \tag{22}$$

On the other hand, inequality (18) implies subordination (13) with best dominant $q(z)$, meaning that $q(z) \prec q_*(z)$. This is a contradiction to the assumption $\lambda_* < \lambda$ which proves the sharpness of the result. □

Previous corollary can be written in the following, equivalent, form that gives conditions for inclusion of the class $\mathcal{M}'(\alpha,\beta)$ into the class $\mathcal{U}(\lambda, -1/\alpha)$.

Corollary 4. *Let* $f \in \mathcal{A}$, $0 < \lambda \le 1$, $\alpha \ne 0$, *and* $\beta = \lambda|\alpha|/(1+\lambda)$. *Also, let* $f(z)/z \ne 0$ *and* $f'(z) \ne 0$ *for all* $z \in \mathbb{D}$. *Then*

$$f \in \mathcal{M}'(\alpha,\beta) \Longrightarrow f \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right). \tag{23}$$

The constant λ, *for the class* $\mathcal{U}(\lambda, -1/\alpha)$, *cannot be replaced by a smaller one such that the inclusion still holds.*

Next result gives the connection between classes $\mathcal{M}(\alpha,\gamma)$ and $\mathcal{U}(\lambda, -1/\alpha)$.

Corollary 5. *Let* $f \in \mathcal{A}$, $0 < \lambda \le 1$, $\alpha \ne 0$, *and*

$$\gamma := \begin{cases} 1 + \alpha - \dfrac{\alpha}{1-\lambda}, & \text{if } \alpha > 0,\ \lambda \ne 1, \\[2mm] 1 + \alpha - \dfrac{\alpha}{1+\lambda}, & \text{if } \alpha < 0. \end{cases} \tag{24}$$

Also, let $f(z)/z \ne 0$ *and* $f'(z) \ne 0$ *for all* $z \in \mathbb{D}$. *If subordination* (13) *holds, then*

$$f \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right) \cap \mathcal{M}(\alpha,\gamma). \tag{25}$$

Proof. It is easy to check that all conditions of Theorem 2 are fulfilled; hence, $f \in \mathcal{U}(\lambda, -1/\alpha)$.

It remains to verify that $f \in \mathcal{M}(\alpha,\gamma)$; that is, subordination (13) implies

$$\operatorname{Re} J(f,\alpha;z) > \gamma, \quad z \in \mathbb{D}. \tag{26}$$

Having in mind the definition of subordination and the fact that $h(z) = 1 + (\alpha\lambda z)/(1+\lambda z)$ is univalent, it is enough to show that $\operatorname{Re}\ h(z) \ge \gamma$ for all $z \in \mathbb{D}$. The last is true because

$$\begin{aligned} \inf\{\operatorname{Re} h(z) : z \in \mathbb{D}\} &= \min\{\operatorname{Re} h(z) : |z| = 1\} \\ &= \min\{h(1), h(-1)\} = \gamma. \end{aligned} \tag{27}$$

□

Remark 6. The case $\alpha > 0$ and $\lambda = 1$ is not covered by the previous corollary since then $\inf\{\operatorname{Re}\ h(z) : z \in \mathbb{D}\} = -\infty$.

3. Examples

Now we will apply the results from the previous section on specific functions $f \in \mathcal{A}$ and receive interesting conclusions.

Example 1. Let $0 < \lambda \le 1$, $\alpha \ne 0$, and $a \in \mathbb{R}$. Consider the following.

(i) Let

$$\gamma := \begin{cases} 1 + \alpha - \dfrac{\alpha}{1-\lambda}, & \text{if } \alpha > 0,\ \lambda \ne 1 \\[2mm] 1 + \alpha - \dfrac{\alpha}{1+\lambda}, & \text{if } \alpha < 0. \end{cases} \tag{28}$$

If $a \ne 0$, $a \ne -1$, $\lambda \le |1+a|$, $a\alpha > 0$, and $|a| > |\alpha|$, then

$$f(z) = z \cdot \left(1 + \frac{\lambda}{1+a} z\right)^a \in \mathcal{U}\left(\lambda, -\frac{1}{\alpha}\right) \cap \mathcal{M}(\alpha,\gamma). \tag{29}$$

(ii) If $|a| < 1$ and one of the following two sets of conditions holds:

$$\alpha > 0, \quad |a|(1+\lambda)(1-|a|+\alpha) < \lambda\alpha(1-|a|) \tag{30}$$

or

$$\alpha < 0, \quad |a|(1+\lambda)(1+|a|+\alpha) < \lambda|\alpha|(1+|a|), \tag{31}$$

then

$$f(z) = z \cdot e^{az} \in \mathscr{U}\left(\lambda, -\frac{1}{\alpha}\right). \tag{32}$$

In both cases, power is taken by its principal value.

Proof. (i) For the function $f(z) = z(1 + (\lambda/(1 + a))z)^a$, we have $f(0) = 0$ and

$$f'(z) = \left(1 + \frac{\lambda}{1+a} z\right)^a \cdot \left(1 + \frac{\alpha\lambda z}{1 + \alpha + \lambda z}\right) \Longrightarrow f'(0) = 1. \tag{33}$$

Condition $\lambda \le |1 + a|$ guarantees that $1 + \alpha + \lambda z \ne 0$ for all $z \in \mathbb{D}$; hence, f is an analytic function and $f \in \mathscr{A}$. For the function f, it is easy to verify that

$$\begin{aligned} J(f, \alpha; z) &\equiv (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &= 1 + \frac{a\lambda z}{1 + \lambda z}. \end{aligned} \tag{34}$$

Further, from the definition of subordination, we have that $az \prec \alpha z$ when $|a| > |\alpha|$ and $a\alpha > 0$; that is, both are positive or both negative. Therefore,

$$J(f, \alpha; z) = 1 + \frac{a\lambda z}{1 + \lambda z} \prec 1 + \frac{\alpha\lambda z}{1 + \lambda z}. \tag{35}$$

So, all conditions of Corollary 5 bring us to the conclusion that $f \in \mathscr{U}(\lambda, -1/\alpha) \cap \mathscr{M}(\alpha, \gamma)$.

(ii) It is easy to verify that $f \in \mathscr{A}$ and that

$$\begin{aligned} J(f, \alpha; z) &\equiv (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &= 1 + az\left(1 + \frac{\alpha}{1 + az}\right). \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} \Delta &\equiv \sup\left\{\left|J(f, \alpha; z) - 1\right| : z \in \mathbb{D}\right\} \\ &= \max\left\{\left|J(f, \alpha; z) - 1\right| : |z| = 1\right\} \\ &= |a| \cdot \max\left\{1 + \frac{\alpha}{1 + az} : z \in \mathbb{D}\right\} \\ &= |a| \cdot \begin{cases} 1 + \dfrac{\alpha}{1 - |a|}, & \text{if } \alpha > 0, \\ 1 + \dfrac{\alpha}{1 + |a|}, & \text{if } \alpha < 0. \end{cases} \end{aligned} \tag{37}$$

Further, if one of the conditions (30) or (31) holds, then $\Delta \le (\lambda|\alpha|)/(1 + \lambda)$; that is, we receive inequality (18). Finally, we have shown that all conditions of Corollary 3 are fulfilled, which leads to $f(z) \in \mathscr{U}(\lambda, -1/\alpha)$. □

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values of λ and/or α.

Example 2. Let $f \in \mathscr{A}$, $0 < \lambda \le 1$ and let $f(z)/z \ne 0$ and $f'(z) \ne 0$ for all $z \in \mathbb{D}$. Consider the following:

(i) $f \in \mathscr{M}'(\alpha, |\alpha|/2) \Rightarrow f \in \mathscr{U}(1, -1/\alpha)$ ($\lambda = 1$ in Corollary 4);

(ii) $f \in \mathscr{U}(1, -1/\alpha) \cap \mathscr{M}(\alpha, 1 + \alpha/2)$ ($\lambda = 1$ and $\alpha < 0$ in Corollary 5).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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