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Research Article

Existence of Solutions for a Coupled System of Second and Fourth Order Elliptic Equations

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The generalized quasilinearization technique is applied to obtain a monotone sequence of iterates converging uniformly and quadratically to a solution of a coupled system of second and fourth order elliptic equations.

1. Introduction

The object of this paper is to study the existence of solutions of the following elliptic system:

$$\begin{aligned} \Delta^2 u &= f_1(x,u,v), \quad \text{in } \Omega, \\ -\Delta v &= f_2(x,u,v), \quad \text{in } \Omega, \\ u &= \Delta u = v = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

This problem is related to a coupled system of second and fourth ordinary differential equations:

$$\begin{aligned} &y'''' = f_1(x,y,z), \quad \text{in } (0,L), \\ &-z'' = f_2(x,y,z), \quad \text{in } (0,L), \\ &y(0) = y(L) = y''(0) = y''(L) = 0, \\ &z(0) = z(L) = 0, \end{aligned} \tag{2}$$

which is yielded from the steady state of the following Lazer-McKenna suspension bridge models proposed in [1]:

$$\begin{aligned} &m_1 y_{tt} + a^2 y_{xxxx} + \delta_1 y_t + k(y-z)^+ \\ &\quad = W(x), \quad \text{in } (0,L)\times R, \\ &m_2 z_{tt} - b^2 z_{xx} + \delta_2 z_t - k(y-z)^+ \\ &\quad = h(x,t), \quad \text{in } (0,L)\times R, \\ &y(0,t) \\ &\quad = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0, \quad t\in R, \\ &z(0,t) \\ &\quad = z(L,t) = 0, \quad t \in R. \end{aligned} \tag{3}$$

Because of the important background, in [2], Ru and An presented the existence of single and multiple positive solutions of the 2p-order and 2q-order nonlinear ordinary differential systems by using the fixed-point theorem of cone expansion and compression type due to Krasnosel'skill; in [3], the authors studied the existence, multiplicity, and nonexistence of positive solutions for 2p-order and 2q-order systems of singular boundary value problems with integral boundary conditions; in [4], the author studied the existence, nonexistence, and multiplicity of solutions of an Ambrosetti-Prodi type problem for a system of second and fourth order ordinary differential equations by the variational method. In addition, in [5], An dealt with the maximum principle for the coupled system of second and fourth order elliptic equations:

$$\begin{aligned} \Delta^2 u &= a(x)u + b(x)v + f_1(x), \quad \text{in } \Omega, \\ -\Delta v &= c(x)u + d(x)v + f_2(x), \quad \text{in } \Omega, \\ u &= \Delta u = v = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

where the functions $b(x), c(x)$ are supposed to be nonnegative such that 2×2-matrix $A = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ is cooperative. In addition, using the fixed point theorem, the author obtained the existence of solution for the following system:

$$\begin{aligned} \Delta^2 u &= a(x)u + b(x)v + f_1(x,u,v), \quad \text{in } \Omega, \\ -\Delta v &= c(x)u + d(x)v + f_2(x,u,v), \quad \text{in } \Omega, \\ u &= \Delta u = v = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{5}$$

Inspired by the above references, the aim of this paper is to study the existence of solutions for the elliptic system (1) by using the method of upper and lower solutions and the generalized quasilinearization technique on the basis of the maximum principle for the linear elliptic system (4) built by An in [5]. The quasilinearization method is pioneered by Bellaman and Kalaba [6] and generalized by Lakshmikantham and Vatsala [7]. The quasilinearization method has been applied to a variety of problems. We refer the readers to [8–12].

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we obtain a monotone sequence of approximate solutions converging uniformly and quadratically to a solution of (1).

2. Preliminaries

Let $\Omega \subset R^N$ be a bounded smooth domain. H denotes the Hilbert space

$$H = L^2(\Omega) \times L^2(\Omega) \tag{6}$$

with the inner production $(u,v)_H = \int_\Omega (u_1v_1 + u_2v_2)dx$. The corresponding norm is $\|u\|_H^2 = \|u_1\|_{L^2}^2 + \|u_1\|_{L^2}^2$; W denotes the Hilbert space

$$W = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) \tag{7}$$

with the inner production $(u,v)_W = \int_\Omega (|\Delta u_1 \Delta v_1| + |\nabla u_2 \nabla v_2|)dx$. The corresponding norm is $\|u\|_W^2 = \int_\Omega (|\Delta u_1|^2 + |\nabla u_2|^2)dx$.

Let λ_1 be the positive first eigenvalue of the second order eigenvalue problem

$$\begin{aligned} -\Delta v &= \lambda v, \quad \text{in } \Omega, \\ v &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{8}$$

Then λ_1^2 is the positive first eigenvalue of the fourth order eigenvalue problem

$$\begin{aligned} \Delta^2 u &= \lambda u, \quad \text{in } \Omega, \\ \Delta u &= u = 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{9}$$

From the Poincare inequality, for all $u \in W$, it follows that

$$\|u\|_W^2 \geq \underline{\lambda}\, \|u\|_H^2, \tag{10}$$

where $\underline{\lambda} = \min\{\lambda_1, \lambda_1^2\}$.

Lemma 1 (see [5]). *Assume that the cooperative matrix* $A = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ *with* $b(x) \geq 0$, $c(x) \geq 0$ *verifies condition*

$$(A\xi, \xi) < \underline{\lambda}\xi^2, \quad \forall \xi \in R^2. \tag{11}$$

Then (4) satisfies the maximum principle. By a maximum principle one means that if $F(x) = (f_1, f_2) \geq 0$, *then (4) has a nonnegative solution* $U = (u,v) \geq 0$.

Lemma 2. *Let* $f_1(x), f_2(x) \in L^\infty(\Omega)$. *Then (4) has a solution* $(u,v) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. *In addition, there exists a constant* C *such that*

$$\|(u,v)\|_{L^\infty} \leq C\left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}\right). \tag{12}$$

Proof. Let Ł denote the operator matrix

$$Ł = \begin{pmatrix} \Delta^2 & 0 \\ 0 & -\Delta \end{pmatrix} \tag{13}$$

and let $A(x)$ denote the coefficient matrix

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}. \tag{14}$$

Thus, (4) can be rewritten as

$$ŁU = A(x)U + F(x), \tag{15}$$

where $F(x) = (f_1(x), f_2(x))^T$. From Lemma 1 it follows that the operator matrix $(Ł - A(x))$ is reversible, and $(Ł - A(x))^{-1}$ is a compact positive operator. So $U = (Ł - A(x))^{-1}F(x)$. Furthermore, we can get $\|(u,v)\|_{L^\infty} \leq C(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty})$ for some constant C. □

3. Main Result

Letting $F(x,U) = \begin{pmatrix} f_1(x,u,v) \\ f_2(x,u,v) \end{pmatrix}$, then (1) is equivalent to

$$ŁU = F(x,U). \tag{16}$$

We introduce the following relation:

$$U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \geq U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{17}$$

which is ordered by the usual positive cone $K \subset E$, where E denotes the Banach space $\{U = (u,v) \in C^2(\overline{\Omega}) \times C^1(\overline{\Omega}) : \Delta u = u = v = 0, \text{ on } \partial\Omega\}$.

Definition 3. A function $U_* = (u_*, v_*) \in E$ is a lower solution of (16) if it satisfies

$$ŁU_* \leq F(x, U_*), \quad \text{in } D'(\Omega). \tag{18}$$

Definition 4. A function $U^* = (u^*, v^*) \in E$ is an upper solution of (16) if it satisfies

$$ŁU^* \geq F(x, U^*), \quad \text{in } D'(\Omega). \tag{19}$$

For convenience, we give some assumptions as follows.

(A1) Equation (16) has a pair of lower and upper solutions $U_*, U^* \in E$ with $U_* \le U^*$.

(A2) $F \in C(\overline{\Omega} \times R^2, R^2)$, $F(x,U)$ is quasimonotone nondecreasing in U and is convex in U uniformly in $x \in \overline{\Omega}$, $F_U(x,U)$ exists and is continuous on $R \times R^2$, and $F_U(x,U)$ is nondecreasing in U for $x \in \overline{\Omega}$.

(A3) The assumptions of Lemma 1 hold for $F_U(x,\Phi(x))$ with $U_*(x) \le \Phi(x) \le U^*(x)$, $x \in \overline{\Omega}$.

(A4) $|F_U(x,V) - F_U(x,U)| \le L|V-U|$, for some positive matrix L.

Theorem 5. *Assume that (A1)–(A4) hold. Then there exist two monotone sequences $\{U_n\}$, $\{V_n\}$, which converge uniformly to the solution of (1) or (16) in $\overline{\Omega}$ and the convergence is quadratic.*

Proof. By (A2), we have

$$F(x,V) \ge F(x,U) + F_U(x,U)(V-U), \tag{20}$$

where $U_* \le U \le V \le U^*$, $x \in \overline{\Omega}$. Define a function $G(x,Y)$ as follows:

$$G(x,Y) = F(x,\Phi(x)) + F_U(x,\Phi(x))(Y - \Phi(x)), \tag{21}$$

where $U_* \le \Phi(x) \le U^*$, $x \in \overline{\Omega}$. It is clear to see that $G(x,Y)$ is quasimonotone nondecreasing in Y for each $x \in \overline{\Omega}$.

From Lemma 1 and (A3), it follows that the linear systems

$$ŁU = F(x,U_*) + F_U(x,U_*)(U-U_*), \tag{22}$$

$$ŁV = F(x,U^*) + F_U(x,U^*)(V-U^*), \tag{23}$$

respectively, have a solution U_1, V_1.

Let $Z = U_1 - U_*$. Then, from (22), we have

$$\begin{aligned} ŁZ &= ŁU_1 - ŁU_* \\ &\ge [F(x,U_*) + F_U(x,U_*)(U_1 - U_*)] \\ &\quad - F(x,U_*) = F_U(x,U_*)Z. \end{aligned} \tag{24}$$

Using Lemma 1 and (A3), it implies $Z \ge 0$; namely, $U_* \le U_1$.

Let $Z = U^* - V_1$. Then, from (23), we get

$$\begin{aligned} ŁZ &= ŁU^* - ŁV_1 \\ &\ge F(x,U^*) \\ &\quad - [F(x,U^*) + F_U(t,x,U_*)(V_1 - U^*)] \\ &= F_U(x,U_*)Z. \end{aligned} \tag{25}$$

Using Lemma 1 and (A3), the above inequality implies $Z \ge 0$; namely, $V_1 \le U^*$.

Finally, let $Z = V_1 - U_1$. Then, using (20), (22), and (23), we can obtain

$$\begin{aligned} ŁZ &= ŁV_1 - ŁU_1 \\ &= F(x,U^*) + F_U(x,U_*)(V_1 - U^*) - F(x,U_*) \\ &\quad - F_U(x,U_*)(U_1 - U_*) \\ &= F_U(x,\xi)(U^* - U_*) + F_U(x,U_*)(V_1 - U^*) \\ &\quad - F_U(x,U_*)(U_1 - U_*) \\ &\ge F_U(x,U_*)(U^* - U_*) + F_U(x,U_*)(V_1 - U^*) \\ &\quad - F_U(x,U_*)(U_1 - U_*) \ge F_U(x,U_*)(V_1 - U_1), \end{aligned} \tag{26}$$

where $U_* \le \xi \le U^*$. Using Lemma 1 and (A3), it yields that $Z \ge 0$; namely, $U_1 \le V_1$. So, from the above discussions, we can get $U_* \le U_1 \le V_1 \le U^*$.

Continuing this process successively, then we can obtain two monotone sequences $\{U_n\}$, $\{V_n\}$ satisfying

$$U_* \le U_1 \le \cdots \le U_n \le V_n \le \cdots \le V_1 \le U^*, \quad \text{in } \overline{\Omega}, \tag{27}$$

where U_n, V_n are, respectively, the solutions of the following systems:

$$\begin{aligned} ŁU_n &= F(x,U_{n-1}) + F_U(x,U_{n-1})(U_n - U_{n-1}), \\ ŁV_n &= F(x,V_{n-1}) + F_U(x,U_{n-1})(V_n - V_{n-1}). \end{aligned} \tag{28}$$

From (27), it is clear to see that $\{U_n\}$, $\{V_n\}$ are uniformly bounded in W. Moreover, $\{U_n\}$ is nondecreasing sequence and therefore there exists a subsequence $\{U_{nk}\}$ which converges uniformly to $U = (u,v)$. By the dominated convergence, it is easy to verify that U is a solution of (1) in the distribution sense. Similarly, convergence holds for $\{V_n\}$.

Now we show that the convergence of $\{U_n\}$ and $\{V_n\}$ is quadratic. For this purpose, let $P_{n+1} = U - U_{n+1} \ge 0$, $Q_{n+1} = V_{n+1} - U \ge 0$. Then, using the mean value theorem, we have

$$\begin{aligned} ŁP_{n+1} &= Ł(U - U_{n+1}) \\ &= F(x,U) - F(x,U_n) \\ &\quad - F_U(x,U_n)(U_{n+1} - U_n) \\ &= \left(\int_0^1 F_U(x, sU + (1-s)U_n)\,ds\right)P_n \\ &\quad - F_U(x,U_n)(U_{n+1} - U + U - U_n) \\ &= \left(\int_0^1 F_U(x, sU + (1-s)U_n)\,ds\right)P_n \\ &\quad - F_U(x,U_n)(-P_{n+1} + P_n) \\ &\le (F_U(x,U) - F_U(x,U_n))P_n \\ &\quad + F_U(x,U_n)P_{n+1}. \end{aligned} \tag{29}$$

By (A3) and (A4), we get the inequality

$$ŁP_{n+1} \le F_U(x,U_n)P_{n+1} + L|P_n|^2, \tag{30}$$

where $|P_n|^2 = (|P_{(1,n)}|^2, |P_{(2,n)}|^2)$.

Let $W(x) = (\omega_1(x), \omega_2(x))$ be the unique solution of the linear problem

$$\begin{gathered} ŁW(x) = F_U(x, U_n) W(x) + L|P_n|^2, \quad \text{in } \Omega \\ \omega_1 = \omega_2 = 0, \\ \Delta\omega_1 = 0 \\ \text{on } \partial\Omega. \end{gathered} \tag{31}$$

From Lemma 1, (30), and (31), we know that $W(x) \geq P_{n+1}(x) \geq 0$. Furthermore, we have $\|P_{n+1}(x)\|_{L^\infty} \leq \|W(x)\|_{L^\infty}$. From Lemma 2, we know that $\|W(x)\|_{L^\infty} \leq C\|P_n\|^2_{L^\infty}$. Furthermore, we can obtain that $\|P_{n+1}\|_{L^\infty} \leq C\|P_n\|^2_{L^\infty}$.

In the similar way, we also have

$$\|Q_{n+1}\|_{L^\infty} \leq C\|Q_n\|^2_{L^\infty}. \tag{32}$$

Therefore, the proof is complete. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] A. C. Lazer and P. J. McKenna, "Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis," *SIAM Review*, vol. 32, no. 4, pp. 537–578, 1990.

[2] Y. Ru and Y. An, "Positive solutions for 2p-order and 2q-order nonlinear ordinary differential systems," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1093–1104, 2006.

[3] P. Kang, J. Xu, and Z. Wei, "Positive solutions for 2p-order and 2q-order systems of singular boundary value problems with integral boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 6, pp. 2767–2786, 2010.

[4] Y. An and J. Feng, "Ambrosetti-Prodi type results in a system of second and fourth-order ordinary differential equations," *Electronic Journal of Differential Equations*, vol. 2008, pp. 1–14, 2008.

[5] Y. An, "Maximum principles for a coupled system of second and fourth order elliptic equations and an application," *Applied Mathematics and Computation*, vol. 161, no. 1, pp. 121–127, 2005.

[6] R. Bellaman and R. Kalaba, *Quasilinearisation and Nonliear Boundary Value Problems*, American Elsevier, New York, NY, USA, 1965.

[7] V. Lakshmikantham and A. S. Vatsala, *Genearalized Quasilinearization for Nonlinear Problems*, Kluwer Academic, Boston, Mass, USA, 1998.

[8] G. A. Afrouzi and M. Alizadeh, "A quasilinearization method for p-Laplacian equations with a nonlinear boundary condition," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2829–2833, 2009.

[9] B. Ahmad and A. Alsaedi, "Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 1, pp. 358–367, 2009.

[10] P. Amster and P. De Nápoli, "A quasilinearization method for elliptic problems with a nonlinear boundary condition," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 10, pp. 2255–2263, 2007.

[11] M. El-Gebeily and D. O'Regan, "Upper and lower solutions and quasilinearization for a class of second order singular nonlinear differential equations with nonlinear boundary conditions," *Nonlinear Analysis: Real World Applications*, vol. 8, no. 2, pp. 636–645, 2007.

[12] V. Lakshmikantham, S. Carl, and S. Heikkilä, "Fixed point theorems in ordered Banach spaces via quasilinearization," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3448–3458, 2009.