Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 670287, 6 pages
http://dx.doi.org/10.1155/2014/670287

Research Article

Periodic Solutions for Second-Order Ordinary Differential Equations with Linear Nonlinearity

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Received 17 February 2013; Accepted 29 October 2013; Published 30 January 2014

Academic Editor: Pei Yu

By using minimax methods in critical point theory, we obtain the existence of periodic solutions for second-order ordinary differential equations with linear nonlinearity.

1. Introduction and Main Results

Consider the second-order ordinary differential systems

$$\begin{aligned} &\ddot{u}(t) + m^2\omega^2 u(t) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T], \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1}$$

where $T > 0$, $\omega = 2\pi/T$, m is a nonnegative integer; and $F : [0, T] \times R^N \to R$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$ and $b \in L^1([0, T], R^+)$ such that

$$|F(t, x)| \le a(|x|)\, b(t), \qquad |\nabla F(t, x)| \le a(|x|)\, b(t), \tag{2}$$

for all $x \in R^N$ and a.e. $t \in [0, T]$, where R^+ is the set of all nonnegative real numbers.

In the case of $m = 0$, the existence of periodic solutions for problem (1) is obtained in articles [1–17] with many solvability conditions, such as the coercive type potential condition (see [1]), the convex type potential condition (see [2]), the periodic type potential conditions (see [3]), the even type potential condition (see [4]), the subquadratic potential condition in Rabinowitz's sense (see [5]), the bounded nonlinearity condition (see [6]), the subadditive condition (see [7]), the sublinear nonlinearity condition (see [9, 15]), and the linear nonlinearity condition (see [13, 14, 16, 17]).

In the case of $m \neq 0$, Mawhin and Willem [6] prove that problem (1) has at least one solution under the bounded nonlinearity condition; that is, $|\nabla F(t, x)| \le g(t)$ for some $g \in L^1(0, T)$, each $x \in R^N$, and a.e. $t \in [0, T]$ when

$$\int_0^T F(t, a\cos m\omega t + b\sin m\omega t)\, dt \longrightarrow +\infty \text{ as } |(a, b)| \longrightarrow \infty \text{ in } R^{2N} \tag{3}$$

or

$$\int_0^T F(t, a\cos m\omega t + b\sin m\omega t)\, dt \longrightarrow -\infty \text{ as } |(a, b)| \longrightarrow \infty \text{ in } R^{2N}. \tag{4}$$

Under the sublinear nonlinearity condition, that is, there exist $f, g \in L^2[0, T]$ and $\alpha \in [0, 1)$, such that

$$|\nabla F(t, x)| \le f(t)\, |x|^{\alpha} + g(t), \tag{5}$$

for $x \in R^N$ and a.e. $t \in [0, T]$, Han [18] proves that problem (1) has at least one solution when

$$|(a, b)|^{-2\alpha} \int_0^T F(t, a\cos m\omega t + b\sin m\omega t)\, dt \longrightarrow +\infty \text{ as } |(a, b)| \longrightarrow \infty \text{ in } R^{2N} \tag{6}$$

or

$$|(a,b)|^{-2\alpha}\int_0^T F(t, a\cos m\omega t + b\sin m\omega t)\,dt \longrightarrow -\infty \text{ as } |(a,b)| \longrightarrow \infty \text{ in } R^{2N}. \quad (7)$$

Recently, when $m = 0$, Zhao and Wu [13, 14] and Meng and Tang [16, 17] also prove the existence of solutions for problem (1) under linear nonlinearity condition; that is, there exist $f, g \in L^1([0,T], R^+)$ such that

$$|\nabla F(t,x)| \le f(t)|x| + g(t). \quad (8)$$

In this paper, motivated by the results mentioned above, we investigate the existence of periodic solutions of problem (1) in the case of $m \ge 1$.

Let H_T^1 be a Hilbert space defined by

$$H_T^1 = \{u : [0,T] \longrightarrow R^N \mid u \text{ is absolutely continuous}, u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T)\}, \quad (9)$$

with the norm

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{1/2}, \quad (10)$$

for $u \in H_T^1$.

Let

$$H^0 = \{a\cos m\omega t + b\sin m\omega t : a \in R^N, b \in R^N\},$$

$$\overline{H} = \left\{\sum_{k=1}^{m-1} a_k\cos k\omega t + b_k \sin k\omega t : a_k \in R^N, b_k \in R^N, 1 \le k \le m-1\right\}, \quad (11)$$

$$\widetilde{H} = \left\{u \in H_T^1 : \int_0^T u(t)\cos k\omega t\,dt = \int_0^T u(t)\sin k\omega t\,dt = 0,\ 1 \le k \le m\right\};$$

then $H_T^1 = H^0 \oplus \overline{H} \oplus \widetilde{H}$ ([6]). For all $u \in H_T^1$, we have $u = u^0 + \overline{u} + \widetilde{u}$, where $u^0 \in H^0$, $\overline{u} \in \overline{H}$, and $\widetilde{u} \in \widetilde{H}$. It is easy to obtain

$$\|\dot{\overline{u}}\|_2^2 \le (m-1)^2\omega^2\|\overline{u}\|_2^2, \quad \forall \overline{u} \in \overline{H}, \quad (12)$$

$$\|\dot{\widetilde{u}}\|_2^2 \ge (m+1)^2\omega^2\|\widetilde{u}\|_2^2, \quad \forall \widetilde{u} \in \widetilde{H}. \quad (13)$$

Furthermore, we have $\|u\|_\infty \le C_0\|u\|$ for some $C_0 > 0$ and all $u(t) \in H_T^1$ (see, [6, Proposition 1.3]).

Our main results are the following theorems.

Theorem 1. *Suppose that (A) and* (8) *hold and*

(i)

$$(2+a)C_0^2\int_0^T f(t)\,dt < \min\left\{\frac{(2m+1)\omega^2}{1+(m+1)^2\omega^2}, \frac{(2m-1)\omega^2}{1+(m-1)^2\omega^2}\right\}, \quad (14)$$

where a is a parameter and satisfies $a > 1/2$;

(ii)

$$\lim_{u\in H^0, \|u\|\to\infty} \inf \|u\|^{-2}\int_0^T F(t,u)\,dt > C_0^2\int_0^T f(t)\,dt + \frac{5C_0^2}{2a-1}\int_0^T f(t)\,dt + \frac{1}{2a-1}. \quad (15)$$

Then problem (1) *has at least one solution.*

Theorem 2. *Suppose that (A),* (8) *and (i) hold and*

(iii)

$$\lim_{u\in H^0, \|u\|\to\infty} \sup \|u\|^{-2}\int_0^T F(t,u)\,dt < -\left[\frac{5C_0^2}{2a-1}\int_0^T f(t)\,dt + C_0^2\int_0^T f(t)\,dt + \frac{m^2\omega^2}{2a-1}\right]. \quad (16)$$

Then problem (1) *has at least one solution.*

Remark 3. (i) It is worth noting that, in the case of $m = 0$, one solution was obtained by Tang [9] and Han [15] under the sublinear nonlinearity condition.

(ii) It is also worth noting that the sublinear nonlinearity condition in [15, 18] is different from that of [9].

2. Proof of Main Results

Let

$$J(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{m^2\omega^2}{2} \times \int_0^T |u(t)|^2 dt - \int_0^T F(t, u(t))\,dt, \quad (17)$$

for any $u \in H_T^1$. It follows from assumption (A) that the functional J on H_T^1 is continuously differentiable; moreover we obtain

$$\langle J'(u), v\rangle = \int_0^T (\dot{u}(t), \dot{v}(t))\,dt - m^2\omega^2 \times \int_0^T (u(t), v(t))\,dt - \int_0^T (\nabla F(t, u(t)), v(t))\,dt \quad (18)$$

for any $u, v \in H_T^1$. It is well known that the solutions of problem (1) correspond to the critical points of J (see [6]).

For the sake of convenience, we denote

$$M_1 = \int_0^T f(t)\,dt, \qquad M_2 = \int_0^T g(t)\,dt. \tag{19}$$

Proof of Theorem 1. Firstly, we assert that the functional J satisfies (PS) condition. Let $\{u_n\}$ be a sequence in H_T^1 such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$. By the proof of [6] Proposition 4.1, we only need to prove that $\{u_n\}$ is bounded. On one hand, we have

$$\begin{aligned}
&\|\overline{u}_n\| \\
&\quad \geq \left\langle J'(\dot{u}_n), -\dot{\overline{u}}_n \right\rangle = -\int_0^T \left[\left(\dot{u}_n, \dot{\overline{u}}_n\right) - m^2\omega^2 (u_n, \overline{u}_n) \right. \\
&\qquad\qquad \left. - (\nabla F(t, u_n), \overline{u}_n) \right] dt \\
&\quad = -\int_0^T \left|\dot{\overline{u}}_n\right|^2 dt + m^2\omega^2 \\
&\qquad \times \int_0^T |\overline{u}_n|^2 dt + \int_0^T (\nabla F(t, u_n), \overline{u}_n)\, dt \\
&\quad \geq \left[m^2 - (m-1)^2\right]\omega^2 \\
&\qquad \times \int_0^T |\overline{u}_n|^2 dt - \int_0^T f(t) \left|u_n^0 + \overline{u}_n + \widetilde{u}_n\right| |\overline{u}_n|\, dt \\
&\qquad - \int_0^T g(t) |\overline{u}_n|\, dt \\
&\quad \geq \frac{(2m-1)\omega^2}{1 + (m-1)^2\omega^2} \|\overline{u}_n\|^2 - C_0^2 M_1 \|\overline{u}_n\|^2 \\
&\qquad - C_0^2 M_1 \|\overline{u}_n\| \|\widetilde{u}_n\| - C_0^2 M_1 \|\overline{u}_n\| \left\|u_n^0\right\| - C_0 M_2 \|\overline{u}_n\| \\
&\quad \geq \left(\frac{(2m-1)\omega^2}{1 + (m-1)^2\omega^2} - 2C_0^2 M_1 \right) \\
&\qquad \times \|\overline{u}_n\|^2 - \frac{C_0^2 M_1}{2} \|\widetilde{u}_n\|^2 - \frac{C_0^2 M_1}{2} \left\|u_n^0\right\|^2 - C_0 M_2 \|\overline{u}_n\|.
\end{aligned} \tag{20}$$

So

$$\begin{aligned}
&\frac{C_0^2 M_1}{2} \left(\|\widetilde{u}_n\|^2 + \left\|u_n^0\right\|^2 \right) \\
&\quad \geq \left(\frac{(2m-1)\omega^2}{1 + (m-1)^2\omega^2} - (2+a) C_0^2 M_1 \right) \\
&\qquad \times \|\overline{u}_n\|^2 - (C_0 M_2 + 1) \|\overline{u}_n\| \\
&\qquad + a C_0^2 M_1 \|\overline{u}_n\|^2 \\
&\quad \geq a C_0^2 M_1 \|\overline{u}_n\|^2 + C_1,
\end{aligned} \tag{21}$$

where $C_1 = \min_{s \in [0,\infty)} \{(((2m-1)\omega^2/(1 + (m-1)^2\omega^2)) - (2 + a)C_0^2 M_1)s^2 - (C_0 M_2 + 1)s\}$.

Since (14), so $-\infty < C_1 < 0$. Then

$$\|\overline{u}_n\|^2 \leq \frac{\|\widetilde{u}_n\|^2}{2a} + \frac{\left\|u_n^0\right\|^2}{2a} + C_2, \tag{22}$$

where $C_2 = -C_1/aC_0^2 M_1 > 0$.

On the other hand, we have

$$\begin{aligned}
\|\widetilde{u}_n\| &\geq \left\langle J'(u_n), \widetilde{u}_n \right\rangle \\
&\geq \left(\frac{(2m+1)\omega^2}{1 + (m+1)^2\omega^2} - 2C_0^2 M_1 \right) \\
&\quad \times \|\widetilde{u}_n\|^2 - \frac{C_0^2 M_1}{2} \|\overline{u}_n\|^2 \\
&\quad - \frac{C_0^2 M_1}{2} \left\|u_n^0\right\|^2 - C_0 M_2 \|\widetilde{u}_n\|.
\end{aligned} \tag{23}$$

So

$$\begin{aligned}
&\frac{C_0^2 M_1}{2} \left(\|\overline{u}_n\|^2 + \left\|u_n^0\right\|^2 \right) \\
&\quad \geq \left(\frac{(2m+1)\omega^2}{1 + (m+1)^2\omega^2} - (2+a) C_0^2 M_1 \right) \\
&\qquad \times \|\widetilde{u}_n\|^2 - (C_0 M_2 + 1) \|\widetilde{u}_n\| \\
&\qquad + a C_0^2 M_1 \|\widetilde{u}_n\|^2 \\
&\quad \geq a C_0^2 M_1 \|\widetilde{u}_n\|^2 + C_3,
\end{aligned} \tag{24}$$

where $0 > C_3 = \min_{s \in [0,\infty)} \{(((2m+1)\omega^2/(1 + (m+1)^2\omega^2)) - (2 + a)C_0^2 M_1)s^2 - (C_0 M_2 + 1)s\}$.

Then

$$\|\widetilde{u}_n\|^2 \leq \frac{\|\overline{u}_n\|^2}{2a} + \frac{\left\|u_n^0\right\|^2}{2a} + C_4, \tag{25}$$

where $C_4 = -C_3/aC_0^2 M_1 > 0$.

From (22) and (25), we have

$$\begin{aligned}
\|\overline{u}_n\|^2 &\leq \frac{1}{2a-1} \left\|u_n^0\right\|^2 + C_5, \\
\|\widetilde{u}_n\|^2 &\leq \frac{1}{2a-1} \left\|u_n^0\right\|^2 + C_5,
\end{aligned} \tag{26}$$

where $C_5 = \max\{(4a^2 C_2 + 2aC_4)/(4a^2 - 1), (4a^2 C_4 + 2aC_2)/(4a^2 - 1)\}$.

By (8), (26) we get

$$\left|\int_0^T F(t,u_n)-F\left(t,u_n^0\right)dt\right|$$
$$=\left|\int_0^T\int_0^1 \nabla F\left(t,u_n^0+s\left(\overline{u}_n+\widetilde{u}_n\right),u_n-u_n^0\right)ds\,dt\right|$$
$$\leq \int_0^T\int_0^1 f(t)\left|u_n^0+s\left(\overline{u}_n+\widetilde{u}_n\right)\right|\left|u_n-u_n^0\right|dt$$
$$+\int_0^T\int_0^1 g(t)\left|u_n-u_n^0\right|dt$$
$$\leq C_0^2M_1\left\|u_n^0\right\|^2+\frac{5}{2}C_0^2M_1\left\|\overline{u}_n\right\|^2$$
$$+\frac{5}{2}C_0^2M_1\left\|\widetilde{u}_n\right\|^2+C_0M_2\left(\left\|\overline{u}_n\right\|+\left\|\widetilde{u}_n\right\|\right)$$
$$\leq\left(C_0^2M_1+\frac{5C_0^2M_1}{2a-1}\right)\left\|u_n^0\right\|^2$$
$$+2C_0M_2\sqrt{\frac{1}{2a-1}}\left\|u_n^0\right\|+5C_5C_0^2M_1+2\sqrt{C_5}C_0M_2. \quad (27)$$

It follows from (26), (27), and the boundedness of $J(u_n)$ that

$$J(u_n)=\frac{1}{2}\int_0^T\left|\dot{u}_n\right|^2dt-\frac{m^2\omega^2}{2}\int_0^T\left|u_n\right|^2dt-\int_0^T F(t,u_n)\,dt$$
$$\leq\frac{1}{2}\left(\left\|\overline{u}_n\right\|^2+\left\|\widetilde{u}_n\right\|^2\right)$$
$$-\int_0^T F(t,u_n)-F\left(t,u_n^0\right)dt-\int_0^T F\left(t,u_n^0\right)dt$$
$$\leq\left(C_0^2M_1+\frac{5C_0^2M_1}{2a-1}+\frac{1}{2a-1}\right)$$
$$\times\left\|u_n^0\right\|^2+2C_0M_2\sqrt{\frac{1}{2a-1}}\left\|u_n^0\right\|$$
$$-\int_0^T F\left(t,u_n^0\right)dt+5C_5C_0^2M_1+2\sqrt{C_5}C_0M_2$$
$$=\left\|u_n^0\right\|^2\left[C_0^2M_1+\frac{5C_0^2M_1}{2a-1}+\frac{1}{2a-1}\right.$$
$$+2C_0M_2\sqrt{\frac{1}{2a-1}}\left\|u_n^0\right\|^{-1}$$
$$\left.-\left\|u_n^0\right\|^{-2}\int_0^T F\left(t,u_n^0\right)dt\right]$$
$$+5C_5C_0^2M_1+2\sqrt{C_5}C_0M_2. \quad (28)$$

The above inequality and (15) imply that $\{u_n^0\}$ is bounded. Hence $\{u_n\}$ is bounded by (26).

Secondly, we assert that

(J_1) $J(u)\to+\infty$ as $\|u\|\to\infty$ in $\widetilde{H}$, which implies that $\inf_{u\in\widetilde{H}}J(u)>-\infty$;

(J_2) $J(u)\to-\infty$ as $\|u\|\to\infty$ in $H^0\oplus\overline{H}$,

for all $u\in H^0\oplus\overline{H}$; that is, $u=u^0+\overline{u}$; then by (8) and (12) we have

$$J(u)=\frac{1}{2}\int_0^T|\dot{u}(t)|^2dt-\frac{m^2\omega^2}{2}\int_0^T|u(t)|^2dt$$
$$-\int_0^T F(t,u(t))\,dt$$
$$=\frac{1}{2}\left(\int_0^T\left|\dot{\overline{u}}(t)\right|^2dt-m^2\omega^2\int_0^T|\overline{u}(t)|^2dt\right)$$
$$-\int_0^T\left[F\left(t,u^0+\overline{u}\right)-F\left(t,u^0\right)\right]dt-\int_0^T F\left(t,u^0\right)dt$$
$$\leq\frac{1}{2}(1-2m)\omega^2\|\overline{u}\|_2^2$$
$$-\int_0^T\int_0^1\left(\nabla F\left(t,u^0+s\overline{u}\right),\overline{u}\right)dt-\int_0^T F\left(t,u^0\right)dt$$
$$\leq\frac{1}{2}(1-2m)\omega^2\|\overline{u}\|_2^2+\int_0^T f(t)|\overline{u}(t)|^2dt$$
$$+\int_0^T f(t)|\overline{u}(t)|\left|u^0\right|dt$$
$$+\int_0^T g(t)|\overline{u}(t)|\,dt-\int_0^T F\left(t,u^0\right)dt$$
$$\leq\frac{1}{2}(1-2m)\omega^2\|\overline{u}\|_2^2+C_0^2M_1\|\overline{u}\|^2$$
$$+C_0^2M_1\left\|u^0\right\|\|\overline{u}\|+C_0M_2\|\overline{u}\|-\int_0^T F\left(t,u^0\right)dt$$
$$\leq\frac{1}{2}(1-2m)\omega^2\|\overline{u}\|_2^2+C_0^2M_1\|\overline{u}\|^2$$
$$+\frac{C_0^2M_1}{2a}\left\|u^0\right\|^2+\frac{aC_0^2M_1}{2}\|\overline{u}\|^2$$
$$+C_0M_2\|\overline{u}\|-\int_0^T F\left(t,u^0\right)dt$$
$$<\frac{1}{2}(1-2m)\omega^2\|\overline{u}\|_2^2$$
$$+\frac{(2+a)C_0^2M_1}{2}\left[1+(m-1)^2\omega^2\right]\|\overline{u}\|_2^2$$
$$+C_0M_2\left[(m-1)\omega+1\right]\|\overline{u}\|_2$$
$$+C_0^2M_1\left\|u^0\right\|^2-\int_0^T F\left(t,u^0\right)dt$$

$$
\begin{aligned}
&= \left\{\frac{1}{2}(1-2m)\,\omega^2 + \frac{(2+a)\,C_0^2 M_1}{2}\left[1+(m-1)^2\omega^2\right]\right\} \\
&\quad \times \|\overline{u}\|_2^2 + C_0 M_2\left[(m-1)\,\omega+1\right]\|\overline{u}\|_2 \\
&\quad + \left\|u^0\right\|^2\left[C_0^2 M_1 - \left\|u^0\right\|^{-2}\int_0^T F\left(t,u^0\right)dt\right],
\end{aligned}
\tag{29}
$$

for $\|u\| \to \infty$ in X if and only if $\|\overline{u}\|_2 \to \infty$ or $\|u^0\| \to \infty$. So, by $m \geq 1$, (14), and (15), we obtain $J(u) \to -\infty$ as $\|u\| \to \infty$ in X.

Let $u \in \widetilde{H}$; then by (8) and (13), we have

$$
\begin{aligned}
J(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{m^2\omega^2}{2}\int_0^T |u(t)|^2 dt \\
&\quad - \int_0^T F(t,u(t))\,dt \\
&\geq \frac{1}{2}\left(1-\frac{m^2\omega^2}{(m+1)^2\omega^2}\right)\int_0^T \left|\dot{\widetilde{u}}(t)\right|^2 dt \\
&\quad - \int_0^T \left[F(t,\widetilde{u}) - F(t,0)\right]dt - \int_0^T F(t,0)\,dt \\
&\geq \frac{1}{2}\frac{2m+1}{(m+1)^2} \times \frac{(m+1)^2\omega^2}{1+(m+1)^2\omega^2}\|\widetilde{u}\|^2 \\
&\quad - \int_0^T\int_0^1 \left(\nabla F(t,s\widetilde{u}),\widetilde{u}\right)dt - \int_0^T F(t,0)\,dt \\
&\geq \frac{1}{2}\frac{(2m+1)\,\omega^2}{1+(m+1)^2\omega^2}\|\widetilde{u}\|^2 \\
&\quad - \int_0^T f(t)\,|\widetilde{u}|^2 dt - \int_0^T g(t)\,|\widetilde{u}|\,dt - \int_0^T F(t,0)\,dt \\
&\geq \left(\frac{1}{2}\frac{(2m+1)\,\omega^2}{1+(m+1)^2\omega^2} - C_0^2 M_1\right) \\
&\quad \times \|\widetilde{u}\|^2 - C_0 M_2\|\widetilde{u}\| - \int_0^T F(t,0)\,dt.
\end{aligned}
\tag{30}
$$

So, by (14), J is bounded from below on $\widetilde{H}$.

Hence, by Rabinowitz's Saddle point Theorem (see [19, Theorem 4.6]), we obtain that the problem (1) has at least one solution. □

Proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 1, so we omit it here. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank Professor Zhang S. Q. for his guidance, Editor Pei Yu for his hard work, and the anonymous referees for their comments and suggestions. This paper is supported partially by NSF of China and by Dr. Start-up fund of Chongqing University of Posts and Telecommunications.

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