Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 638784, 7 pages
http://dx.doi.org/10.1155/2014/638784

Research Article

On Strongly Irregular Points of a Brouwer Homeomorphism Embeddable in a Flow

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Received 25 February 2014; Accepted 25 July 2014; Published 16 October 2014

Academic Editor: Douglas R. Anderson

We study the set of all strongly irregular points of a Brouwer homeomorphism f which is embeddable in a flow. We prove that this set is equal to the first prolongational limit set of any flow containing f. We also give a sufficient condition for a class of flows of Brouwer homeomorphisms to be topologically conjugate.

1. Introduction

In this part we recall the requisite definitions and results concerning Brouwer homeomorphisms and flows of such homeomorphisms.

By a *Brouwer homeomorphism* we mean an orientation preserving homeomorphism of the plane onto itself which has no fixed points. By a *flow* we mean a group of homeomorphisms of the plane onto itself $\{f^t : t \in \mathbb{R}\}$ under the operation of composition which satisfies the following conditions:

(1) the function $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, $F(x,t) = f^t(x)$ is continuous,

(2) $f^0(x) = x$ for $x \in \mathbb{R}^2$,

(3) $f^t(f^s(x)) = f^{t+s}(x)$ for $x \in \mathbb{R}^2, t, s \in \mathbb{R}$.

We say that a Brouwer homeomorphism f is *embeddable in a flow* if there exists a flow $\{f^t : t \in \mathbb{R}\}$ such that $f = f^1$. Then for each $t \in \mathbb{R} \setminus \{0\}$, f^t is a Brouwer homeomorphism.

For any sequence of subsets $(A_n)_{n\in\mathbb{Z}_+}$ of the plane we define *limes superior* $\limsup_{n\to\infty} A_n$ as the set of all points $p \in \mathbb{R}^2$ such that any neighbourhood of p has common points with infinitely many elements of the sequence $(A_n)_{n\in\mathbb{Z}_+}$. For any subset B of the plane we define the *positive limit set* $\omega_f(B)$ as the limes superior of the sequence of its iterates $(f^n(B))_{n\in\mathbb{Z}_+}$ and *negative limit set* $\alpha_f(B)$ as the limes superior of the sequence $(f^{-n}(B))_{n\in\mathbb{Z}_+}$. Under the assumption that B is compact, Nakayama [1] proved that

$$\begin{aligned}
&\omega_f(B) \\
&\quad = \Big\{q \in \mathbb{R}^2 : \text{there exist sequences } \left(p_j\right)_{j\in\mathbb{Z}_+}, \\
&\qquad \left(n_j\right)_{j\in\mathbb{Z}_+} \text{ such that } p_j \in B, n_j \in \mathbb{Z}_+, n_j \longrightarrow +\infty, \\
&\qquad f^{n_j}\left(p_j\right) \longrightarrow q \text{ as } j \longrightarrow +\infty\Big\}, \\
&\alpha_f(B) \\
&\quad = \Big\{q \in \mathbb{R}^2 : \text{there exist sequences } \left(p_j\right)_{j\in\mathbb{Z}_+}, \\
&\qquad \left(n_j\right)_{j\in\mathbb{Z}_+} \text{ such that } p_j \in B, n_j \in \mathbb{Z}_+, n_j \longrightarrow +\infty, \\
&\qquad f^{-n_j}\left(p_j\right) \longrightarrow q \text{ as } j \longrightarrow +\infty\Big\}.
\end{aligned} \tag{1}$$

A point p is called *positively irregular* if $\omega_f(B) \neq \emptyset$ for each Jordan domain B containing p in its interior and *negatively irregular* if $\alpha_f(B) \neq \emptyset$ for each Jordan domain B containing p in its interior, where by a Jordan domain we mean the union of a Jordan curve J and the Jordan region determined by J (i.e., the bounded component of $\mathbb{R}^2 \setminus J$). A point which is not positively irregular is said to be *positively regular*. Similarly, a point which is not negatively irregular is called *negatively* *regular*. A point which is positively or negatively irregular is called *irregular*, otherwise it is *regular*.

We say that a set $U \subset \mathbb{R}^2$ is *invariant* under f if $f(U) = U$. An invariant simply connected region $U \subset \mathbb{R}^2$ is said to be *parallelizable* if there exists a homeomorphism φ_U mapping U onto $\mathbb{R}^2$ such that

$$f(x) = \varphi_U^{-1}(\varphi_U(x) + (1,0)) \quad \text{for } x \in U. \tag{2}$$

The homeomorphism φ_U occurring in this equality is called a *parallelizing homeomorphism* of $f|_U$. On account of the Brouwer Translation Theorem, for each $p \in \mathbb{R}^2$ there exists a parallelizable region U containing p (see [2]).

Homma and Terasaka [3] proved a theorem describing the structure of an arbitrary Brouwer homeomorphism. The theorem can be formulated in the following way.

Theorem 1 (see [3], First Structure Theorem). *Let f be a Brouwer homeomorphism. Then the plane is divided into at most three kinds of pairwise disjoint sets: $\{O_i : i \in I\}$, where $I = \mathbb{Z}_+$ or $I = \{1, \ldots, n\}$ for a positive integer n, $\{O_i' : i \in \mathbb{Z}_+\}$, and F. The sets $\{O_i : i \in I\}$ and $\{O_i' : i \in \mathbb{Z}_+\}$ are the components of the set of all regular points such that each O_i is a parallelizable unbounded simply connected region and each O_i' is a simply connected region satisfying the condition $O_i' \cap f^n(O_i') = \emptyset$ for $n \in \mathbb{Z} \setminus \{0\}$. The set F is invariant and closed and consists of all irregular points.*

For an irregular point p of a Brouwer homeomorphism f the set $P^+(p)$ is defined as the intersection of all $\omega_f(B)$ and the set $P^-(p)$ as the intersection of all $\alpha_f(B)$, where B is a Jordan domain containing p in its interior. An irregular point p is *strongly positively irregular* if $P^+(p) \neq \emptyset$, otherwise it is *weakly positively irregular*. Similarly, p is *strongly negatively irregular* if $P^-(p) \neq \emptyset$, otherwise it is *weakly negatively irregular*. We say that p is *strongly irregular* if it is strongly positively irregular or strongly negatively irregular. Otherwise, an irregular point p is said to be *weakly irregular*.

Homma and Terasaka [3] proved that for all $p, q \in \mathbb{R}^2$

$$q \in P^+(p) \Longleftrightarrow p \in P^-(q). \tag{3}$$

Nakayama [4] showed that for any Brouwer homeomorphism the set of strongly irregular points has no interior points. The set of weakly irregular points consists of all cluster points of the set of strongly irregular points which are not strongly irregular points (see [3]).

A counterpart of Theorem 1 for a Brouwer homeomorphism embeddable in a flow has been given in [5]. Namely, if a Brouwer homeomorphism is embeddable in a flow, then the set of regular points is a union of pairwise disjoint parallelizable unbounded simply connected regions.

2. Strongly Irregular Points

In this section we study the structure of the set of all irregular points for Brouwer homeomorphisms embeddable in a flow.

Let f be a Brouwer homeomorphism. Assume that there exists a flow $\{f^t : t \in \mathbb{R}\}$ such that $f^1 = f$. Let $U \subset \mathbb{R}^2$ be a simply connected region such that $f^t(U) = U$ for $t \in \mathbb{R}$. We say that U is a *parallelizable region* of the flow if there exists a homeomorphism φ_U mapping U onto $\mathbb{R}^2$ such that

$$f^t(x) = \varphi_U^{-1}(\varphi_U(x) + (t,0)) \quad \text{for } x \in U,\ t \in \mathbb{R}. \tag{4}$$

Such a homeomorphism φ_U will be called a *parallelizing homeomorphism* of the flow $\{f^t|_U : t \in \mathbb{R}\}$. It is known that for any simply connected region U which is invariant under the flow $\{f^t : t \in \mathbb{R}\}$ the existence of a parallelizing homeomorphism of $f|_U$ is equivalent to the existence of a parallelizing homeomorphism of $\{f^t|_U : t \in \mathbb{R}\}$ (see [6]).

By the trajectory of a point $p \in \mathbb{R}^2$ we mean the set $C_p := \{f^t(p) : t \in \mathbb{R}\}$. It is known that a region U is parallelizable if and only if there exists a *topological line* K in U (i.e., a homeomorphic image of a straight line that is a closed set in U) such that K has exactly one common point with every trajectory of $\{f^t : t \in \mathbb{R}\}$ contained in U (see [7], page 49). Such a set K we will call a *section* in U (or a *local section* of $\{f^t : t \in \mathbb{R}\}$). On account of the Whitney-Bebutov Theorem (see [7], page 52), for each $p \in \mathbb{R}^2$ there exists a parallelizable region U_p containing p. Without loss of generality we can assume that the parallelizing homeomorphism φ_{U_p} satisfies the condition $\varphi_{U_p}(p) = (0,0)$. Then $K_{\varphi_{U_p}} := \varphi_{U_p}^{-1}(\{0\} \times \mathbb{R})$ is a local section containing p.

For a flow $\{f^t : t \in \mathbb{R}\}$ and a point $p \in \mathbb{R}^2$ we define the *first positive prolongational limit set* and the *first negative prolongational limit set* of p by

$$\begin{aligned} J^+(p) := \{ q \in \mathbb{R}^2 :\ & \text{there exist sequences } (p_n)_{n\in\mathbb{Z}_+}, \\ & (t_n)_{n\in\mathbb{Z}_+} \text{ such that } p_n \longrightarrow p,\ t_n \longrightarrow +\infty, \\ & f^{t_n}(p_n) \longrightarrow q \text{ as } n \longrightarrow +\infty \}, \\ J^-(p) := \{ q \in \mathbb{R}^2 :\ & \text{there exist sequences } (p_n)_{n\in\mathbb{Z}_+}, \\ & (t_n)_{n\in\mathbb{Z}_+} \text{ such that } p_n \longrightarrow p,\ t_n \longrightarrow -\infty, \\ & f^{t_n}(p_n) \longrightarrow q \text{ as } n \longrightarrow +\infty \}. \end{aligned} \tag{5}$$

The set $J(p) := J^+(p) \cup J^-(p)$ is called the *first prolongational limit set* of p. For a subset $H \subset \mathbb{R}^2$ we define

$$J(H) := \bigcup_{p\in H} J(p). \tag{6}$$

The set $J(\mathbb{R}^2)$ will be called the *first prolongational limit set* of the flow $\{f^t : t \in \mathbb{R}\}$. For all $p, q \in \mathbb{R}^2$ we have

$$q \in J^+(p) \Longleftrightarrow p \in J^-(q). \tag{7}$$

In [5] it has been proven that for each point $p \in \mathbb{R}^2$ the set $P^+(p)$ is contained in $J^+(p)$. Now we prove the converse inclusion.

Theorem 2. *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$ and let $p \in \mathbb{R}^2$. Then $J^+(p) \subset P^+(p)$.*

Proof. Let $q \in J^+(p)$. Denote by S_{pq} the strip between trajectories C_p and C_q of points p and q, respectively. Then for each $z \in S_{pq}$ the trajectory C_p is contained in the strip S_{qz} between trajectories C_q and C_z of points q and z, respectively, and the trajectories C_q and C_z are subsets of the same component of $S_{qz} \setminus C_p$ (see [8]). Let K_0 and L_0 be local sections of $\{f^t : t \in \mathbb{R}\}$ such that $p \in K_0$ and $q \in L_0$.

Let B be a Jordan domain containing p in its interior. If $K_0 \cap \mathrm{bd}B \neq \emptyset$, then by compactness of $\mathrm{bd}B$, there exists a $p_0 \in K_0 \cap S_{pq}$ such that p_0 is the only common point of $\mathrm{bd}B$ with the subarc K of K_0 having p and p_0 as its endpoints. If $K_0 \cap \mathrm{bd}B = \emptyset$, then we put $K := K_0 \cap (S_{pq} \cup C_p)$. Take an $r_B > 0$ such that $B(p, r_B) \subset \mathrm{int}\, B$ and $B(p, r_B) \cap S_{pq}$ is contained in the union of all trajectories having a common point with K, where $B(p, r_B)$ denotes the ball with centre p and radius r_B. Fix a $T > 0$ and an $r_q > 0$. Without loss of generality we can assume that $B(q, r_q) \cap B(p, r_B) = \emptyset$.

Now we take an $r \in (0, r_q)$ for which there exists a $y \in L_0 \cap S_{pq}$ such that $\mathrm{dist}(q, y) > r$, where dist denotes the Euclidean metric on the plane. Then $\mathrm{bd}B(q, r) \cap L_0 \cap S_{pq} \neq \emptyset$. By compactness of $\mathrm{bd}B(q, r)$, there exists a $q_0 \in L_0 \cap S_{pq}$ such that q_0 is the only common point of $\mathrm{bd}B(q, r)$ with the subarc L of L_0 having q and q_0 as its endpoints. Denote by W the union of all trajectories having a common point with the arc L. Since $q_0 \in S_{pq}$, each trajectory contained in W is a subset of the component of $\mathrm{cl}\, S_{qq_0} \setminus C_p$ which contains C_q and C_{q_0}, where S_{qq_0} denotes the strip between trajectories C_q and C_{q_0} of points q and q_0.

By the assumption that $q \in J^+(p)$, there exist sequences $(p_n)_{n \in \mathbb{Z}_+}$ and $(t_n)_{n \in \mathbb{Z}_+}$ such that $p_n \to p$, $t_n \to +\infty$, $f^{t_n}(p_n) \to q$ as $n \to +\infty$. Thus there exists an $n_0 \in \mathbb{Z}_+$ such that for all $n > n_0$ we have $t_n > T$, $p_n \in B(p, r_B)$ and $f^{t_n}(p_n) \in B(q, r) \cap W$. Then, for every $n > n_0$ there exists $\alpha_n \in \mathbb{R}$ such that $f^{t_n + \alpha_n}(p_n) \in L$. Moreover, by the definition of r_B, for every $n > n_0$ there exists $x_n \in K$ and $\beta_n \in \mathbb{R}$ such that $f^{\beta_n}(x_n) = p_n$. Thus $f^{t_n + \alpha_n + \beta_n}(x_n) \in L$ for $n > n_0$.

Fix any $n > n_0$ and take a positive integer k_n such that $k_n > t_n + \alpha_n + \beta_n$ and $k_n > T$. Then x_n and $f^{k_n}(x_n)$ belong to different components of $W \setminus L$, since L is a section in W. By continuity of f^{k_n} at p there exists a $y_n \in f^{k_n}(K)$ such that x_n and y_n belong to the same component of $W \setminus L$, since any neighbourhood of $f^{k_n}(p)$ must contain a point from $f^{k_n}(K)$. Thus $f^{k_n}(K)$ has a common point w_n with L. Then $w_n \in B(q, r)$ and hence $w_n \in B(q, r_q)$. Taking $z_n = f^{-k_n}(w_n)$ we have $z_n \in B$, since $K \subset B$. Consequently, for each $n > n_0$ we have $k_n > T$ and $f^{k_n}(z_n) \in B(q, r)$. Hence $k_n \to +\infty$ and $f^{k_n}(z_n) \to q$ as $n \to +\infty$, which implies that $q \in \omega_f(B)$. Consequently $q \in P^+(p)$. □

Since an analogous reasoning can be applied to the set of strongly negatively irregular points and the first negative prolongational limit set, our considerations can be summarized in the following way.

Corollary 3. *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$ and let $p \in \mathbb{R}^2$. Then $P^+(p) = J^+(p)$ and $P^-(p) = J^-(p)$, and consequently the set of all strongly irregular points of f is equal to the first prolongational limit set of the flow $\{f^t : t \in \mathbb{R}\}$.*

Corollary 4. *Let f be a Brouwer homeomorphism which is embeddable in a flow. Then, for each flow containing f, the first prolongational limit set is the same.*

After a reparametrization of the flow $\{f^t : t \in \mathbb{R}\}$ containing f each element f^t of the flow, for $t \in \mathbb{R} \setminus \{0\}$ or $t > 0$, respectively, can be treated as f.

Corollary 5. *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then the set of all strongly irregular points of f^t is the same for all $t \in \mathbb{R} \setminus \{0\}$. Moreover, the set of all strongly positive irregular points of f^t and the set of all strongly negative irregular points of f^t are the same for all $t > 0$.*

3. Flows of Brouwer Homeomorphisms

In this section we describe the form of any flow of Brouwer homeomorphisms. To give a sufficient condition for the topological conjugacy of flows of Brouwer homeomorphisms one can use covers of the plane by maximal parallelizable regions. We will study the functions which express the relations between parallelizing homeomorphisms of such regions.

It is known that a simply connected region U is parallelizable if and only if $J(U) \cap U = \emptyset$. Hence for every parallelizable region U we have $J(U) \subset \mathrm{bd}U$. In the case where U is a maximal parallelizable region (i.e., U is not contained properly in any parallelizable region), the boundary of U consists of strongly irregular points. It follows from the fact that for each maximal parallelizable region U the equality $J(U) = \mathrm{bd}U$ holds. The proof of this fact can be found in [9]. For the convenience of the reader, we outline the essential ideas in that proof.

Let U be a parallelizable region. Assume that there exists a point $p \in \mathrm{bd}U$ such that $p \notin J(U)$. Denote by D_1 the component of $\mathbb{R}^2 \setminus C_p$ which has a common point with U and by D_2 the other component of $\mathbb{R}^2 \setminus C_p$. Let V be a parallelizable region which contains p and put $V_1 := V \cap D_2$. Let $U_1 := U \cup C_p \cup V_1$. We show that $J(q) \cap U_1 = \emptyset$ for each $q \in U_1$, which means that U_1 is a parallelizable region. To see this we consider three cases. First, let us consider the case where $q \in U$. Then $J(q) \subset \mathrm{cl}\, D_1$, since $q \in D_1$. Hence by parallelizability of U, we have $J(q) \cap U = \emptyset$ and by the assumption that $p \notin J(U)$, we get $J(q) \cap C_p = \emptyset$. Thus $J(q) \cap U_1 = \emptyset$. Now, let $q \in V_1$. Then $J(q) \subset \mathrm{cl}\, D_2$. Hence $J(q) \cap U_1 = \emptyset$, since by parallelizability of V we have $J(q) \cap (C_p \cup V_1) = \emptyset$. Finally, let $q \in C_p$. Then, as in the previous case, $J(q) \cap (C_p \cup V_1) = \emptyset$, and by the assumption that $p \notin J(U)$, we get $J(q) \cap U = \emptyset$. Thus we proved that $J(U_1) \cap U_1 = \emptyset$, which means that U_1 is parallelizable. Since U is contained properly in U_1, we obtain that U cannot be a maximal parallelizable region.

For any distinct trajectories C_{p_1}, C_{p_2}, and C_{p_3} of $\{f^t : t \in \mathbb{R}\}$ one of the following two possibilities must be satisfied: exactly one of the trajectories C_{p_1}, C_{p_2}, and C_{p_3} is contained in the strip between the other two or each of the trajectories C_{p_1}, C_{p_2}, and C_{p_3} is contained in the strip between the other two. In the first case if C_{p_j} is the trajectory which lies in the strip between C_{p_i} and C_{p_k} we will write $C_{p_i}|C_{p_j}|C_{p_k}$ (i, j, $k \in \{1,2,3\}$ and i, j, k are different). In the second case we will write $|C_{p_i}, C_{p_j}, C_{p_k}|$ (cf. [10]).

Let X be a nonempty set. Denote by $X^{<\omega}$ the set of all finite sequences of elements of X. A subset T of $X^{<\omega}$ is called a *tree* on X if it is closed under initial segments; that is, for all positive integers m, n such that $n > m$ if $(x_1, \dots, x_m, \dots, x_n) \in T$, then $(x_1, \dots, x_m) \in T$. Let $\alpha = (x_1, \dots, x_n) \in X^{<\omega}$. Then, for any $x \in X$ by $\alpha^\frown x$ we denote the sequence $(x_1, \dots, x_n, x)$. A node $\alpha = (x_1, \dots, x_n) \in T$ of a tree T is said to be *terminal* if there is no node of properly extending it; that is, there is no element $x \in X$ such that $\alpha^\frown x \in T$.

A tree $A^+ \subset \mathbb{Z}_+^{<\omega}$ will be termed *admissible* if the following conditions hold:

(i) A^+ contains the sequence 1 and no other one-element sequence;

(ii) if $\alpha^\frown k$ is in A^+ and $k > 1$, then so also is $\alpha^\frown(k-1)$.

A tree $A^- \subset \mathbb{Z}_-^{<\omega}$ will be termed *admissible* if the following conditions hold:

(iii) A^- contains the sequence -1 and no other one-element sequence;

(iv) if $\alpha^\frown k$ is in A^- and $k < -1$, then so also is $\alpha^\frown(k+1)$.

The set $A := A^+ \cup A^-$ will be said to be *admissible class of finite sequences*, where A^+ and A^- are some admissible classes of finite sequences of positive and negative integers, respectively.

Now we recall results describing the flows of Brouwer homeomorphisms.

Theorem 6 (see [11]). *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then there exists a family of trajectories $\{C_\alpha : \alpha \in A\}$ and a family of maximal parallelizable regions $\{U_\alpha : \alpha \in A\}$, where $A = A^+ \cup A^-$ is an admissible class of finite sequences, such that $U_1 = U_{-1}$, $C_1 = C_{-1}$, and*

$$\begin{gathered} C_\alpha \subset U_\alpha \quad \text{for } \alpha \in A, \\ \bigcup_{\alpha \in A} U_\alpha = \mathbb{R}^2, \\ U_\alpha \cap U_{\alpha^\frown i} \neq \emptyset \quad \text{for } \alpha^\frown i \in A, \\ C_{\alpha^\frown i} \subset bdU_\alpha \quad \text{for } \alpha^\frown i \in A, \\ \left|C_\alpha, C_{\alpha^\frown i_1}, C_{\alpha^\frown i_2}\right| \quad \text{for } \alpha^\frown i_1, \alpha^\frown i_2 \in A,\ i_1 \neq i_2, \\ C_\alpha \left|C_{\alpha^\frown i}\right| C_{\alpha^\frown i^\frown j} \quad \text{for } \alpha^\frown i^\frown j \in A. \end{gathered} \tag{8}$$

Proposition 7 (see [11]). *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then there exists a family of the parallelizing homeomorphisms $\{\varphi_\alpha : \alpha \in A^+\}$, where $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$, U_α are those occurring in Theorem 6, and for each $\alpha^\frown i \in A^+$*

$$\begin{gathered} \varphi_{\alpha^\frown i}\left(U_\alpha \cap U_{\alpha^\frown i}\right) = \mathbb{R} \times \left(c_{\alpha^\frown i}, 0\right), \\ \varphi_\alpha\left(U_\alpha \cap U_{\alpha^\frown i}\right) = \mathbb{R} \times \left(c_\alpha, d_\alpha\right), \end{gathered} \tag{9}$$

where $c_\alpha \in \mathbb{R} \cup \{-\infty\}$, $d_\alpha \in \mathbb{R} \cup \{+\infty\}$, and $c_{\alpha^\frown i} \in [-\infty, 0)$ are some constants such that $c_\alpha < d_\alpha$ and at least one of the constants c_α, d_α is finite. Moreover, there exists a continuous function $\mu_{\alpha^\frown i} : (c_\alpha, d_\alpha) \to \mathbb{R}$ and a homeomorphism $\nu_{\alpha^\frown i} : (c_\alpha, d_\alpha) \to (c_{\alpha^\frown i}, 0)$ such that the homeomorphism

$$h_{\alpha^\frown i} : \mathbb{R} \times \left(c_\alpha, d_\alpha\right) \longrightarrow \mathbb{R} \times \left(c_{\alpha^\frown i}, 0\right) \tag{10}$$

given by the relation $h_{\alpha^\frown i} := \varphi_{\alpha^\frown i} \circ (\varphi_{\alpha|_{U_\alpha \cap U_{\alpha^\frown i}}})^{-1}$ has the form

$$h_{\alpha^\frown i}(t, s) = \left(\mu_{\alpha^\frown i}(s) + t, \nu_{\alpha^\frown i}(s)\right), \quad t \in \mathbb{R}, s \in \left(c_\alpha, d_\alpha\right). \tag{11}$$

The above proposition is formulated for $\alpha \in A^+$, but the analogous result holds for $\alpha \in A^-$. The admissible class of finite sequences occurring in Theorem 6 is not unique for a given flow, so we can usually choose a convenient A when solving a problem of topological conjugacy.

The homeomorphisms $\nu_{\alpha^\frown i}$ occurring in Proposition 7 can be either increasing or decreasing. For each $\alpha^\frown i \in A$ denote by $C^\alpha_{\alpha^\frown i}$ the unique trajectory contained in $U_\alpha \cap J(C_{\alpha^\frown i})$ (the uniqueness has been proven in [8]). From the construction of the families $\{C_\alpha : \alpha \in A\}$ and $\{U_\alpha : \alpha \in A\}$ occurring in Theorem 6 we obtain that, in case $C_\alpha|C^\alpha_{\alpha^\frown i}|C_{\alpha^\frown i}$ or $C_\alpha = C^\alpha_{\alpha^\frown i}$, the homeomorphism $\nu_{\alpha^\frown i}$ is decreasing and $c_\alpha > 0$ or $c_\alpha = 0$, respectively. However, in case $|C_\alpha, C^\alpha_{\alpha^\frown i}, C_{\alpha^\frown i}|$, the homeomorphism $\nu_{\alpha^\frown i}$ is increasing and $d_\alpha > 0$ (see [11]).

The continuous functions $\mu_{\alpha^\frown i}$ describe the time needed for the flow $\{f^t : t \in \mathbb{R}\}$ to move from the point with coordinates $(0, \nu_{\alpha^\frown i}(s))$ in the chart $\varphi_{\alpha^\frown i}$ until it reaches the point with coordinates $(0, s)$ in the chart φ_α. In other words, $\mu_{\alpha^\frown i}$ describe the time needed for the flow to move from a point from the section $K_{\varphi_{\alpha^\frown i}}$ in $U_{\alpha^\frown i}$ to a point from the section K_{φ_α} in U_α.

Proposition 8. *The functions $\mu_{\alpha^\frown i}$ occurring in Proposition 7 satisfy the condition*

$$\lim_{s \to c_\alpha} \mu_{\alpha^\frown i}(s) = \begin{cases} -\infty & \text{if } C_{\alpha^\frown i} \subset J^+\left(C^\alpha_{\alpha^\frown i}\right), \\ +\infty & \text{if } C_{\alpha^\frown i} \subset J^-\left(C_{\alpha^\frown i}\right) \end{cases} \tag{12}$$

in the case where $C_\alpha|C^\alpha_{\alpha^\frown i}|C_{\alpha^\frown i}$ or $C_\alpha = C^\alpha_{\alpha^\frown i}$ or the condition

$$\lim_{s \to d_\alpha} \mu_{\alpha^\frown i}(s) = \begin{cases} -\infty & \text{if } C_{\alpha^\frown i} \subset J^+\left(C^\alpha_{\alpha^\frown i}\right), \\ +\infty & \text{if } C_{\alpha^\frown i} \subset J^-\left(C_{\alpha^\frown i}\right) \end{cases} \tag{13}$$

in the case where $|C_\alpha, C^\alpha_{\alpha^\frown i}, C_{\alpha^\frown i}|$.

Proof. Let us consider the case where $C_\alpha|C^\alpha_{\alpha^\frown i}|C_{\alpha^\frown i}$ or $C_\alpha = C^\alpha_{\alpha^\frown i}$ and assume that $C_{\alpha^\frown i} \subset J^+(C^\alpha_{\alpha^\frown i})$. The other cases are similar. Denote by p and q the points for which $\varphi_\alpha(p) = (0, c_\alpha)$ and $\varphi_{\alpha^\frown i}(q) = (0,0)$; that is, $p \in K_{\varphi_\alpha} \cap C^\alpha_{\alpha^\frown i}$ and $q \in K_{\varphi_{\alpha^\frown i}} \cap C_{\alpha^\frown i}$. Then $q \in J^+(p)$. Thus there exist sequences $(p_n)_{n \in \mathbb{Z}_+}$ and $(t_n)_{n \in \mathbb{Z}_+}$ such that $p_n \to p$, $t_n \to +\infty$, and $f^{t_n}(p_n) \to q$ as $n \to +\infty$. This means that there exist sequences $(u_n)_{n \in \mathbb{Z}_+}$, $(s_n)_{n \in \mathbb{Z}_+}$ such that $u_n \to 0$, $s_n \to c_\alpha$, where $\varphi_\alpha(p_n) = (u_n, s_n)$. Hence $\varphi_\alpha(f^{t_n}(p_n)) = \varphi_\alpha(p_n) + (t_n, 0) = (t_n + u_n, s_n)$ and by (11)

$$h_{\alpha^\frown i}\left(t_n + u_n, s_n\right) = \left(\mu_{\alpha^\frown i}\left(s_n\right) + t_n + u_n, \nu_{\alpha^\frown i}\left(s_n\right)\right). \tag{14}$$

Thus $\mu_{\alpha^\frown i}(s_n)+t_n+u_n \to 0$ as $n \to +\infty$, since $f^{t_n}(p_n) \to q$ as $n \to +\infty$. Hence $\mu_{\alpha^\frown i}(s_n) \to -\infty$, since $t_n \to +\infty$ and $u_n \to 0$. Consequently, $\liminf_{s\to c_\alpha}\mu_{\alpha^\frown i}(s) = -\infty$.

Suppose, on the contrary, that there exists a sequence $(s_n)_{n\in\mathbb{Z}_+}$ such that $s_n \to c_\alpha$ and $\mu_{\alpha^\frown i}(s_n) \to c$ for some $c \in \mathbb{R}$. Consider the sequence $(p_n)_{n\in\mathbb{Z}_+}$ such that $\varphi_\alpha(p_n) = (0, s_n)$. Then each element of the sequence $(p_n)_{n\in\mathbb{Z}_+}$ belongs to K_{φ_α}. Moreover, the sequence $(p_n)_{n\in\mathbb{Z}_+}$ tends to the point p such that $\varphi_\alpha(p) = (0, c_\alpha)$. Hence $p \in K_{\varphi_\alpha} \cap C^\alpha_{\alpha^\frown i}$. On the other hand, by (11)

$$\varphi_{\alpha^\frown i}(p_n) = h_{\alpha^\frown i}(0, s_n) = (\mu_{\alpha^\frown i}(s_n), \nu_{\alpha^\frown i}(s_n)). \tag{15}$$

Hence $\lim_{n\to+\infty}\varphi_{\alpha^\frown i}(p_n) = (c, 0)$. Consequently $\lim_{n\to+\infty}p_n = \tilde{q}$, where $\tilde{q}$ is a point such that $\varphi_{\alpha^\frown i}(\tilde{q}) = (c, 0)$; that is, $\tilde{q} \in C_{\alpha^\frown i}$. But this is impossible, since $C_{\alpha^\frown i} \cap C^\alpha_{\alpha^\frown i} = \emptyset$. □

By the fact that $\nu_{\alpha^\frown i} : (c_\alpha, d_\alpha) \to (c_{\alpha^\frown i}, 0)$ is a homeomorphism, the function $\tilde{h}_{\alpha^\frown i} : \mathbb{R} \times (c_{\alpha^\frown i}, 0) \to \mathbb{R} \times (c_{\alpha^\frown i}, 0)$ defined by

$$\tilde{h}_{\alpha^\frown i}(t, s) := \left(\left(\mu_{\alpha^\frown i} \circ \nu^{-1}_{\alpha^\frown i}\right)(\nu) + t, \nu\right), \quad t \in \mathbb{R},\ \nu \in (c_{\alpha^\frown i}, 0) \tag{16}$$

is continuous. Moreover, putting $s := \nu^{-1}_{\alpha^\frown i}(\nu)$ in Proposition 8 we obtain the following result.

Corollary 9. *The functions $\eta_{\alpha^\frown i} : (c_{\alpha^\frown i}, 0) \to \mathbb{R}$ given by*

$$\eta_{\alpha^\frown i} := \mu_{\alpha^\frown i} \circ \nu^{-1}_{\alpha^\frown i}, \tag{17}$$

where $\mu_{\alpha^\frown i}$ and $\nu_{\alpha^\frown i}$ are those occurring in Proposition 7, satisfy the condition

$$\lim_{\nu\to 0}\eta_{\alpha^\frown i}(\nu) = \begin{cases} -\infty & \text{if } C_{\alpha^\frown i} \subset J^+\left(C^\alpha_{\alpha^\frown i}\right), \\ +\infty & \text{if } C_{\alpha^\frown i} \subset J^-\left(C_{\alpha^\frown i}\right). \end{cases} \tag{18}$$

4. Topological Conjugacy of Generalized Reeb Flows

In this section we consider the problem of topological conjugacy of a class of flows of Brouwer homeomorphisms. To prove our result we use the form of such flows.

We say that flows $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$, where $f^t, g^t : \mathbb{R}^2 \to \mathbb{R}^2$, are *topologically conjugate* if there exists a homeomorphism Φ of the plane onto itself such that

$$g^t = \Phi^{-1} \circ f^t \circ \Phi, \quad t \in \mathbb{R}. \tag{19}$$

In [12] a lemma can be found which says that the set of strongly irregular points (called the set of singular pairs there) is invariant with respect to topological conjugacy of flows. Thus, by Corollary 3, we have the following result.

Proposition 10. *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically conjugate flows of Brouwer homeomorphisms and let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism which conjugates the flows. Then $\Phi(J_{\{f^t\}}(\mathbb{R}^2)) = J_{\{g^t\}}(\mathbb{R}^2)$, where $J_{\{f^t\}}(\mathbb{R}^2)$ and $J_{\{g^t\}}(\mathbb{R}^2)$ denote the first prolongational limit set of $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$, respectively.*

Put

$$\begin{aligned} P_0 &:= \left\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\right\}, \\ P_1 &:= \left\{(x, y) \in \mathbb{R}^2 : x < 0, y > 0\right\}, \\ P_2 &:= \left\{(x, y) \in \mathbb{R}^2 : x > 0, y < 0\right\}, \\ L_x &:= \left\{(x, 0) \in \mathbb{R}^2 : x > 0\right\}, \\ L_y &:= \left\{(0, y) \in \mathbb{R}^2 : y > 0\right\} \end{aligned} \tag{20}$$

and $U := P_0 \cup P_1 \cup P_2 \cup L_x \cup L_y$. Consider the flow $\{g^t : t \in \mathbb{R}\}$, where for each $t \in \mathbb{R}$ the homeomorphism $g^t : U \to U$ is defined by

$$g^t(x, y) := \begin{cases} (2^t x, 2^{-t} y) & \text{if } (x, y) \in P_0 \cup L_x \cup L_y, \\ (x, 2^{-t} y) & \text{if } (x, y) \in P_1, \\ (2^t x, y) & \text{if } (x, y) \in P_2. \end{cases} \tag{21}$$

Then $J^+(U) = L_x$ and $J^-(U) = L_y$.

Put $A^+ := \{1, (1, 1)\}$, $U_1 := P_1 \cup L_y \cup P_0$, $U_{(1,1)} := P_0 \cup L_x \cup P_2$, $C_1 := L_y$, $C_{(1,1)} := L_x$ and $A^- := \{-1\}$, $U_{-1} := U_1$, $C_{-1} := C_1$. Then $C^1_{(1,1)} := C_1$ and $C_{(1,1)} \subset J^+(C_1)$. Let

$$K_{\varphi_1} := \{(s, 1) : s \in \mathbb{R}\}, \qquad K_{\varphi_{(1,1)}} := \{(1, s) : s \in \mathbb{R}\}. \tag{22}$$

Note that the trajectories of $\{g^t : t \in \mathbb{R}\}$ contained in P_0 are given by the equation $xy = s$ for $s \in (0, +\infty)$. Hence

$$\mu_{(1,1)} : (0, +\infty) \longrightarrow \mathbb{R}, \qquad \mu_{(1,1)}(s) = \log_2 s, \tag{23}$$

since $g^{\log_2 s}(1, s) = (2^{\log_2 s}, 2^{-\log_2 s}s) = (s, 1)$. Moreover,

$$\nu_{(1,1)} : (0, +\infty) \longrightarrow (-\infty, 0), \qquad \nu_{(1,1)}(s) = -s. \tag{24}$$

For each flow $\{f^t : t \in \mathbb{R}\}$, where $f^t : U \to U$ for $t \in \mathbb{R}$, having the same trajectories (including the orientation) as the flow $\{g^t : t \in \mathbb{R}\}$ given by (21), one can consider the function $\mu_{\{f^t\},(1,1)} : (0, +\infty) \to \mathbb{R}$ occurring in Proposition 7 which describes the time needed to move from each point $p \in P_0 \cap K_{\varphi_{(1,1)}}$ to the point of K_{φ_1} belonging to the trajectory of p, that is, from the point of the form $(1, s)$ to the point of the form $(s, 1)$ for some $s \in (0, +\infty)$. Then by Proposition 8

$$\lim_{s\to 0}\mu_{\{f^t\},(1,1)}(s) = -\infty. \tag{25}$$

Consider a constant $\sigma(\mu_{\{f^t\},(1,1)}) \in [0, +\infty]$ defined by

$$\sigma\left(\mu_{\{f^t\},(1,1)}\right) := \limsup_{\nu\to 0}\mu^*_{\{f^t\},(1,1)}(\nu), \tag{26}$$

where $\mu^*_{\{f^t\},(1,1)} : (0, 1] \to [0, +\infty)$ is given by

$$\mu^*_{\{f^t\},(1,1)}(\nu) := \mu_{\{f^t\},(1,1)}(\nu) - \min\left\{\mu_{\{f^t\},(1,1)}(s) : s \in [\nu, 1]\right\} \tag{27}$$

(cf. [13, 14]). Then the flow $\{f^t : t \in \mathbb{R}\}$ is topologically conjugate to the flow $\{g^t : t \in \mathbb{R}\}$ given by (21) if and only if

$\sigma(\mu_{\{f^t\},(1,1)}) = 0$ (see [13]). In particular, this condition holds in the case where $\mu_{\{f^t\},(1,1)}$ is increasing.

Now we introduce a class of flows of Brouwer homeomorphisms. Put $\alpha_1 := 1$ and $\alpha_{n+1} := \alpha_n{}^\frown 1$ for $n \in \mathbb{Z}_+$. For any positive integer k we define $A_k := \{\alpha_n : 1 \le n \le k\}$ and $A_{+\infty} := \{\alpha_n : n \in \mathbb{Z}_+\}$. Similarly, put $\alpha_{-1} := -1$, $\alpha_{n-1} := \alpha_n{}^\frown -1$ for $n \in \mathbb{Z}_-$ and for any negative integer k let $A_k := \{\alpha_n : k \le n \le -1\}$ and $A_{-\infty} := \{\alpha_n : n \in \mathbb{Z}_-\}$.

Consider a flow $\{h^t : t \in \mathbb{R}\}$ of Brouwer homeomorphisms $h^t : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A = A^+ \cup A^-$ can be given in one of the following forms:

(a) $A^- = \{-1\}$ and $A^+ = A_k$ for some $k \in \mathbb{Z}_+$,

(b) $A^- = \{-1\}$ and $A^+ = A_{+\infty}$,

(c) $A^- = A_{-\infty}$ and $A^+ = A_{+\infty}$.

We assume that $U_{\alpha_n} \cap J(\mathbb{R}^2) = C_{\alpha_n}$ for each $\alpha_n \in A$, where $\{U_{\alpha_n} : \alpha_n \in A\}$ and $\{C_{\alpha_n} : \alpha_n \in A\}$ are whose occurring in Theorem 6. Then $C^{\alpha_n}_{\alpha_{n+1}} = C_{\alpha_n}$, since $C^{\alpha_n}_{\alpha_{n+1}} \subset U_{\alpha_n} \cap J(\mathbb{R}^2)$ for every $\alpha_{n+1} \in A^+$. Similarly, $C^{\alpha_n}_{\alpha_{n-1}} = C_{\alpha_n}$ for every $\alpha_{n-1} \in A^-$.

Fix an $\alpha_{n+1} \in A^+$. Denote by $V_{\alpha_{n+1}}$ the strip between C_{α_n} and $C_{\alpha_{n+1}}$. Then $V_{\alpha_{n+1}} \subset U_{\alpha_n}$ and $|C, C_{\alpha_n}, C_{\alpha_{n+1}}|$ for every trajectory $C \subset V_{\alpha_{n+1}}$ (see [8]). In particular, if C_{α_n} is equal to the vertical line $\{(n-1, y) : y \in \mathbb{R}\}$ for each $\alpha_n \in A^+$, then $V_{\alpha_{n+1}}$ is a vertical strip for each $\alpha_{n+1} \in A^+$. In a similar way we define the strip $V_{\alpha_{n-1}}$ for $\alpha_{n-1} \in A^-$.

Let us assume that for each $\alpha_n \in A \setminus \{1, -1\}$ there exists a homeomorphism $\psi_{\alpha_n} : \mathrm{cl}\, V_{\alpha_n} \to P_0 \cup L_x \cup L_y$ such that

$$h^t = \psi_{\alpha_n}^{-1} \circ g^t \circ \psi_{\alpha_n}, \quad t \in \mathbb{R}, \tag{28}$$

where $\{g^t : t \in \mathbb{R}\}$ is given by (21). If $C_{\alpha_n} \subset J^+(C_{\alpha_{n-1}})$, then $\psi_{\alpha_n}(C_{\alpha_{n-1}}) = L_y$ and $\psi_{\alpha_n}(C_{\alpha_n}) = L_x$. In case $C_{\alpha_n} \subset J^-(C_{\alpha_{n-1}})$ we have $\psi_{\alpha_n}(C_{\alpha_{n-1}}) = L_x$ and $\psi_{\alpha_n}(C_{\alpha_n}) = L_y$. The flow $\{h^t : t \in \mathbb{R}\}$ described above will be called a *standard generalized Reeb flow*.

A standard generalized Reeb flow can have either a finite number of maximal parallelizable regions or an infinite number of such regions. The first case holds if the set of indices A of the flow is of the form (a). However, the second case holds if this set is of the form (b) or (c). The trajectories of a standard generalized Reeb flow with an infinite number of maximal parallelizable regions are shown in Figures 1 and 2 for the set A of the forms (b) and (c), respectively.

Consider a flow $\{f^t : t \in \mathbb{R}\}$ of Brouwer homeomorphisms which has the same trajectories as a standard generalized Reeb flow. For $\alpha_n \in A \setminus \{1, -1\}$ and $s \in (0, +\infty)$ denote by $C^{\alpha_n}_s$ the image of the trajectory $\{(x, y) \in P_0 : xy = s\}$ of $\{g^t : t \in \mathbb{R}\}$ under $\psi_{\alpha_n}^{-1}$. For each $\alpha_n \in A \setminus \{1, -1\}$ consider the function $\mu_{\{f^t\},\alpha_n} : (0, +\infty) \to \mathbb{R}$ taking as $\mu_{\{f^t\},\alpha_n}(s)$ the time needed to move from the unique point of the set $C^{\alpha_n}_s \cap \psi_{\alpha_n}^{-1}(K_{\varphi_{(1,1)}})$ to the unique point of $C^{\alpha_n}_s \cap \psi_{\alpha_n}^{-1}(K_{\varphi_1})$. Define $\mu^*_{\{f^t\},\alpha_n} : (0, 1] \to [0, +\infty)$ by

$$\mu^*_{\{f^t\},\alpha_n}(v) := \mu_{\{f^t\},\alpha_n}(v) - \min\left\{\mu_{\{f^t\},\alpha_n}(s) : s \in [v, 1]\right\} \tag{29}$$

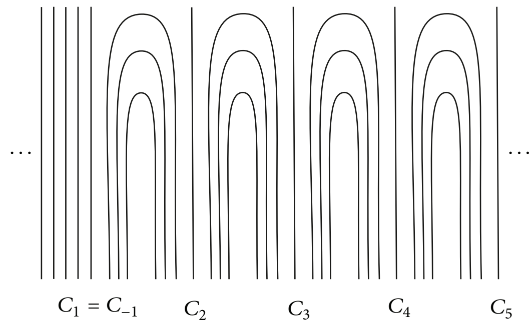

Figure 1: A generalized Reeb flow with $A^- = \{-1\}$ and $A^+ = A_{+\infty}$.

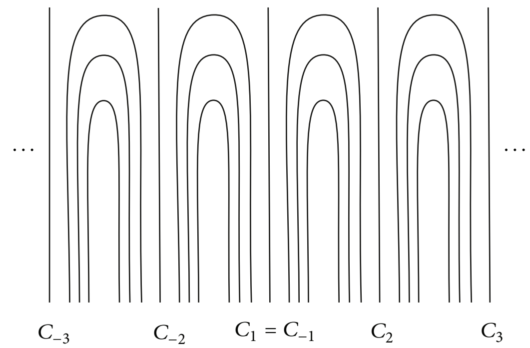

Figure 2: A generalized Reeb flow with $A^- = A_{-\infty}$ and $A^+ = A_{+\infty}$.

in case $C_{\alpha_{n+1}} \subset J^+(C_{\alpha_n})$, and by

$$\mu^*_{\{f^t\},\alpha_n}(v) := \max\left\{\mu_{\{f^t\},\alpha_n}(s) : s \in [v, 1]\right\} - \mu_{\{f^t\},\alpha_n}(v) \tag{30}$$

in case $C_{\alpha_{n+1}} \subset J^-(C_{\alpha_n})$. Put

$$\sigma\left(\mu_{\{f^t\},\alpha_n}\right) := \limsup_{v \to 0} \mu^*_{\{f^t\},\alpha_n}(v). \tag{31}$$

Now we can prove the following conjugacy result.

Theorem 11. *Let $\{h^t : t \in \mathbb{R}\}$ be a standard generalized Reeb flow. Let A be an admissible class of finite sequences satisfying one of the conditions (a)–(c). Assume that $\{f^t : t \in \mathbb{R}\}$ is a flow of Brouwer homeomorphisms having the same trajectories including orientation as $\{h^t : t \in \mathbb{R}\}$. If $\sigma(\mu_{\{f^t\},\alpha_n}) = 0$ for all $\alpha_n \in A \setminus \{1, -1\}$, then the flows $\{f^t : t \in \mathbb{R}\}$ and $\{h^t : t \in \mathbb{R}\}$ are topologically conjugate.*

Proof. Assume that one of the conditions (a) and (b) holds. First, let us note that there exists a topological conjugacy $\Phi_1 : U_1 \to U_1$ of flows $\{f^t|_{U_1} : t \in \mathbb{R}\}$ and $\{h^t|_{U_1} : t \in \mathbb{R}\}$, since U_1 is a parallelizable region of each of these flows. More precisely, if $\varphi_{\{f^t\},1} : U_1 \to \mathbb{R}^2$ and $\varphi_{\{h^t\},1} : U_1 \to \mathbb{R}^2$ are parallelizing homeomorphisms for $\{f^t|_{U_1} : t \in \mathbb{R}\}$ and $\{h^t|_{U_1} : t \in \mathbb{R}\}$, respectively, then for every $p \in U_1$ we put $\Phi_1(p) := (\varphi^{-1}_{\{h^t\},1} \circ \varphi_{\{f^t\},1})(p)$.

Fix an $\alpha_{n+1} \in A^+$. Assume that we have defined a homeomorphism Φ_n which conjugates $\{f^t : t \in \mathbb{R}\}$ and $\{h^t : t \in \mathbb{R}\}$ on the set $\bigcup_{i=1}^n U_{\alpha_i}$. Define $F^t : P_0 \cup L_x \cup L_y \to$

$P_0 \cup L_x \cup L_y$ by $F^t := \psi_{\alpha_{n+1}} \circ f^t \circ \psi_{\alpha_{n+1}}^{-1}$ for $t \in \mathbb{R}$, where $\psi_{\alpha_{n+1}}$ satisfies (28). Then $\sigma(\mu_{\{F^t\},(1,1)}) = \sigma(\mu_{\{f^t\},\alpha_n})$. Hence $\sigma(\mu_{\{F^t\},(1,1)}) = 0$, since by the assumption $\sigma(\mu_{\{f^t\},\alpha_n}) = 0$. Thus $\{F^t : t \in \mathbb{R}\}$ and $\{g^t|_{P_0 \cup L_x \cup L_y} : t \in \mathbb{R}\}$ are topologically conjugate. Consequently $\{f^t|_{W_{\alpha_{n+1}}} : t \in \mathbb{R}\}$ and $\{h^t|_{W_{\alpha_{n+1}}} : t \in \mathbb{R}\}$ are topologically conjugate, where $W_{\alpha_{n+1}} := \operatorname{cl} V_{\alpha_{n+1}}$. Denote by $\phi_{\alpha_{n+1}}$ the homeomorphism which conjugates these flows.

Fix any $p_0 \in C_{\alpha_n}$ and put $q_1 := \Phi_n(p_0)$, $q_2 := \phi_{\alpha_{n+1}}(p_0)$. Take $t_0 \in \mathbb{R}$ such that $h^{t_0}(q_2) = q_1$ and define $\Phi_{\alpha_{n+1}} := h^{t_0}|_{W_{\alpha_{n+1}}} \circ \phi_{\alpha_{n+1}}$. Then $\Phi_{\alpha_{n+1}}$ conjugates the flows $\{f^t|_{W_{\alpha_{n+1}}} : t \in \mathbb{R}\}$ and $\{h^t|_{W_{\alpha_{n+1}}} : t \in \mathbb{R}\}$, since

$$\begin{aligned}
&\Phi_{\alpha_{n+1}} \circ f^t|_{W_{\alpha_{n+1}}} \\
&\quad = h^{t_0}|_{W_{\alpha_{n+1}}} \circ \phi_{\alpha_{n+1}} \circ f^t|_{W_{\alpha_{n+1}}} \\
&\quad = h^{t_0}|_{W_{\alpha_{n+1}}} \circ h^t|_{W_{\alpha_{n+1}}} \circ \phi_{\alpha_{n+1}} \\
&\quad = h^t|_{W_{\alpha_{n+1}}} \circ h^{t_0}|_{W_{\alpha_{n+1}}} \circ \phi_{\alpha_{n+1}} = h^t|_{W_{\alpha_{n+1}}} \circ \Phi_{\alpha_{n+1}}.
\end{aligned} \tag{32}$$

Moreover $\Phi_{\alpha_{n+1}}(p_0) = q_1$, since

$$\Phi_{\alpha_{n+1}}(p_0) = h^{t_0}\left(\phi_{\alpha_{n+1}}(p_0)\right) = h^{t_0}(q_2) = q_1. \tag{33}$$

Hence $\Phi_{\alpha_{n+1}}|_{C_{\alpha_n}} = \Phi_n|_{C_{\alpha_n}}$. Thus we can define Φ_{n+1} by

$$\Phi_{n+1}(p) := \begin{cases} \Phi_n(p), & p \in \bigcup_{i=1}^{n} U_{\alpha_i} \setminus V_{\alpha_{n+1}}, \\ \Phi_{\alpha_{n+1}}(p), & p \in V_{\alpha_{n+1}}. \end{cases} \tag{34}$$

Then Φ_{n+1} conjugates $\{f^t : t \in \mathbb{R}\}$ and $\{h^t : t \in \mathbb{R}\}$ on the set $\bigcup_{i=1}^{n} U_{\alpha_i} \cup C_{\alpha_{n+1}}$. Since $\{f^t : t \in \mathbb{R}\}$ and $\{h^t : t \in \mathbb{R}\}$ are parallelizable on $U_{\alpha_{n+1}}$ we can extend the topological conjugacy Φ_{n+1} on the component of $U_{\alpha_{n+1}} \setminus C_{\alpha_{n+1}}$ which do not contain C_{α_n} (see [12]). Such an extension is really needed in case of (a) to obtain the conjugacy on the whole plane. In case of (c), for any $\alpha_{n-1} \in A^-$ we extend Φ_n from $\bigcup_{i=-1}^{n} U_{\alpha_i}$ to Φ_{n-1} defined on $\bigcup_{i=-1}^{n} U_{\alpha_i} \cup C_{\alpha_{n-1}}$ in a similar way. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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