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Research Article

Asymptotic Properties of Solutions to Third-Order Nonlinear Neutral Differential Equations

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The aim of this work is to discuss asymptotic properties of a class of third-order nonlinear neutral functional differential equations. The results obtained extend and improve some related known results. Two examples are given to illustrate the main results.

1. Introduction

In this work, we study the asymptotic behavior of solutions of the third-order neutral differential equation

$$\begin{aligned}&\Big(a(t)\big(b(t)(x(t)+p(t)x(\sigma(t)))'\big)'\Big)'\\&\quad+q(t)f(x(\tau(t)))g\big(x'(t)\big)=0.\end{aligned} \tag{1}$$

We always assume that the following conditions hold:

(H_1) $a(t), b(t), p(t), q(t) \in C([t_0,\infty),[0,\infty))$, $0 \le p(t) \le p_0 < 1$;

(H_2) $\sigma(t), \tau(t) \in C([t_0,\infty),[0,\infty))$, $\sigma(t) \le t$, $\tau(t) \le t$, $\lim_{t\to\infty}\sigma(t) = \lim_{t\to\infty}\tau(t) = \infty$;

(H_3) $f \in C(R,R)$, $f(x)/x \ge K > 0$, for all $x \ne 0$;

(H_4) $g \in C(R,[L,\infty))$, $L > 0$;

(H_5) $\int_{t_0}^{\infty}(1/a(t))dt = \int_{t_0}^{\infty}(1/b(t))dt = \infty$.

Set $z(t) = x(t) + p(t)x(\sigma(t))$. By a solution of (1), we mean a nontrivial function $x(t) \in C([T_x,\infty),R)$, $T_x \ge t_0$, which has the properties $z(t) \in C^1([T_x,\infty),R)$, $b(t)z'(t) \in C^1([T_x,\infty))$, and $a(t)(b(t)z'(t))' \in C^1([T_x,\infty))$ and satisfies (1) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (1) which satisfies $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (1) possesses such a solution. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is called nonoscillatory.

Recently, great attention has been devoted to the oscillation of various classes of differential equations. See, for example, [1–19]. Hartman and Wintner [1] and Erbe et al. [3] studied the third-order differential equation

$$x'''(t) + q(t)x(t) = 0. \tag{2}$$

Paper [5] studied the oscillation of third-order trinomial delay differential equation

$$x'''(t) + p(t)x'(t) + g(t)x(\tau(t)) = 0. \tag{3}$$

Li et al. [7] discussed (1) with $f(x(\tau(t))) = x(\tau(t))$ and $g(x'(t)) = 1$. Han [8] examined the oscillation of (1) with $b(t) = 1$.

In this work, we establish some oscillation criteria for (1) which extend and improve the results in [7, 8].

2. Main Results

In the following, all functional inequalities considered are assumed to hold eventually for all t large enough. Without loss of generality, we deal only with the positive solutions of (1).

Theorem 1. *Suppose that*

$$\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{v}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} q(s)\,ds\,du\,dv = \infty. \tag{4}$$

If for some function $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_2 > t_1 > t_0$, *one has*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\rho(s) Q(s) - \frac{(\rho'(s))^2 a(s)}{4\rho(s)}\right] ds = \infty, \tag{5}$$

where

$$Q(t) = \frac{KL(1-p_0)\int_{t_2}^{\tau(t)} \int_{t_1}^{v} (1/(a(u)/b(v)))\,du\,dv}{\int_{t_1}^{t} (1/b(u))\,du}, \tag{6}$$

then all solutions of (1) *are oscillatory or convergent to zero asymptotically.*

Proof. Assume that x is a positive solution of (1). Based on condition (H_5), there are two possible cases:

(1) $z(t) > 0$, $z'(t) > 0$, $(b(t)z'(t))' > 0$, $[a(t)(b(t)z'(t))']' < 0$;

(2) $z(t) > 0$, $z'(t) < 0$, $(b(t)z'(t))' > 0$, $[a(t)(b(t)z'(t))']' < 0$.

First, consider that $z(t)$ satisfies (1). We have

$$x(t) = z(t) - p(t)x(\sigma(t)) \geq (1-p_0)z(t). \tag{7}$$

From (1), (H_3), and (H_4), we get

$$[a(t)(b(t)z'(t))']' \leq -KL(1-p_0)q(t)z(\tau(t)) \leq 0. \tag{8}$$

Define a function ω by

$$\omega(t) = \rho(t)\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}, \quad t \geq t_1; \tag{9}$$

we obtain $\omega(t) > 0$. Then

$$\begin{aligned}
\omega'(t) &= \rho'(t)\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)} + \rho(t)\frac{(a(t)(b(t)z'(t))')'}{b(t)z'(t)} - \rho(t)\frac{a(t)\left[(b(t)z'(t))'\right]^2}{(b(t)z'(t))^2} \\
&= \rho(t)\frac{\left(a(t)(b(t)z'(t))'\right)'}{b(t)z'(t)} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\omega^2(t)}{\rho(t)a(t)} \\
&\leq -\frac{KL(1-p_0)\rho(t)q(t)z(\tau(t))}{b(t)z'(t)} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\omega^2(t)}{\rho(t)a(t)} \\
&\leq -\frac{KL(1-p_0)\rho(t)q(t)z(\tau(t))}{b(t)z'(t)} - \left[\frac{\omega(t)}{\sqrt{\rho(t)a(t)}} - \frac{1}{2}\sqrt{\frac{a(t)}{\rho(t)}}\rho'(t)\right]^2 + \frac{(\rho'(t))^2 a(t)}{4\rho(t)}.
\end{aligned} \tag{10}$$

By the proof of [7, Theorem 2.1], we have

$$\omega'(t) \leq -\left[\rho(t)Q(t) - \frac{(\rho'(t))^2 a(t)}{4\rho(t)}\right], \tag{11}$$

where $Q(t)$ is defined as in (6). We obtain

$$\int_{t_1}^{t}\left[\rho(s)Q(s) - \frac{(\rho'(s))^2 a(s)}{4\rho(s)}\right] ds \leq -\int_{t_1}^{t}\omega'(s)\,ds. \tag{12}$$

That is,

$$\begin{aligned}
&\int_{t_1}^{t}\left[\rho(s)Q(s) - \frac{(\rho'(s))^2 a(s)}{4\rho(s)}\right] ds \\
&\quad \leq \omega(t_1) - \omega(t) < \omega(t_1) < \infty,
\end{aligned} \tag{13}$$

which contradicts (5). Assume that case (2) holds. Using the similar proof of [8, Lemma 4], we can get $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Based on Theorem 1, we present a Kamenev-type criterion for (1).

Theorem 2. *Assume that* (4) *holds. If for some function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 > t_0$, *one has*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n\left[\rho(s)Q(s)-\frac{(\rho'(s))^2a(s)}{4\rho(s)}\right]ds=\infty, \tag{14}$$

then all solutions of (1) *are oscillatory or convergent to zero asymptotically.*

Proof. Assume that $x(t)$ is a positive solution of (1). Then by the proof of Theorem 1, we have cases (1) and (2). Let case (1) hold. Proceeding as in the proof of Theorem 1, we have (11). Then we have

$$\begin{aligned}\int_{t_1}^{t}(t-s)^n\left[\rho(s)Q(s)-\frac{(\rho'(s))^2a(s)}{4\rho(s)}\right]ds\\ \le -\int_{t_1}^{t}(t-s)^n\omega'(s)\,ds.\end{aligned} \tag{15}$$

That is,

$$\begin{aligned}&\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n\left[\rho(s)Q(s)-\frac{(\rho'(s))^2a(s)}{4\rho(s)}\right]ds\\ &\quad\le -\frac{n}{t^n}\int_{t_1}^{t}(t-s)^{n-1}\omega(s)\,ds+\left(1-\frac{t_1}{t}\right)^n\omega(t_1)\\ &\quad<\left(1-\frac{t_1}{t}\right)^n\omega(t_1)<\infty,\end{aligned} \tag{16}$$

which contradicts (14). Assume that case (2) holds. We can get $\lim_{t\to\infty}x(t)=0$. The proof is completed. □

Next, we present a Philos-type criterion for (1). Let

$$D=\{(t,s):t\ge s\ge t_0\},\qquad D_0=\{(t,s):t>s\ge t_0\}. \tag{17}$$

We say that a function $H\in C(D,R)$ belongs to a function class P, if it satisfies

(i) $H(t,t)=0$, $t\ge t_0$; $H(t,s)>0$, $(t,s)\in D_0$;

(ii) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable, and such that

$$-\frac{\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)},\quad (t,s)\in D_0. \tag{18}$$

Theorem 3. *Assume that* (4) *holds. If for some function* $\rho\in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1>t_0$, *one has*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\rho(s)Q(s)-\frac{h_1^2(t,s)}{4B(s)}\right]ds=\infty, \tag{19}$$

where $Q(t)$ *is defined as in* (6), $B(t)=1/\rho(t)a(t)$, *and*

$$h_1(t,s)=h(t,s)-\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}, \tag{20}$$

then all solutions of (1) *are oscillatory or convergent to zero asymptotically.*

Proof. Assume that $x(t)$ is a positive solution of (1), and $z(t)$ has the case of (1); $\omega(t)$ is defined as in (9). Then

$$\omega'(t)\le-\rho(t)Q(t)+\frac{\rho'(t)}{\rho(t)}\omega(t)-\frac{1}{a(t)\rho(t)}\omega^2(t). \tag{21}$$

Let $B(t)=1/\rho(t)a(t)$, we have

$$\rho(t)Q(t)\le-\omega'(t)+\frac{\rho'(t)}{\rho(t)}\omega(t)-B(t)\omega^2(t). \tag{22}$$

We obtain

$$\begin{aligned}&\int_{t_1}^{t}H(t,s)\rho(s)Q(s)\,ds\\ &\quad\le\int_{t_1}^{t}H(t,s)\left[-\omega'(s)+\frac{\rho'(s)}{\rho(s)}\omega(s)-B(s)\omega^2(s)\right]ds\\ &\quad=H(t,t_1)\omega(t_1)\\ &\qquad-\int_{t_1}^{t}\left[\left(h(t,s)\sqrt{H(t,s)}-H(t,s)\frac{\rho'(s)}{\rho(s)}\right)\omega(s)\right.\\ &\qquad\qquad\left.+H(t,s)B(s)\omega^2(s)\right]ds\\ &\quad=H(t,t_1)\omega(t_1)\\ &\qquad-\int_{t_1}^{t}\left[\sqrt{H(t,s)}h_1(t,s)\omega(s)+H(t,s)B(s)\omega^2(s)\right]ds\\ &\quad=H(t,t_1)\omega(t_1)\\ &\qquad-\int_{t_1}^{t}\left[\sqrt{H(t,s)B(s)}\omega(s)+\frac{h_1(t,s)}{2\sqrt{B(s)}}\right]^2ds\\ &\qquad+\int_{t_1}^{t}\frac{h_1^2(t,s)}{4B(s)}ds<H(t,t_1)\omega(t_1)+\int_{t_1}^{t}\frac{h_1^2(t,s)}{4B(s)}ds.\end{aligned} \tag{23}$$

That is,

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\rho(s)Q(s)-\frac{h_1^2(t,s)}{4B(s)}\right]ds\le\omega(t_1), \tag{24}$$

which contradicts (19). Assume that (2) holds. We can get $\lim_{t\to\infty}x(t)=0$. The proof is completed. □

3. Examples

In this section, we will present two examples to illustrate the main results.

Example 4. Consider the third-order nonlinear neutral differential equation:

$$\left(t\left(x(t)+p_1 x\left(\frac{t}{2}\right)\right)''\right)' + \frac{\lambda}{t^2} x(t)\left(1+\left(x'(t)\right)^2\right) = 0, \quad \lambda > 0,\ t \geq 1, \tag{25}$$

where $p_1 \in [0,1)$, and $K = L = 1$. Let $\rho(t) = t$. It follows from Theorem 1 that every solution $x(t)$ of (25) is oscillatory or convergent to zero asymptotically.

Example 5. Consider the third-order nonlinear neutral differential equation:

$$\begin{aligned}&\left(\frac{1}{t}\left(t^{1/2}\left(x(t)+\frac{1}{2}x\left(t-\frac{1}{2}\right)\right)'\right)'\right)' \\ &\quad + t^{\lambda}\left(\lambda\frac{2-\cos t}{t}+2+\sin t\right) \\ &\quad \times x(t-1)\left(1+x^2(t-1)\right)\left(1+\left(x'(t)\right)^2\right) = 0,\end{aligned} \tag{26}$$

where $\lambda > 0$, $t \geq 1$. We have

$$\begin{aligned}\int_{t_0}^{t} q(s)\,\mathrm{d}s &= \int_{t_1}^{t} s^{\lambda}\left(\lambda\frac{2-\cos s}{s}+2+\sin s\right)\mathrm{d}s \\ &\geq \int_{t_1}^{t} s^{\lambda}\left(\lambda\frac{2-\cos s}{s}+\sin s\right)\mathrm{d}s \\ &= \int_{t_1}^{t} d\left[s^{\lambda}(2-\cos s)\right] \\ &= t^{\lambda}(2-\cos t) - t_0^{\lambda}(2-\cos t_0) \\ &\geq t^{\lambda} - K_0 \longrightarrow \infty\ (t \longrightarrow \infty);\end{aligned} \tag{27}$$

we see that (4) and (H_1)–(H_5) hold. Let $H(t,s) = (t-s)^2$, $\rho(t) = 1$. Then $h_1(t,s) = 2$. It follows, from Theorem 3, that the solutions of (26) are oscillatory or convergent to zero asymptotically.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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