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Research Article

Oscillation Theorems for Second-Order Half-Linear Neutral Delay Dynamic Equations with Damping on Time Scales

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We establish the oscillation criteria of Philos type for second-order half-linear neutral delay dynamic equations with damping on time scales by the generalized Riccati transformation and inequality technique. Our results are new even in the continuous and the discrete cases.

1. Introduction

In reality, it is known that the movement in the vacuum or ideal state is rare, while the movement with damping and disturbance is extensive. In recent years, the study of the oscillation of the second-order dynamic equations with damping on time scales is emerging; see [1–7], for example. Besides, the study of the oscillation for the second-order linear and nonlinear or semilinear dynamic equations can be found in [8–23] and of the oscillation for the high-order dynamic equations can be found in [24–33]. Then, inspired by the above work, this paper will study the oscillatory behavior of all solutions of a more extensive second-order half-linear neutral delay dynamic equation with damping, which is given as follows:

$$\begin{aligned}\left(a(t)\,\Phi\left(z^{\Delta}(t)\right)\right)^{\Delta} + p(t)\,\Phi\left(z^{\Delta}(t)\right) \\ + q(t)\,f(\Phi(x(\tau(t)))) = 0, \quad t \in \mathbb{T},\ t \geq t_0,\end{aligned} \tag{1}$$

where $\Phi(s) = |s|^{\gamma-2}s$, $z(t) = x(t) + r(t)x(\tau(t))$, $\gamma > 1$.

Here, we give the following hypotheses at first.

(H_1) $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above and when $t_0 \in \mathbb{T}$ with $t_0 > 0$, we define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$.

(H_2) $a, r, p, q : \mathbb{T} \to \mathbb{R}$ are positive rd-continuous functions such that $0 \leq r(t) < 1$, $-p/a \in \mathscr{R}^{+}$, where $\mathscr{R}$ is defined as the set of all regressive and rd-continuous functions and $\mathscr{R}^{+}$ is all positively regressive elements of $\mathscr{R}$.

(H_3) $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing and differentiable function such that

$$\tau(t) \leq t, \qquad \lim_{t\to\infty} \tau(t) = \infty, \qquad \tau(\mathbb{T}) = \mathbb{T}. \tag{2}$$

(H_4) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for some positive constant L,

$$\frac{f(x)}{x} \geq L \quad \text{for all } x \neq 0. \tag{3}$$

The solution of (1) defines a nontrivial real-valued function x satisfying (1) for $t \in \mathbb{T}$. A solution x of (1) is called oscillatory if it is neither eventually positive nor negative; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory. Here, we pay attention to those solutions of (1) which are not the eventually identical zero.

The purpose of this paper is to establish the oscillation criteria of Philos [34] for (1). The two famous results of Philos [34] about oscillation of second-order linear differential equations are extended to (1), while it satisfies

$$\int_{t_0}^{\infty} \left[\frac{1}{a(t)} e_{-p/a}(t,t_0)\right]^{1/(\gamma-1)} \Delta t = \infty. \tag{4}$$

Besides, two criteria of (1) about the fact that each solution is either oscillatory or converges to zero are obtained when

$$\int_{t_0}^{\infty}\left[\frac{1}{a(t)}e_{-p/a}(t,t_0)\right]^{1/(\gamma-1)}\Delta t<\infty. \tag{5}$$

The paper is organized as follows. In Section 2, we present some basic definitions and results about the theory of calculus on time scales. In Section 3, we give some lemmas. Section 4 introduces the main results of this paper. We established four new oscillatory criteria when conditions (4) and (5) hold, respectively, for the solutions of (1) in this paper.

2. Some Preliminaries

On the time scale $\mathbb{T}$ we define the forward and backward jump operators by

$$\begin{aligned}\sigma(t)&=\inf\{s\in\mathbb{T}:s>t\},\\ \rho(t)&=\sup\{s\in\mathbb{T}:s<t\}.\end{aligned} \tag{6}$$

A point $t\in\mathbb{T}$ is said to be left-dense if it satisfies $\rho(t)=t$, right-dense if it satisfies $\sigma(t)=t$, left-scattered if it satisfies $\rho(t)<t$, and right-scattered if it satisfies $\sigma(t)>t$. The graininess function $\mu:\mathbb{T}\to[t_0,\infty)$ of the time scale is defined by $\mu(t)=\sigma(t)-t$. For a function $f:\mathbb{T}\to\mathbb{R}$, the (delta) derivative is defined by

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\sigma(t)-t}, \tag{7}$$

if f is continuous at t and t is right-scattered. If t is right-dense, then the derivative is defined by

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}, \tag{8}$$

provided this limit exists. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at every left-dense point. Denote by $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ the set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$, and denote by $C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ the set of functions f which is Δ-differentiable and the derivative f^{Δ} is rd-continuous. The derivative f^{Δ} of f, the shift f^{σ} of f, and the graininess function μ are related by the following formula:

$$f^{\sigma}=f+\mu f^{\Delta}\quad\text{where } f^{\sigma}=f\circ\sigma. \tag{9}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g:

$$\begin{aligned}(fg)^{\Delta}(t)&=f^{\Delta}(t)\,g(t)+f(\sigma(t))\,g^{\Delta}(t)\\ &=f(t)\,g^{\Delta}(t)+f^{\Delta}(t)\,g(\sigma(t)),\end{aligned} \tag{10}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)\,g(t)-f(t)\,g^{\Delta}(t)}{g(t)\,g(\sigma(t))}\quad\text{if } gg^{\sigma}\neq0. \tag{11}$$

For $b,c\in\mathbb{T}$, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\,\Delta t=f(c)-f(b). \tag{12}$$

The integration by parts formula reads

$$\begin{aligned}\int_b^c f^{\Delta}(t)\,g(t)\,\Delta t&=f(c)\,g(c)-f(b)\,g(b)\\ &\quad-\int_b^c f^{\sigma}(t)\,g^{\Delta}(t)\,\Delta t,\end{aligned} \tag{13}$$

and the infinite integral is defined by

$$\int_b^{\infty}f(s)\,\Delta s=\lim_{t\to\infty}\int_b^t f(s)\,\Delta s. \tag{14}$$

For more details, see [8, 9].

3. Several Lemmas

In this section, we present six lemmas that are needed in Section 4. The first lemma is well known, and it can be found in Chapter 2 of [8]. Lemma 2 is Theorem 1.93 of [8]; Lemma 3 is the simple corollary of Theorem 1.90 in [8]; Lemma 4 is Theorem 41 in [35]; and Lemma 5 is Theorem 3 in [36].

Lemma 1. *If $g\in\mathcal{R}^+$, that is, $g:\mathbb{T}\to\mathbb{R}$ is rd-continuous, such that $1+\mu(t)g(t)>0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$, then the initial value problem $y^{\Delta}=g(t)y$, $y(t_0)=y_0\in\mathbb{R}$ has a unique and positive solution on $[t_0,\infty)_{\mathbb{T}}$, denoted by $e_g(\cdot,t_0)$. This "exponential function" satisfies the semigroup property $e_g(a,b)e_g(b,c)=e_g(a,c)$.*

Lemma 2. *Assume that $\nu:\mathbb{T}\to\mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}}:=\nu(\mathbb{T})$ is a time scale. Let $w:\widetilde{\mathbb{T}}\to\mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\widetilde{\Delta}}(\nu(t))$ exist on $\mathbb{T}^k$, where*

$$\mathbb{T}^k=\begin{cases}\mathbb{T}\setminus(\rho(\sup\mathbb{T}),\sup\mathbb{T}], & \text{if }\ \sup\mathbb{T}<\infty,\\ \mathbb{T}, & \text{if }\ \sup\mathbb{T}=\infty,\end{cases} \tag{15}$$

then

$$(w\circ\nu)^{\Delta}=\left(w^{\widetilde{\Delta}}\circ\nu\right)\nu^{\Delta}. \tag{16}$$

Lemma 3. *If x is differentiable, then*

$$(x^{\gamma})^{\Delta}=\gamma x^{\Delta}\int_0^1\left[hx^{\sigma}+(1-h)\,x\right]^{\gamma-1}dh. \tag{17}$$

Lemma 4. *Assume that X and Y are nonnegative real numbers; then*

$$\lambda XY^{\lambda-1}-X^{\lambda}\le(\lambda-1)\,Y^{\lambda}\quad\text{for all }\lambda>1, \tag{18}$$

where the equality holds if and only if $X=Y$.

Lemma 5. *Let $a,b\in\mathbb{T}$ and $a<b$. Then for positive rd-continuous functions $f,g:[a,b]\to\mathbb{R}$ we have*

$$\int_a^b|f(s)\,g(s)|\,\Delta s\le\left(\int_a^b|f(s)|^p\Delta s\right)^{1/p}\left(\int_a^b|g(s)|^q\Delta s\right)^{1/q}, \tag{19}$$

where $p>1$ and $(1/p)+(1/q)=1$.

Lemma 6. *Assume that* (H_1)*–*(H_4) *and* (4) *hold. Let* $x(t)$ *be an eventually positive solution of* (1). *Then there exists* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$z^{\Delta}(t) > 0, \quad \left(a(t)\left|z^{\Delta}(t)\right|^{\gamma-2} z^{\Delta}(t)\right)^{\Delta} < 0. \tag{20}$$

Proof. Suppose that $x(t)$ is an eventually positive solution of (1). There exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. From the definition of $z(t)$, we get $z(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, and at the same time for $t \in [t_1,\infty)_{\mathbb{T}}$, from (1), we get

$$\left(a(t)\left|z^{\Delta}(t)\right|^{\gamma-2} z^{\Delta}(t)\right)^{\Delta} + p(t)\left|z^{\Delta}(t)\right|^{\gamma-2} z^{\Delta}(t) < 0. \tag{21}$$

Hence, from Lemma 1 and (11) we obtain

$$\begin{aligned}
&\left[\frac{a\left|z^{\Delta}\right|^{\gamma-2} z^{\Delta}}{e_{-p/a}(\cdot,t_0)}\right]^{\Delta} \\
&= \frac{\left(a\left|z^{\Delta}\right|^{\gamma-2} z^{\Delta}\right)^{\Delta} e_{-p/a}(\cdot,t_0) - e^{\Delta}_{-p/a}(\cdot,t_0)\, a\left|z^{\Delta}\right|^{\gamma-2} z^{\Delta}}{e_{-p/a}(\cdot,t_0)\, e^{\sigma}_{-p/a}(\cdot,t_0)} \\
&= \frac{\left(a\left|z^{\Delta}\right|^{\gamma-2} z^{\Delta}\right)^{\Delta} + p\left|z^{\Delta}\right|^{\gamma-2} z^{\Delta}}{e^{\sigma}_{-p/a}(\cdot,t_0)} < 0
\end{aligned} \tag{22}$$

for $[t_1,\infty)_{\mathbb{T}}$. So

$$\frac{a\left|z^{\Delta}\right|^{\gamma-2} z^{\Delta}}{e_{-p/a}(\cdot,t_0)} \tag{23}$$

is decreasing. By Lemma 1, $z^{\Delta}(t)$ is either eventually positive or eventually negative. Therefore, for arbitrary $t \in [t_1,\infty)_{\mathbb{T}}$, we have

$$z^{\Delta}(t) > 0. \tag{24}$$

Otherwise, we assume that (24) is not satisfied; then there exits $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) < 0$ for all $t \in [t_2,\infty)_{\mathbb{T}}$. Because (23) is decreasing, from Lemma 1 we have

$$\begin{aligned}
\frac{a(t)\left|z^{\Delta}(t)\right|^{\gamma-2} z^{\Delta}(t)}{e_{-p/a}(t,t_0)} &\le \frac{a(t_2)\left|z^{\Delta}(t_2)\right|^{\gamma-2} z^{\Delta}(t_2)}{e_{-p/a}(t_2,t_0)} \\
&= -\frac{M^{\gamma-1}}{e_{-p/a}(t_2,t_0)}
\end{aligned} \tag{25}$$

for $t \in [t_2,\infty)_{\mathbb{T}}$, where $M = a(t_2)^{1/(\gamma-1)}|z^{\Delta}(t_2)| > 0$. By (25) and Lemma 1, we get

$$-\left(z^{\Delta}(t)\right)^{\gamma-1} \ge \frac{M^{\gamma-1}}{a(t)} e_{-p/a}(t,t_2), \quad t \in [t_2,\infty)_{\mathbb{T}}; \tag{26}$$

that is,

$$z^{\Delta}(t) \le -M\left[\frac{1}{a(t)} e_{-p/a}(t,t_2)\right]^{1/(\gamma-1)}, \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{27}$$

After integrating the two sides of inequality (27) from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
z(t) \le z(t_2) - M\int_{t_2}^{t}\left[\frac{1}{a(s)} e_{-p/a}(s,t_2)\right]^{1/(\gamma-1)}\Delta s, \\
t \in [t_2,\infty)_{\mathbb{T}}.
\end{aligned} \tag{28}$$

Next, we find the limits of the two sides of (28) when $t \to \infty$. From (4), we get $\lim_{t\to\infty} z(t) = -\infty$. Therefore, $z(t)$ is eventually negative, which is contradictory to $z(t) > 0$. So the inequality (24) holds.

From (24) and (21), it is obvious that the second inequality of (20) holds. This completes the proof. □

4. Main Results

Firstly, the two famous results of Philos [24] about oscillation of second-order linear differential equations are extended to (1) when condition (4) is satisfied.

Theorem 7. *Assume that* (H_1)*–*(H_4) *and* (4) *hold. Let* $H : D_{\mathbb{T}} \equiv \{(t,s) : t \ge s \ge t_0, t,s \in [t_0,\infty)_{\mathbb{T}}\} \to \mathbb{R}$ *be rd-continuous function, such that*

$$\begin{aligned}
&H(t,t) = 0, \quad t \ge t_0; \\
&H(t,s) > 0, \quad t > s \ge t_0, \; t,s \in [t_0,\infty)_{\mathbb{T}},
\end{aligned} \tag{29}$$

and H *has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies* (31). *Let* $h : D_{\mathbb{T}} \to \mathbb{R}$ *be a rd-continuous function and satisfies*

$$-H^{\Delta_s}(t,s) = h(t,s)(H(t,s))^{(\gamma-1)/\gamma}, \quad (t,s) \in D_{\mathbb{T}}, \tag{30}$$

$$0 < \inf_{s\ge T_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T_0)}\right] \le \infty, \quad T_0 \in [t_0,\infty)_{\mathbb{T}}. \tag{31}$$

If there exist a positive and differentiable function $\delta : \mathbb{T} \to \mathbb{R}$ *such that* $\delta^{\Delta}(t) \ge 0$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$ *and a real rd-continuous function* $\Psi : [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{a(\tau(s))}{(\delta(s)\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\Delta s < \infty, \tag{32}$$

$$\int_{T_0}^{\infty}\frac{\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{1/(\gamma-1)}}\left(\frac{\Psi_+(\sigma(s))}{\delta(\sigma(s))}\right)^{\gamma/(\gamma-1)}\Delta s = \infty, \tag{33}$$

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\Bigg[&LH(t,s)\delta(s)q(s)(1-r(\tau(s)))^{\gamma-1} \\
&-\frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\Bigg]\Delta s \\
\ge \Psi(T),&
\end{aligned} \tag{34}$$

where $G(t,s) = (\delta^{\Delta}(s) - (p(s)/a(s))\delta(s))(H(t,s))^{1/\gamma} - \delta(s)h(t,s)$, $G_+(t,s) = \max\{0, G(t,s)\}$, $\Psi_+(t) = \max\{0,\Psi(t)\}$, *and* $T \in [T_0,\infty)_{\mathbb{T}}$, *then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$, such that $x(t)>0$ and $x[\tau(t)]>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$. By the definition of $z(t)$, it follows

$$\begin{aligned} x(t) &= z(t) - r(t)\,x(\tau(t)) \\ &\geq z(t) - r(t)\,z(\tau(t)) \\ &\geq (1-r(t))\,z(t), \quad t\in[t_1,\infty)_{\mathbb{T}}. \end{aligned} \tag{35}$$

Since it satisfies $\lim_{t\to\infty}\tau(t)=\infty$, there exists $T_0\in[t_0,\infty)_{\mathbb{T}}$ such that $\tau(t)\geq t_1$ for all $t\in[T_0,\infty)_{\mathbb{T}}$. Then if it satisfies $t\in[T_0,\infty)_{\mathbb{T}}$, we have

$$x(\tau(t)) \geq (1-r(\tau(t)))\,z(\tau(t)). \tag{36}$$

By Lemma 6 and (H_3), we obtain that

$$\frac{1}{z\circ\tau} \geq \frac{1}{z\circ\tau^\sigma}, \qquad a\left(z^\Delta\right)^{\gamma-1} \geq a^\sigma\left(z^{\Delta\sigma}\right)^{\gamma-1} \tag{37}$$

on $[T_0,\infty)_{\mathbb{T}}$ (where $(z^\Delta)^\sigma$ is short hand for $z^{\Delta\sigma}$), and

$$z^\Delta\circ\tau \geq \frac{(a^\sigma)^{1/(\gamma-1)}}{(a\circ\tau)^{1/(\gamma-1)}}z^{\Delta\sigma} \tag{38}$$

holds. Moreover, using Lemmas 3 and 6, it follows that

$$\begin{aligned} \left[(z\circ\tau)^{\gamma-1}\right]^\Delta &= (\gamma-1)\,(z\circ\tau)^\Delta \\ &\quad\times\int_0^1 \left[h\left(z\circ\tau^\sigma\right)+(1-h)\,(z\circ\tau)\right]^{\gamma-2}\mathrm{d}h \\ &\geq (\gamma-1)\,(z\circ\tau)^\Delta \\ &\quad\times\int_0^1 [h(z\circ\tau)+(1-h)\,(z\circ\tau)]^{\gamma-2}\mathrm{d}h \\ &= (\gamma-1)\,(z\circ\tau)^{\gamma-2}(z\circ\tau)^\Delta. \end{aligned} \tag{39}$$

In Lemma 2, let $v=\tau, w=z$, and $\mathbb{T}$ is unbounded above by (H_1), so $\mathbb{T}^k=\mathbb{T}$, and $\widetilde{\mathbb{T}}=v(\mathbb{T})=\tau(\mathbb{T})=\mathbb{T}$ by (H_3); using Lemma 2, we get

$$(z\circ\tau)^\Delta = \left(z^\Delta\circ\tau\right)\tau^\Delta. \tag{40}$$

Thus

$$\left[(z\circ\tau)^{\gamma-1}\right]^\Delta \geq (\gamma-1)\,(z\circ\tau)^{\gamma-2}\left(z^\Delta\circ\tau\right)\tau^\Delta. \tag{41}$$

By the above inequality and the first inequality in (37), we obtain that

$$\frac{\left[(z\circ\tau)^{\gamma-1}\right]^\Delta}{(z\circ\tau)^{\gamma-1}} \geq \frac{(\gamma-1)\left(z^\Delta\circ\tau\right)\tau^\Delta}{z\circ\tau^\sigma} \tag{42}$$

holds on $[T_0,\infty)_{\mathbb{T}}$. Now we define the function W by

$$W = \delta\frac{a\left(z^\Delta\right)^{\gamma-1}}{(z\circ\tau)^{\gamma-1}}. \tag{43}$$

Then we have $W>0$ on $[T_0,\infty)_{\mathbb{T}}$, and

$$\begin{aligned} W^\Delta &\overset{(10)}{=} \frac{\delta}{(z\circ\tau)^{\gamma-1}}\left[a\left(z^\Delta\right)^{\gamma-1}\right]^\Delta \\ &\quad + a^\sigma\left(z^{\Delta\sigma}\right)^{\gamma-1}\frac{(z\circ\tau)^{\gamma-1}\delta^\Delta-\delta\left[(z\circ\tau)^{\gamma-1}\right]^\Delta}{(z\circ\tau)^{\gamma-1}(z\circ\tau^\sigma)^{\gamma-1}} \\ &\overset{(1)(H_4)}{\leq} -\frac{Lq\delta(x\circ\tau)^{\gamma-1}}{(z\circ\tau)^{\gamma-1}} - \frac{p\delta}{(z\circ\tau)^{\gamma-1}}\left(z^\Delta\right)^{\gamma-1} \\ &\quad + a^\sigma\left(z^{\Delta\sigma}\right)^{\gamma-1}\frac{(z\circ\tau)^{\gamma-1}\delta^\Delta-\delta\left[(z\circ\tau)^{\gamma-1}\right]^\Delta}{(z\circ\tau)^{\gamma-1}(z\circ\tau^\sigma)^{\gamma-1}} \\ &\overset{(36)}{\leq} -Lq\delta(1-r\circ\tau)^{\gamma-1} - \frac{p\delta}{(z\circ\tau)^{\gamma-1}}\left(z^\Delta\right)^{\gamma-1} \\ &\quad + a^\sigma\left(z^{\Delta\sigma}\right)^{\gamma-1}\frac{(z\circ\tau)^{\gamma-1}\delta^\Delta-\delta\left[(z\circ\tau)^{\gamma-1}\right]^\Delta}{(z\circ\tau)^{\gamma-1}(z\circ\tau^\sigma)^{\gamma-1}} \\ &\overset{(43)}{\leq} -Lq\delta(1-r\circ\tau)^{\gamma-1} - \frac{p}{a}W + \frac{\delta^\Delta}{\delta^\sigma}W^\sigma \\ &\quad - \frac{\delta a^\sigma\left(z^{\Delta\sigma}\right)^{\gamma-1}\left[(z\circ\tau)^{\gamma-1}\right]^\Delta}{(z\circ\tau)^{\gamma-1}(z\circ\tau^\sigma)^{\gamma-1}} \\ &\overset{(37)}{\leq} -Lq\delta(1-r\circ\tau)^{\gamma-1} - \frac{p\delta}{a\delta^\sigma}W^\sigma + \frac{\delta^\Delta}{\delta^\sigma}W^\sigma \\ &\quad - \frac{\delta a^\sigma\left(z^{\Delta\sigma}\right)^{\gamma-1}\left[(z\circ\tau)^{\gamma-1}\right]^\Delta}{(z\circ\tau)^{\gamma-1}(z\circ\tau^\sigma)^{\gamma-1}} \\ &\overset{(42)}{\leq} -Lq\delta(1-r\circ\tau)^{\gamma-1} + \left(\delta^\Delta-\frac{p}{a}\delta\right)\frac{W^\sigma}{\delta^\sigma} \\ &\quad - \frac{(\gamma-1)\,\delta a^\sigma\left(z^{\Delta\sigma}\right)^{\gamma-1}\left(z^\Delta\circ\tau\right)\tau^\Delta}{(z\circ\tau^\sigma)^\gamma} \\ &\overset{(38)}{\leq} -Lq\delta(1-r\circ\tau)^{\gamma-1} + \left(\delta^\Delta-\frac{p}{a}\delta\right)\frac{W^\sigma}{\delta^\sigma} \\ &\quad - \frac{(\gamma-1)\,\delta\tau^\Delta(a^\sigma)^{\gamma/(\gamma-1)}\left(z^{\Delta\sigma}\right)^\gamma}{(a\circ\tau)^{1/(\gamma-1)}(z\circ\tau^\sigma)^\gamma} \\ &\overset{(43)}{\leq} -Lq\delta(1-r\circ\tau)^{\gamma-1} + \left(\delta^\Delta-\frac{p}{a}\delta\right)\frac{W^\sigma}{\delta^\sigma} \\ &\quad - \frac{(\gamma-1)\,\delta\tau^\Delta}{(a\circ\tau)^{1/(\gamma-1)}(\delta^\sigma)^{\gamma/(\gamma-1)}}(W^\sigma)^{\gamma/(\gamma-1)}; \end{aligned} \tag{44}$$

then we obtain

$$\begin{aligned} W^\Delta(t) &\leq -Lq(t)\,\delta(t)\,(1-r(\tau(t)))^{\gamma-1} \\ &\quad + \frac{\overline{\delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ &\quad - \frac{(\gamma-1)\,\delta(t)\,\tau^\Delta(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^\lambda}(W(\sigma(t)))^\lambda \end{aligned} \tag{45}$$

on $[T_0,\infty)_{\mathbb{T}}$, where $\lambda = \gamma/(\gamma-1)$, $\overline{\delta}(t) = \delta^{\Delta}(t) - (p(t)/a(t))\delta(t)$. Thus, for every $t, T \in [T_0,\infty)_{\mathbb{T}}$ with $t \geq T \geq T_0$, by (13), we get

$$\begin{aligned}
&\int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\
&\leq H(t,T)\,W(T) \\
&\quad - \int_T^t \left(-H^{\Delta_s}(t,s)\right) W(\sigma(s))\,\Delta s \\
&\quad + \int_T^t H(t,s)\,\frac{\overline{\delta}(s)}{\delta(\sigma(s))} W(\sigma(s))\,\Delta s \\
&\quad - \int_T^t H(t,s)\,\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}} (W(\sigma(s)))^{\lambda}\Delta s \\
&\overset{(30)}{=} H(t,T)\,W(T) \\
&\quad + \int_T^t \frac{\overline{\delta}(s)\,H(t,s) - \delta(\sigma(s))\,h(t,s)\,H^{1/\lambda}(t,s)}{\delta(\sigma(s))} W(\sigma(s))\,\Delta s \\
&\quad - \int_T^t H(t,s)\,\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}} (W(\sigma(s)))^{\lambda}\Delta s \\
&\leq H(t,T)\,W(T) \\
&\quad + \int_T^t \frac{\overline{\delta}(s)\,H^{(\lambda-1)/\lambda}(t,s) - \delta(s)\,h(t,s)}{\delta(\sigma(s))} \\
&\qquad \times H^{1/\lambda}(t,s)\,W(\sigma(s))\,\Delta s \\
&\quad - \int_T^t H(t,s)\,\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}} (W(\sigma(s)))^{\lambda}\Delta s \\
&\leq H(t,T)\,W(T) \\
&\quad + \int_T^t \frac{G_+(t,s)}{\delta(\sigma(s))} H^{1/\lambda}(t,s)\,W(\sigma(s))\,\Delta s \\
&\quad - \int_T^t H(t,s)\,\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}} (W(\sigma(s)))^{\lambda}\Delta s,
\end{aligned} \tag{46}$$

where $G(t,s) = \overline{\delta}(s)H^{(\lambda-1)/\lambda}(t,s) - \delta(s)h(t,s) = (\delta^{\Delta}(s) - (p(s)/a(s))\delta(s))(H(t,s))^{1/\gamma} - \delta(s)h(t,s)$, $G_+(t,s) = \max\{0, G(t,s)\}$. So using Lemma 4, let

$$X = \left[H(t,s)\frac{(\gamma-1)\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right]^{1/\lambda} W(\sigma(s)), \tag{47}$$

$$Y = \left[\frac{G_+(t,s)}{\lambda\delta(\sigma(s))}\left(\frac{(\gamma-1)\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right)^{-1/\lambda}\right]^{1/(\lambda-1)}. \tag{48}$$

Using the inequality (18), we have

$$\begin{aligned}
&\frac{G_+(t,s)}{\delta(\sigma(s))} H^{1/\lambda}(t,s)\,W(\sigma(s)) - H(t,s) \\
&\qquad \times \frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}} (W(\sigma(s)))^{\lambda} \\
&\quad \leq C\left(\frac{G_+(t,s)}{\delta(\sigma(s))}\right)^{\lambda/(\lambda-1)} \left(\frac{\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right)^{-1/(\lambda-1)},
\end{aligned} \tag{49}$$

where $C = (\lambda-1)\lambda^{-\lambda/(\lambda-1)}(\gamma-1)^{-1/(\lambda-1)} = 1/\gamma^{\gamma}$. Thus

$$\begin{aligned}
&\frac{G_+(t,s)}{\delta(\sigma(s))} H^{1/\lambda}(t,s)\,W(\sigma(s)) \\
&\quad - H(t,s)\,\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}} (W(\sigma(s)))^{\lambda} \\
&\quad \leq \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s).
\end{aligned} \tag{50}$$

From (46) and (50), we obtain

$$\begin{aligned}
&\int_T^t \left[LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\right. \\
&\qquad \left. - \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\right]\Delta s \leq H(t,T)\,W(T);
\end{aligned} \tag{51}$$

that is,

$$\begin{aligned}
&\frac{1}{H(t,T)} \int_T^t \left[LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\right. \\
&\qquad \left. - \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\right]\Delta s \leq W(T).
\end{aligned} \tag{52}$$

From condition (34), we have

$$\Psi(T) \leq W(T), \quad T \in [T_0,\infty)_{\mathbb{T}}, \tag{53}$$

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\
&\quad \geq \Psi(T).
\end{aligned} \tag{54}$$

By (46), we have

$$\begin{aligned}
&\frac{1}{H(t,T)}\int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s\\
&\quad\le W(T)+\frac{1}{H(t,T)}\int_T^t\frac{G_+(t,s)}{\delta(\sigma(s))}H^{1/\lambda}(t,s)\,W(\sigma(s))\,\Delta s\\
&\qquad-\frac{1}{H(t,T)}\int_T^t H(t,s)\,\frac{(\gamma-1)\,\delta(s)\,\tau^\Delta(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^\lambda}\\
&\qquad\times(W(\sigma(s)))^\lambda\Delta s,
\end{aligned}\tag{55}$$

and from the above inequality, let $T=T_0$, and denote that

$$\begin{aligned}
A(t)&=\frac{1}{H(t,T_0)}\int_{T_0}^t\frac{G_+(t,s)}{\delta(\sigma(s))}H^{1/\lambda}(t,s)\,W(\sigma(s))\,\Delta s,\\
B(t)&=\frac{1}{H(t,T_0)}\int_{T_0}^t H(t,s)\,\frac{(\gamma-1)\,\delta(s)\,\tau^\Delta(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^\lambda}\\
&\qquad\times(W(\sigma(s)))^\lambda\Delta s;
\end{aligned}\tag{56}$$

meanwhile noting (54), we obtain

$$\begin{aligned}
&\liminf_{t\to\infty}[B(t)-A(t)]\\
&\quad\le W(T_0)-\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\\
&\qquad\times\int_{T_0}^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s\\
&\quad\le W(T_0)-\Psi(T_0)<\infty.
\end{aligned}\tag{57}$$

Now we assert that

$$\int_{T_0}^\infty\frac{\delta(s)\,\tau^\Delta(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^\lambda}(W(\sigma(s)))^\lambda\Delta s<\infty\tag{58}$$

holds. Suppose to the contrary that

$$\int_{T_0}^\infty\frac{\delta(s)\,\tau^\Delta(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^\lambda}(W(\sigma(s)))^\lambda\Delta s=\infty.\tag{59}$$

By (31), there exists a constant $\varepsilon>0$ such that

$$\inf_{s\ge T_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T_0)}\right]>\varepsilon>0.\tag{60}$$

From (59), there exists a $T\in[T_0,\infty)_{\mathbb{T}}$ for arbitrary real number $M>0$ such that

$$\int_{T_0}^t\frac{\delta(s)\,\tau^\Delta(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^\lambda}(W(\sigma(s)))^\lambda\Delta s\ge\frac{M}{(\gamma-1)\,\varepsilon},\tag{61}$$

for $t\in[T,\infty)_{\mathbb{T}}$. By (13), we have

$$\begin{aligned}
B(t)&=\frac{1}{H(t,T_0)}\int_{T_0}^t\Bigg\{(\gamma-1)\,H(t,s)\\
&\qquad\times\left(\int_{T_0}^s\frac{\delta(u)\,\tau^\Delta(u)}{(a(\tau(u)))^{\lambda-1}(\delta(\sigma(u)))^\lambda}\right.\\
&\qquad\left.\times\,(W(\sigma(u)))^\lambda\Delta u\right)^{\Delta_s}\Bigg\}\Delta s\\
&=\frac{1}{H(t,T_0)}\int_{T_0}^t\Big\{\big[-(\gamma-1)\,H^{\Delta_s}(t,s)\big]\\
&\qquad\times\int_{T_0}^{\sigma(s)}\frac{\delta(u)\,\tau^\Delta(u)}{(a(\tau(u)))^{\lambda-1}(\delta(\sigma(u)))^\lambda}\\
&\qquad\times(W(\sigma(u)))^\lambda\Delta u\Big\}\Delta s\\
&\ge\frac{1}{H(t,T_0)}\int_T^t\Big\{\big[-(\gamma-1)\,H^{\Delta_s}(t,s)\big]\\
&\qquad\times\int_{T_0}^{s}\frac{\delta(u)\,\tau^\Delta(u)}{(a(\tau(u)))^{\lambda-1}(\delta(\sigma(u)))^\lambda}\\
&\qquad\times(W(\sigma(u)))^\lambda\Delta u\Big\}\Delta s\\
&\ge\frac{1}{H(t,T_0)}\int_T^t\big[-(\gamma-1)\,H^{\Delta_s}(t,s)\big]\frac{M}{(\gamma-1)\,\varepsilon}\Delta s\\
&=\frac{M}{\varepsilon}\frac{H(t,T)}{H(t,T_0)}.
\end{aligned}\tag{62}$$

From (60), there exists $t_2\in[T,\infty)_{\mathbb{T}}$ such that $H(t,T)/H(t,T_0)\ge\varepsilon$ for $t\in[t_2,\infty)_{\mathbb{T}}$, so $B(t)\ge M$. Since M is arbitrary, we have

$$\lim_{t\to\infty}B(t)=\infty.\tag{63}$$

Selecting a sequence $\{T_n\}_{n=1}^\infty : T_n\in[T_0,\infty)_{\mathbb{T}}$ with $\lim_{n\to\infty}T_n=\infty$ satisfying

$$\lim_{n\to\infty}[B(T_n)-A(T_n)]=\liminf_{t\to\infty}[B(t)-A(t)]<\infty,\tag{64}$$

then there exists a constant $M_0>0$ such that

$$B(T_n)-A(T_n)\le M_0\tag{65}$$

for sufficiently large positive integer n. From (63), we can easily see

$$\lim_{n\to\infty}B(T_n)=\infty,\tag{66}$$

and (65) implies that

$$\lim_{n\to\infty}A(T_n)=\infty.\tag{67}$$

From (65) and (66), we have

$$\frac{A(T_n)}{B(T_n)} - 1 \geq -\frac{M_0}{B(T_n)} > -\frac{M_0}{2M_0} = -\frac{1}{2}; \tag{68}$$

that is,

$$\frac{A(T_n)}{B(T_n)} > \frac{1}{2} \tag{69}$$

for sufficiently large positive integer n, which together with (67) implies

$$\lim_{n\to\infty} \frac{[A(T_n)]^\gamma}{[B(T_n)]^{\gamma-1}} = \lim_{n\to\infty} \left[\frac{A(T_n)}{B(T_n)}\right]^{\gamma-1} A(T_n) = \infty. \tag{70}$$

On the other hand, by Lemma 5, we obtain

$$\begin{aligned}
A(T_n) &= \frac{1}{H(T_n,T_0)} \int_{T_0}^{T_n} \frac{G_+(T_n,s)}{\delta(\sigma(s))} H^{1/\lambda}(T_n,s) W(\sigma(s)) \Delta s \\
&= \int_{T_0}^{T_n} \left\{ \left[\frac{(\gamma-1) H(T_n,s)\delta(s)\tau^\Delta(s)}{H(T_n,T_0)}\right]^{(\gamma-1)/\gamma} \right.\\
&\qquad \left. \times \frac{W(\sigma(s))}{[a(\tau(s))]^{1/\gamma}\delta(\sigma(s))} \right\} \\
&\quad \times \left\{ \frac{[a(\tau(s))]^{1/\gamma} G_+(T_n,s)}{H(T_n,T_0)} H^{(\gamma-1)/\gamma}(T_n,s) \right.\\
&\qquad \left. \times \left[\frac{(\gamma-1) H(T_n,s)\delta(s)\tau^\Delta(s)}{H(T_n,T_0)}\right]^{(1-\gamma)/\gamma} \right\} \Delta s \\
&\leq \left\{ \int_{T_0}^{T_n} \frac{(\gamma-1) H(T_n,s)\delta(s)\tau^\Delta(s)}{H(T_n,T_0)} \right.\\
&\qquad \left. \times \left[\frac{W(\sigma(s))}{(a(\tau(s)))^{1/\gamma}\delta(\sigma(s))}\right]^{\gamma/(\gamma-1)} \Delta s \right\}^{(\gamma-1)/\gamma} \\
&\quad \times \left\{ \int_{T_0}^{T_n} \frac{a(\tau(s)) G_+^\gamma(T_n,s)}{H^\gamma(T_n,T_0)} H^{\gamma-1}(T_n,s) \right.\\
&\qquad \left. \times \left[\frac{(\gamma-1) H(T_n,s)\delta(s)\tau^\Delta(s)}{H(T_n,T_0)}\right]^{1-\gamma} \Delta s \right\}^{1/\gamma} \\
&= [B(T_n)]^{(\gamma-1)/\gamma} \\
&\quad \times \left\{ \frac{(\gamma-1)^{1-\gamma}}{H(T_n,T_0)} \int_{T_0}^{T_n} a(\tau(s)) G_+^\gamma(T_n,s) \right.\\
&\qquad \left. \times \left[\delta(s)\tau^\Delta(s)\right]^{1-\gamma} \Delta s \right\}^{1/\gamma}.
\end{aligned} \tag{71}$$

The above inequality shows that

$$\frac{[A(T_n)]^\gamma}{[B(T_n)]^{\gamma-1}} \leq \frac{(\gamma-1)^{1-\gamma}}{H(T_n,T_0)} \int_{T_0}^{T_n} \frac{a(\tau(s))}{(\delta(s)\tau^\Delta(s))^{\gamma-1}} G_+^\gamma(T_n,s) \Delta s. \tag{72}$$

Hence, (70) implies

$$\lim_{n\to\infty} \frac{1}{H(T_n,T_0)} \int_{T_0}^{T_n} \frac{a(\tau(s))}{(\delta(s)\tau^\Delta(s))^{\gamma-1}} G_+^\gamma(T_n,s) \Delta s = \infty, \tag{73}$$

which contradicts (32). Therefore (58) holds. Noting $\Psi(T) \leq W(T)$ for $T \in [T_0,\infty)_{\mathbb{T}}$, by using (58), we obtain

$$\begin{aligned}
&\int_{T_0}^{\infty} \frac{\delta(s)\tau^\Delta(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^\lambda} (\Psi_+(\sigma(s)))^\lambda \Delta s \\
&\quad \leq \int_{T_0}^{\infty} \frac{\delta(s)\tau^\Delta(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^\lambda} (W(\sigma(s)))^\lambda \Delta s < \infty,
\end{aligned} \tag{74}$$

which is contradicting with (33). This completes the proof. □

Remark 8. From Theorem 7, we can obtain different conditions for oscillation of all solutions of (1) with different choices of $\delta(t)$ and $H(t,s)$. For example, $H(t,s) = (t-s)^m$ or $H(t,s) = (\ln((t+1)/(s+1)))^m$.

Theorem 9. *Assume that* (H_1)*–*(H_4)*,* (4), (30)-(31), *and* (33) *hold, where* H, h, δ, *and* Ψ *are defined in Theorem 7. Assume that*

$$\liminf_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} LH(t,s)\delta(s) q(s)(1-r(\tau(s)))^{\gamma-1} \Delta s < \infty, \tag{75}$$

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} &\left[LH(t,s)\delta(s) q(s)(1-r(\tau(s)))^{\gamma-1} \right.\\
&\left. -\frac{a(\tau(s))}{\gamma^\gamma(\delta(s)\tau^\Delta(s))^{\gamma-1}} G_+^\gamma(t,s) \right] \Delta s \\
&\geq \Psi(T)
\end{aligned} \tag{76}$$

holds, where $T \in [T_0,\infty)_{\mathbb{T}}$, $G(t,s) = (\delta^\Delta(s) - (p(s)/a(s))\delta(s))(H(t,s))^{1/\gamma} - \delta(s)h(t,s), G_+(t,s) = \max\{0, G(t,s)\}$. *Then* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. So $z(t) > 0$ and there exists $T_0 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$x(\tau(t)) \geq (1-r(\tau(t))) z(\tau(t)) \tag{77}$$

for $t \in [T_0, \infty)_{\mathbb{T}}$. Define the function W by

$$W = \delta \frac{a(z^{\Delta})^{\gamma-1}}{(z \circ \tau)^{\gamma-1}}, \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{78}$$

We proceed as in the proof of Theorem 7 to obtain (46) and (50), so that

$$\frac{1}{H(t,T)} \int_T^t \left[LH(t,s)\,\delta(s)\,q(s)\,(1 - r(\tau(s)))^{\gamma-1} - \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s) \right] \Delta s \leq W(T). \tag{79}$$

Hence, (76) implies

$$\Psi(T) \leq W(T), \quad T \in [T_0, \infty)_{\mathbb{T}}, \tag{80}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1 - r(\tau(s)))^{\gamma-1} \Delta s \geq \Psi(T); \tag{81}$$

then we have

$$\begin{aligned}\Psi(T) &\leq \liminf_{t\to\infty} \frac{1}{H(t,T)} \times \int_T^t \left[LH(t,s)\,\delta(s)\,q(s)\,(1 - r(\tau(s)))^{\gamma-1} - \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s) \right] \Delta s \\ &\leq \liminf_{t\to\infty} \frac{1}{H(t,T)} \times \int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1 - r(\tau(s)))^{\gamma-1} \Delta s \\ &\quad - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s.\end{aligned} \tag{82}$$

From the above inequality and (75), we have

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{a(\tau(s))}{(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s < \infty. \tag{83}$$

Therefore, there exists a sequence $\{T_n\}_{n=1}^{\infty} : T_n \in [T_0, \infty)_{\mathbb{T}}$ with $\lim_{n\to\infty} T_n = \infty$ such that

$$\lim_{n\to\infty} \frac{1}{H(T_n,T)} \int_T^{T_n} \frac{a(\tau(s))}{(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s < \infty. \tag{84}$$

Definitions of $A(t)$ and $B(t)$ are as in Theorem 7; from (46), and noting (81), we have

$$\begin{aligned}\limsup_{t\to\infty} [B(t) - A(t)] &\leq W(T_0) - \liminf_{t\to\infty} \frac{1}{H(t,T_0)} \times \int_{T_0}^t LH(t,s)\,\delta(s)\,q(s)\,(1 - r(\tau(s)))^{\gamma-1} \Delta s \\ &\leq W(T_0) - \Psi(T_0) < \infty.\end{aligned} \tag{85}$$

For the above sequence $\{T_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} [B(T_n) - A(T_n)] \leq \limsup_{t\to\infty} [B(t) - A(t)] < \infty. \tag{86}$$

We obtain (58) by using reductio ad absurdum. The rest of the proof is similar to that of Theorem 7 and hence is omitted. This completes the proof. □

If (4) is not satisfied, that is, if condition (5) holds, we can obtain the following results.

Theorem 10. *Assume that (H_1)–(H_4), (5), and (30)–(34) hold, where H, h, δ, and Ψ are defined in Theorem 7. Assume that*

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)} \int_{t_0}^t e_{-p/a}(t,\sigma(s))\,q(s) \times (1 - r(\tau(s)))^{\gamma-1} \Delta s \right)^{1/(\gamma-1)} \Delta t = \infty \tag{87}$$

holds. Then every solution $x(t)$ of (1) is either oscillatory or converges to zero on $[t_0, \infty)_{\mathbb{T}}$.

Proof. As the proof of Theorem 7, assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. So $z(t) > 0$ and there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$x(\tau(t)) \geq (1 - r(\tau(t)))\,z(\tau(t)) \tag{88}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. In the proof of Lemma 6, we find that $z^{\Delta}(t)$ is either eventually positive or eventually negative. Thus, we will distinguish the following two cases:

(I) $z^{\Delta}(t) > 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$;

(II) $z^{\Delta}(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$.

Case (I). When $z^{\Delta}(t)$ is an eventually positive and the proof is similar to that of the proof of Theorem 7, we can obtain that (1) is oscillatory.

Case (II). When $z^{\Delta}(t)$ is an eventually negative, $z(t)$ is decreasing and $\lim_{t\to\infty} z(t) =: b \geq 0$ exists. Therefore, there exists $T_0 \in [t_2, \infty)_{\mathbb{T}}$ such that

$$z(\tau(t)) > z(t) > z(\sigma(t)) \geq b \geq 0 \tag{89}$$

for $t \in [T_0, \infty)_{\mathbb{T}}$. Define the function $u(t) = a(t)|z^{\Delta}(t)|^{\gamma-2}z^{\Delta}(t) = -a(t)|z^{\Delta}(t)|^{\gamma-1}$. Equations (1) and (89) yield

$$\begin{aligned} u^{\Delta}(t) &= -\frac{p(t)}{a(t)}u(t) - q(t) f\left[(x(\tau(t)))^{\gamma-1}\right] \\ &\le -\frac{p(t)}{a(t)}u(t) - Lb^{\gamma-1}q(t)(1-r(\tau(t)))^{\gamma-1}, \\ &\qquad t \in [T_0, \infty)_{\mathbb{T}}. \end{aligned} \tag{90}$$

The inequality (90) is the assumed inequality of [8, Theorem 6.1] (see also [37, Lemma 1]). All assumptions of [8, Theorem 6.1], for example, $-p/a \in \mathscr{R}^{+}$, are satisfied as well. Hence the conclusion of [8, Theorem 6.1] holds; that is,

$$\begin{aligned} u(t) &\le u(T_0) e_{-p/a}(t, T_0) - Lb^{\gamma-1} \\ &\quad \times \int_{T_0}^{t} e_{-p/a}(t, \sigma(s)) q(s)(1-r(\tau(s)))^{\gamma-1}\Delta s \\ &< -Lb^{\gamma-1}\int_{T_0}^{t} e_{-p/a}(t, \sigma(s)) q(s)(1-r(\tau(s)))^{\gamma-1}\Delta s \end{aligned} \tag{91}$$

for all $t \in [T_0, \infty)_{\mathbb{T}}$, and thus

$$\begin{aligned} &\int_{T_0}^{l} z^{\Delta}(t)\Delta t \\ &\quad < -bL^{1/(\gamma-1)}\int_{T_0}^{l}\left(\frac{1}{a(t)}\int_{T_0}^{t} e_{-p/a}(t, \sigma(s)) q(s)\right. \\ &\qquad\qquad \left.\times (1-r(\tau(s)))^{\gamma-1}\Delta s\right)^{1/(\gamma-1)}\Delta t \end{aligned} \tag{92}$$

for all $l \in [T_0, \infty)_{\mathbb{T}}$. Assuming $b > 0$ and using (87) in (92), we can get $\lim_{l\to\infty} z(l) = -\infty$, and this is a contradiction to the fact that $z(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Thus $b = 0$; that is, $\lim_{t\to\infty} z(t) = 0$. Then, it follows from $(1-r(t))z(t) \le x(t) \le z(t)$ that $\lim_{t\to\infty} x(t) = 0$ holds. This completes the proof. □

Using the same method as in the proofs of Theorems 9 and 10, we can easily obtain the following results.

Theorem 11. *Assume that* (H_1)–(H_3), (5), (30)-(31), (33), (75)-(76), *and* (87) *hold, where* H, h, δ, *and* Ψ *are defined in Theorem 9. Then every solution* $x(t)$ *of* (1) *is either oscillatory or converges to zero on* $[t_0, \infty)_{\mathbb{T}}$.

Remark 12. The theorems in this paper are new even for the cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Example 13. Consider a second-order half-linear delay 2-difference equation with damping

$$\begin{aligned} &\left[\frac{1}{t^2}\left|z^{\Delta}(t)\right| z^{\Delta}(t)\right]^{\Delta} + \frac{1}{t^4}\left|z^{\Delta}(t)\right| z^{\Delta}(t) + \frac{1}{t^3}\left|x\left(\frac{t}{2}\right)\right| x\left(\frac{t}{2}\right) \\ &\quad = 0, \quad t \in \overline{2^{\mathbb{Z}}},\ t \ge t_0 := 2, \end{aligned} \tag{93}$$

where $z(t) = x(t) + (1/2)x(t/2)$. Here, we have

$$\begin{gathered} a(t) = \frac{1}{t^2}, \qquad r(t) = \frac{1}{2}, \qquad p(t) = \frac{1}{t^4}, \\ q(t) = \frac{1}{t^3}, \qquad f(u) = u, \qquad \tau(t) = \frac{t}{2}, \qquad \gamma = 3. \end{gathered} \tag{94}$$

Then $\mathbb{T} = \overline{2^{\mathbb{Z}}}$ is unbounded above, $\sigma(t) = 2t$, and $\mu(t) = t$. Conditions (H_1) and (H_3) are clearly satisfied, (H_4) holds with $L = 1$, and (H_2) is satisfied as

$$\begin{gathered} 1 - \mu(t)\frac{p(t)}{a(t)} = 1 - t\cdot\frac{1/t^4}{1/t^2} = 1 - \frac{1}{t} > 0 \\ \forall t \ge 2. \end{gathered} \tag{95}$$

Next, by [37, Lemma 2] and (H_2), we obtain

$$\begin{gathered} e_{-p/a}(t, 2) \ge 1 - \int_{2}^{t}\frac{p(s)}{a(s)}\Delta s = 1 - \int_{2}^{t} s^{-2}\Delta s = \frac{2}{t} \\ \forall t \ge 2, \end{gathered} \tag{96}$$

so

$$\begin{aligned} &\int_{2}^{t}\left[\frac{1}{a(s)}e_{-p/a}(s, 2)\right]^{1/(\gamma-1)}\Delta s \\ &\quad \ge \int_{2}^{t}\left[s^2\cdot\frac{2}{s}\right]^{1/2}\Delta s \\ &\quad = \int_{2}^{t}\sqrt{2}s^{1/2}\Delta s \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \end{aligned} \tag{97}$$

Hence (4) is satisfied. Now let $H(t, s) = (t-s)^2$; then

$$\begin{aligned} H^{\Delta_s}(t, s) &= \frac{(t-2s)^2 - (t-s)^2}{s} \\ &= \frac{(2t-3s)\cdot(-s)}{s} = -(2t-3s) < 0, \\ &\qquad \forall t > s \ge t_0 := 2. \end{aligned} \tag{98}$$

Since

$$\begin{aligned} -H^{\Delta_s}(t, s) = 2t - 3s &= \frac{2t-3s}{(t-s)^{4/3}}\left[(t-s)^2\right]^{2/3} \\ &= \frac{2t-3s}{(t-s)^{4/3}}\left[H(t, s)\right]^{(\gamma-1)/\gamma}, \end{aligned} \tag{99}$$

let $h(t,s) = (2t-3s)/(t-s)^{4/3}$; then condition (30) holds. We have

$$0 < \inf_{s\ge T_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T_0)}\right] = \inf_{s\ge T_0}\left[\liminf_{t\to\infty}\frac{(t-s)^2}{(t-T_0)^2}\right] = 1 < \infty, \quad \forall T_0 \in [t_0,\infty)_{\mathbb{T}}, \tag{100}$$

so condition (31) holds. Let $\delta(t) = t$ as $t \ge 2$; then $\delta^{\Delta}(t) = 1$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, and

$$\begin{aligned} G(t,s) &= \left(\delta^{\Delta}(s) - \frac{p(s)}{a(s)}\delta(s)\right)(H(t,s))^{1/\gamma} - \delta(s)h(t,s) \\ &= \left(1-\frac{1}{s}\right)H^{1/3}(t,s) - \frac{s(2t-3s)}{H^{2/3}(t,s)} \\ &= H^{1/3}(t,s) - \frac{H^{1/3}(t,s)}{s} - \frac{s(2t-3s)}{H^{2/3}(t,s)} < H^{1/3}(t,s), \end{aligned} \tag{101}$$

for all $t > s \ge 2$. Hence

$$\begin{aligned} &\int_{T_0}^{t}\frac{a(\tau(s))}{(\delta(s)\tau^{\Delta}(s))^{\gamma-1}}G_+^{\gamma}(t,s)\Delta s \\ &\quad< \int_{T_0}^{t}\frac{(s/2)^{-2}}{(s\cdot(1/2))^2}\left(H^{1/3}(t,s)\right)^3\Delta s \\ &\quad= 16\int_{T_0}^{t}\frac{(t-s)^2}{s^4}\Delta s \\ &\quad= 16\left[-\frac{8}{7t}+\frac{8}{3t}-\frac{2}{t}\right] \\ &\qquad - 16\left[-\frac{8t^2}{7T_0^3}+\frac{8t}{3T_0^2}-\frac{2}{T_0}\right]. \end{aligned} \tag{102}$$

We get

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{a(\tau(s))}{(\delta(s)\tau^{\Delta}(s))^{\gamma-1}}G_+^{\gamma}(t,s)\Delta s \\ &\le \limsup_{t\to\infty}\left(\left(16\left[-\frac{8}{7t}+\frac{8}{3t}-\frac{2}{t}\right]-16\left[-\frac{8t^2}{7T_0^3}+\frac{8t}{3T_0^2}-\frac{2}{T_0}\right]\right)\right. \\ &\qquad\left.\times(t-T_0)^{-2}\right) \\ &= \frac{128}{7}\frac{1}{T_0^3} < \infty; \end{aligned} \tag{103}$$

thus condition (32) holds. Let $\Psi(t) = 1/4t$; then

$$\begin{aligned} &\int_{T_0}^{\infty}\frac{\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{1/(\gamma-1)}}\left(\frac{\Psi_+(\sigma(s))}{\delta(\sigma(s))}\right)^{\gamma/(\gamma-1)}\Delta s \\ &\quad= \int_{T_0}^{\infty}\frac{s\cdot(1/2)}{\left((s/2)^{-2}\right)^{1/2}}\left(\frac{1/8s}{2s}\right)^{3/2}\Delta s \\ &\quad= \int_{T_0}^{\infty}\frac{s^2}{4}\left(\frac{1}{16s^2}\right)^{3/2}\Delta s \\ &\quad= \frac{1}{256}\int_{T_0}^{\infty}\frac{1}{s}\Delta s = \frac{1}{256}\frac{\ln s}{\ln 2}\bigg|_{T_0}^{\infty} = \infty; \end{aligned} \tag{104}$$

that is, condition (33) holds. Since

$$\begin{aligned} &\int_{T}^{t}LH(t,s)\delta(s)q(s)(1-r(\tau(s)))^{\gamma-1}\Delta s \\ &\quad= \frac{1}{4}\int_{T}^{t}\frac{(t-s)^2\cdot s}{s^3}\Delta s \\ &\quad= \frac{1}{4}\int_{T}^{t}\left(\frac{t^2}{s^2}-\frac{2t}{s}+1\right)\Delta s \\ &\quad= \frac{1}{4}\left[-\frac{2t^2}{s}-\frac{2t\ln s}{\ln 2}+s\right]_{T}^{t} \\ &\quad= \frac{1}{4}\left[-2t-\frac{2t\ln t}{\ln 2}+t\right] \\ &\qquad - \frac{1}{4}\left[-\frac{2t^2}{T}-\frac{2t\ln T}{\ln 2}+T\right], \end{aligned} \tag{105}$$

then

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}LH(t,s)\delta(s)q(s)(1-r(\tau(s)))^{\gamma-1}\Delta s = \frac{1}{2T}. \tag{106}$$

Moreover, (103) implies

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\tau^{\Delta}(s))^{\gamma-1}}G_+^{\gamma}(t,s)\Delta s \le \frac{128}{63}\frac{1}{T^3}. \tag{107}$$

Thus, when T is enough large, we have

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[LH(t,s)\delta(s)q(s)(1-r(\tau(s)))^{\gamma-1}\right. \\ &\qquad\qquad\left. - \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\tau^{\Delta}(s))^{\gamma-1}}G_+^{\gamma}(t,s)\right]\Delta s \\ &\quad\ge \frac{1}{2T}-\frac{128}{63}\frac{1}{T^3} \ge \frac{1}{4T} = \Psi(T), \end{aligned} \tag{108}$$

so (34) is satisfied. By Theorem 7, (93) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$. Similarly, conditions (75) and (76) are satisfied as well. By Theorem 9, we can also obtain that (93) is oscillatory. But the other known results cannot be applied in (93).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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