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Research Article

Natural Filtrations of Infinite-Dimensional Modular Contact Superalgebras

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The natural filtration of the infinite-dimensional contact superalgebra over an algebraic closed field of positive characteristic is proved to be invariant under automorphisms by characterizing ad-nilpotent elements and the subalgebras generated by certain ad-nilpotent elements. Moreover, we obtain an intrinsic characterization of contact superalgebras and a property of automorphisms of these Lie superalgebras.

1. Introduction

Filtration structures play an important role in the classification of modular Lie algebras (see [1, 2]) and nonmodular Lie superalgebras (see [3, 4]), respectively. We know that the Lie algebras and Lie superalgebras of Cartan type possess a natural filtration structure. The natural filtrations of finite-dimensional modular Lie algebras of Cartan type were proved to be invariant in [5, 6]. In the infinite-dimensional case, the same conclusion was proved in [7], by determining ad-nilpotent elements. In the case of Lie superalgebras of Cartan type, the invariance of the natural filtrations of some Lie superalgebras was proved in [8, 9]. Similar results for Lie superalgebras of generalized Cartan type were obtained in [10–12], respectively.

In this paper, we consider the infinite-dimensional modular contact superalgebra $K(2r+1,n)$, which is analogous to the one in the nonmodular situation (see [13]). But since the principal $\mathbb{Z}$-gradations of Lie superalgebras of Cartan type are different (see [13]), most results and proofs for other Lie superalgebras cannot be applied to contact superalgebras. Therefore the corresponding results and proofs for contact superalgebras have to be established separately. By determining the ad-nilpotent elements and subalgebras generated by certain ad-nilpotent elements, we prove the main result of this paper.

Theorem 1. *The natural filtration of the infinite-dimensional contact Lie superalgebra is invariant under automorphisms.*

Thereby, one obtains the following theorems.

Theorem 2. *Suppose that r, r', n, n' are positive integers. Then $K(2r+1,n) \cong K(2r'+1,n')$ if and only if $(r,n) = (r',n')$.*

Theorem 3. *Let ϕ, ψ be automorphisms of $K(2r+1,n)$. Then $\phi = \psi$ if and only if $\phi|_{K_{[-1]}} = \psi|_{K_{[-1]}}$.*

The paper is organized as follows. In Section 2, we recall the necessary definitions concerning the modular contact superalgebra $K(2r+1,n)$. In Section 3, we study the ad-nilpotent elements of $K(2r+1,n)$. In Section 4, we complete the proofs of Theorems 1–3.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ denotes an algebraic closed field of characteristic $p > 2$ and $\mathbb{Z}_2 = \{\overline{0},\overline{1}\}$ the ring of integers modulo 2. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and nonnegative integers, respectively. Given $m \in \mathbb{N}$, $m > 1$ and $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_m) \in \mathbb{N}_0^m$, we put $|\alpha| = \sum_{i=1}^m \alpha_i$. Let $\mathcal{O}(m)$ denote the divided power algebra over $\mathbb{F}$ with basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{N}_0^m\}$. For $\varepsilon_i = (\delta_{i1},\delta_{i2},\ldots,\delta_{im})$, $i = 1,2,\ldots,m$, we abbreviate $x^{(\varepsilon_i)}$ to x_i. Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in n variables $x_{m+1}, x_{m+2}, \ldots, x_s$, where $s = m + n$. Denote by $\mathcal{O}(m,n)$ the tensor product $\mathcal{O}(m)\otimes\Lambda(n)$. The trivial $\mathbb{Z}_2$-gradation of $\mathcal{O}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$ induce a $\mathbb{Z}_2$-gradation of $\mathcal{O}(m,n)$ such that $\mathcal{O}(m,n)$ is an associative superalgebra. For $g \in \mathcal{O}(m)$ and $f \in \Lambda(n)$, we abbreviate $g \otimes f$ to gf. For $\alpha, \beta \in \mathbb{N}_0^m$ and $i, j = m+1, m+2, \ldots, s$, the following formulas in $\mathcal{O}(m,n)$ hold:

$$x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}, \qquad x_i x_j = -x_j x_i,$$
$$x^{(\alpha)} x_j = x_j x^{(\alpha)}, \tag{1}$$

where

$$\binom{\alpha+\beta}{\alpha} = \prod_{i=1}^{m} \binom{\alpha_i+\beta_i}{\alpha_i}. \tag{2}$$

Put $Y_0 = \{1, 2, \ldots, m\}$, $Y_1 = \{m+1, m+2, \ldots, s\}$, and $Y = Y_0 \cup Y_1$. Let

$$\mathbb{B}_k = \{(i_1, i_2, \ldots, i_k) \mid m+1 \le i_1 < i_2 < \cdots < i_k \le s\},$$
$$\mathbb{B}(n) = \bigcup_{k=0}^{n} \mathbb{B}_k, \tag{3}$$

where $\mathbb{B}_0 = \varnothing$. Given $u = (i_1, i_2, \ldots, i_k) \in \mathbb{B}_k$, set $|u| = k$, $\{u\} = \{i_1, i_2, \ldots, i_k\}$, and $x^u = x_{i_1} x_{i_2} \cdots x_{i_k}$ ($|\varnothing| = 0$, $x^{\varnothing} = 1$). Then $\{x^{(\alpha)} x^u \mid \alpha \in \mathbb{N}_0^m, u \in \mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of the infinite-dimensional superalgebra $\mathcal{O}(m,n)$.

Let $D_1, D_2, \ldots, D_s$ be the linear transformations of $\mathcal{O}(m,n)$ such that

$$D_i\left(x^{(\alpha)} x^u\right) = \begin{cases} x^{(\alpha-\varepsilon_i)} x^u & i \in Y_0 \\ x^{(\alpha)} \cdot \dfrac{\partial x^u}{\partial x_i} & i \in Y_1. \end{cases} \tag{4}$$

Then $D_1, D_2, \ldots, D_s$ are superderivations of the superalgebra $\mathcal{O}(m,n)$. Let

$$W(m,n) = \left\{ \sum_{i=1}^{s} a_i D_i \mid a_i \in \mathcal{O}(m,n), i \in Y \right\}. \tag{5}$$

Then $W(m,n)$ is an infinite-dimensional Lie superalgebra contained in $\mathrm{Der}(\mathcal{O}(m,n))$ (see [14]). If $\deg(x)$ appears in some expression in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $\deg(x)$ as the $\mathbb{Z}_2$-degree of x. Then $\deg(D_i) = \mu(i)$, where

$$\mu(i) = \begin{cases} \bar{0} & i \in Y_0 \\ \bar{1} & i \in Y_1. \end{cases} \tag{6}$$

The following formula holds in $W(m,n)$ (see [14]):

$$\left[aD_i, bD_j\right] = aD_i(b) D_j - (-1)^{\deg(aD_i)\deg(bD_j)} bD_j(a) D_i, \tag{7}$$

where $a, b \in \mathcal{O}(m,n)$ and $i, j \in Y$.

Hereafter let r be a positive integer and let $m = 2r+1$. Put $J = Y \setminus \{m\}$ and $J_0 = Y_0 \setminus \{m\}$. For $i \in J$, define

$$i' = \begin{cases} i+r & 1 \le i \le r \\ i-r & r < i \le 2r \\ i & m < i \le s, \end{cases} \qquad \sigma(i) = \begin{cases} 1 & 1 \le i \le r \\ -1 & r < i \le 2r \\ 1 & i \in Y_1. \end{cases} \tag{8}$$

Let $D_K : \mathcal{O}(m,n) \to W(m,n)$ be the linear mapping such that

$$D_K(f) = \sum_{i=1}^{s} f_i D_i, \tag{9}$$

where

$$f_i = (-1)^{\mu(i)\deg(f)} \left(x_i D_m(f) + \sigma(i') D_{i'}(f)\right), \quad \forall i \in J,$$
$$f_m = 2f - \sum_{i\in J} x_i D_i(f). \tag{10}$$

Then

$$\left[D_K(f), D_K(g)\right] = D_K(\llbracket f, g \rrbracket), \tag{11}$$

where $\llbracket f, g \rrbracket = D_K(f)(g) - 2D_m(f)g$ (see [14]). It follows directly from (11) and the injectivity of D_K that $\llbracket , \rrbracket$ defines a Lie multiplication on $\mathcal{O}(m,n)$. This Lie superalgebra, denoted by $K(2r+1, n)$, is called the infinite-dimensional contact superalgebra. In the sequel, we simply write K for $K(2r+1, n)$.

The following formula holds in K (see [14]):

$$\llbracket f, g \rrbracket = \sum_{i\in J} \sigma(i) (-1)^{\mu(i)\deg(f)} D_i(f) D_{i'}(g) + \left(2f - \sum_{i\in J} x_i D_i(f)\right) D_m(g) - (-1)^{\deg(f)\deg(g)} \left(2g - \sum_{i\in J} x_i D_i(g)\right) D_m(f). \tag{12}$$

Then $K = \bigoplus_{i=-2}^{\infty} K_{[i]}$ is a $\mathbb{Z}$-graded Lie superalgebra, where

$$K_{[i]} = \mathrm{span}_{\mathbb{F}} \left\{ x^{(\alpha)} x^u \mid |\alpha| + \alpha_m + |u| = i+2 \right\}. \tag{13}$$

Let $K_j = \bigoplus_{i\ge j} K_{[i]}$ for $j \ge -2$. Then $K = K_{-2} \supset K_{-1} \supset K_0 \supset \cdots$ is referred to as the natural filtration of K.

Lemma 4. *$K = \bigoplus_{i=-2}^{\infty} K_{[i]}$ is transitively graded.*

Proof. Assume the contrary, then there exists $y \in K_{[\ell]}$ such that $\llbracket y, x_j \rrbracket = 0$ for all $j \in J$, where $\ell \ge 0$. Suppose that the largest exponent of x_m among the nonzero summands in the expression of y is equal to t, and write

$$y = \sum_{\alpha,u,\alpha_m=t} c_{\alpha,u} x^{(\alpha)} x^u + \sum_{\beta,v,\beta_m<t} d_{\beta,v} x^{(\beta)} x^v, \tag{14}$$

where $c_{\alpha,u}, d_{\beta,v} \in \mathbb{F}$. Hence, for $j \in J_0$,

$$0 = [\![y, x_j]\!] = \sum_{\alpha,u,\alpha_m=t} \sigma(j') c_{\alpha,u} x^{(\alpha-\varepsilon_j)} x^u + h, \quad (15)$$

where each exponent of x_m of all nonzero summands in the expression of h is less than t. Then $\sum_{\alpha,u,\alpha_m=t} \sigma(j') c_{\alpha,u} x^{(\alpha-\varepsilon_j)} x^u = 0$. Note that all nonzero summands of $\sum_{\alpha,u,\alpha_m=t} \sigma(j') c_{\alpha,u} x^{(\alpha-\varepsilon_j)} x^u$ are $\mathbb{F}$-linear independent. It follows that each $x^{(\alpha-\varepsilon_j)} x^u$ is equal to 0. Hence the exponents of x_j in each $x^{(\alpha)} x^u$ are equal to 0. Similarly, for $j \in Y_1$, we can prove that x_j does not appear in each $x^{(\alpha)} x^u$. Consequently, we see that all $x^{(\alpha)} x^u$ are of the form $x^{(t\varepsilon_m)}$. If $t = 0$, then $y \in K_{[-2]}$, contradicting $\ell \geq 0$. Hence $t > 0$, and we can write

$$y = c_{t\varepsilon_m} x^{(t\varepsilon_m)} + \sum_{\alpha,u,\alpha_m=t-1} c_{\alpha,u} x^{(\alpha)} x^u + \sum_{\beta,v,\beta_m<t-1} d_{\beta,v} x^{(\beta)} x^v, \quad (16)$$

where $c_{t\varepsilon_m} \neq 0$. Note that, for $j \in Y_1$,

$$0 = [\![y, x_j]\!] = -c_{t\varepsilon_m} x^{((t-1)\varepsilon_m)} x_j + \sum_{\alpha,u,\alpha_m=t-1} (-1)^{|u|} c_{\alpha,u} x^{(\alpha)} D_j(x^u) + h, \quad (17)$$

where each exponent of x_m of the nonzero summands in the expression of h is less than $t-1$. Therefore,

$$-c_{t\varepsilon_m} x^{((t-1)\varepsilon_m)} x_j + \sum_{\alpha,u,\alpha_m=t-1} (-1)^{|u|} c_{\alpha,u} x^{(\alpha)} D_j(x^u) = 0. \quad (18)$$

Since x_j appears in $x^{((t-1)\varepsilon_m)} x_j$ and does not appear in $\sum_{\alpha,u,\alpha_m=t-1} (-1)^{|u|} c_{\alpha,u} x^{(\alpha)} D_j(x^u)$, we conclude that $t = 0$, a contradiction. □

3. ad-Nilpotent Elements

Recall that $y \in K$ is called ad-nilpotent if there exists $t \in \mathbb{N}$ such that $(\mathrm{ad}\, y)^t(K) = 0$. For a subset R of K, let $\mathrm{nil}(R)$ denote the set of ad-nilpotent elements in R, and let $\mathrm{Nil}(R)$ denote the subalgebra of K generated by $\mathrm{nil}(R)$.

Lemma 5. *Suppose that $y_{[i]} \in K_{[i]}$ for $i \geq -2$. The following statements hold.*

(1) *If $y = \sum_{i=k}^{t} y_{[i]} \in \mathrm{nil}(K)$, then $y_{[k]} \in \mathrm{nil}(K)$.*

(2) *If $y = \sum_{i=-2}^{t} y_{[i]} \in \mathrm{nil}(K)$, then $y_{[-2]} = 0$.*

(3) *If $y = \sum_{i=-2}^{t} y_{[i]} \in \mathrm{nil}(K_{\bar{0}})$, then $y_{[-1]} = 0$.*

(4) *If $y = \sum_{i=-2}^{t} y_{[i]} \in \mathrm{nil}(K_{\bar{0}})$, then $y_{[0]} \in \mathrm{nil}(K_{\bar{0}})$.*

(5) *If $y = \sum_{i=-2}^{t} y_{[i]} \in \mathrm{nil}(K)$, then $y_{[-1]} \in \mathrm{span}_{\mathbb{F}}\{x_j \mid j \in Y_1\}$.*

Proof. (1) See [15, Lemma 5.1].

(2) Suppose that $y_{[-2]} \neq 0$. As y is ad-nilpotent, $y_{[-2]}$ is ad-nilpotent by (1). Note that

$$(\mathrm{ad}\, 1)^k \left(x^{(k\varepsilon_m)}\right) = (\mathrm{ad}\, 1)^{k-1} \left(2x^{((k-1)\varepsilon_m)}\right) = 2^k \neq 0, \quad (19)$$

for all $k > 0$. This shows that $y_{[-2]}$ is not ad-nilpotent, a contradiction.

(3) By (1), we see that $y_{[-1]}$ is ad-nilpotent. Suppose that $y_{[-1]} = \sum_{i \in J_0} a_i x_i \neq 0$, where $a_i \in \mathbb{F}$. Then there exists some $a_j \neq 0$. A direct calculation shows that

$$\begin{aligned}(\mathrm{ad}\, y_{[-1]})^k \left(x^{(k\varepsilon_{j'})}\right) &= (\mathrm{ad}\, y_{[-1]})^{k-1} \left(\sigma(j) a_j x^{((k-1)\varepsilon_{j'})}\right) \\ &= \sigma(j)^k a_j^k \neq 0,\end{aligned} \quad (20)$$

for all $k > 0$. It follows that $y_{[-1]}$ is not ad-nilpotent.

(4) is an immediate consequence of (2), (3), and (1). (5) follows from (2), (1), and the proof of (3). □

Let $a \in \mathbb{N}_0$ and $a = \sum_{l=0}^{\infty} a_l p^l$ be the p-adic expression of a, where $0 \leq a_l < p$. Then,

$$\mathrm{pad}(a) = (\mathrm{pad}_0(a), \mathrm{pad}_1(a), \mathrm{pad}_2(a), \ldots) \quad (21)$$

is said to be the p-adic sequence of a, where $\mathrm{pad}_j(a) = a_j$ for all $j \in \mathbb{N}_0$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m$, define the p-adic matrix of α to be

$$\mathrm{pad}(\alpha) = \begin{pmatrix} \mathrm{pad}(\alpha_1) \\ \mathrm{pad}(\alpha_2) \\ \vdots \\ \mathrm{pad}(\alpha_m) \end{pmatrix}. \quad (22)$$

As $\mathrm{pad}(\alpha)$ is an $m \times \infty$ matrix with finitely many nonzero elements,

$$\mathrm{ht}(\alpha) = \max \left\{ j \in \mathbb{N}_0 \mid \exists i \in Y_0 : \mathrm{pad}_j(\alpha_i) \neq 0 \right\} \quad (23)$$

is well defined. Let

$$\|\alpha\|_{b,c} = \sum_{i=1}^{m} \sum_{j=b}^{c} \mathrm{pad}_j(\alpha_i), \quad (24)$$

for $b \in \mathbb{N}_0$ and $c \in \mathbb{N}$. We abbreviate $\|\alpha\|_{0,q}$ to $\|\alpha\|_q$.

Suppose that $y = \sum_{\alpha,u} c_{\alpha,u} x^{(\alpha)} x^u$ is a nonzero element of K, where $c_{\alpha,u} \in \mathbb{F}$. Define

$$\mathrm{ht}(y) = \max \{\mathrm{ht}(\alpha) \mid c_{\alpha,u} \neq 0\}. \quad (25)$$

Given that $q > 0$ and $x^{(\alpha)} x^u \in K$, we define

$$\mathfrak{F}_q \left(x^{(\alpha)} x^u\right) = \|\alpha\|_q + 2\|\alpha\|_{1,q} + |u| + \mathrm{pad}_0(\alpha_m). \quad (26)$$

Lemma 6. *Let $\alpha, \beta \in \mathbb{N}_0^m$, $i \in Y_0$, and $q \in \mathbb{N}$. Then,*

(1) *$x^{(\alpha)} x^{(\beta)} \neq 0$ if and only if $\mathrm{pad}(\alpha) + \mathrm{pad}(\beta) = \mathrm{pad}(\alpha + \beta)$;*

(2) *If $\beta_i \neq 0$, then $\|\beta - \varepsilon_i\|_q + 2\|\beta - \varepsilon_i\|_{1,q} \geq \|\beta\|_q + 2\|\beta\|_{1,q} - 1$;*

(3) *If* $x^{(\beta)}x^{v} \in K_1$ *and* $q \geq \mathrm{ht}(x^{(\beta)}x^{v})$, *then* $\mathfrak{F}_q(x^{(\beta)}x^{v}) \geq 3$.

Proof. (1) See [7, Lemma 2.5].

(2) First consider the case $\mathrm{pad}_0(\beta_i) \neq 0$. Then,

$$\mathrm{pad}(\beta_i - 1) = (\mathrm{pad}_0(\beta_i) - 1, \mathrm{pad}_1(\beta_i), \ldots). \quad (27)$$

It follows that $\|\beta - \varepsilon_i\|_q = \|\beta\|_q - 1$ and $\|\beta - \varepsilon_i\|_{1,q} = \|\beta\|_{1,q}$, and thus (2) holds.

Next consider the case $\mathrm{pad}_0(\beta_i) = 0$. We may assume that

$$\begin{aligned} &\mathrm{pad}(\beta_i) \\ &\quad = (0, \ldots, 0, \mathrm{pad}_t(\beta_i), \mathrm{pad}_{t+1}(\beta_i), \ldots), \end{aligned} \quad (28)$$

where $\mathrm{pad}_t(\beta_i) \neq 0$ and $t \geq 1$. Hence,

$$\begin{aligned} &\mathrm{pad}(\beta_i - 1) \\ &\quad = (p-1, \ldots, p-1, \mathrm{pad}_t(\beta_i) - 1, \mathrm{pad}_{t+1}(\beta_i), \ldots). \end{aligned} \quad (29)$$

If $q < t$, then

$$\begin{aligned} &\|\beta - \varepsilon_i\|_q + 2\|\beta - \varepsilon_i\|_{1,q} \\ &\quad = (q+1)(p-1) + 2q(p-1) \\ &\quad > -1 \\ &\quad = \|\beta\|_q + 2\|\beta\|_{1,q} - 1. \end{aligned} \quad (30)$$

If $q = t$, noting that $p > 2$, then

$$\begin{aligned} &\|\beta - \varepsilon_i\|_q + 2\|\beta - \varepsilon_i\|_{1,q} \\ &\quad = t(p-1) + 2(t-1)(p-1) + 3(\mathrm{pad}_t(\beta_i) - 1) \\ &\quad \geq 2 + 3\mathrm{pad}_t(\beta_i) - 3 \\ &\quad = 3\mathrm{pad}_t(\beta_i) - 1 \\ &\quad = \|\beta\|_q + 2\|\beta\|_{1,q} - 1. \end{aligned} \quad (31)$$

If $q > t$, then

$$\begin{aligned} &\|\beta - \varepsilon_i\|_q + 2\|\beta - \varepsilon_i\|_{1,q} \\ &\quad = \|\beta - \varepsilon_i\|_t + 2\|\beta - \varepsilon_i\|_{1,t} + 3\|\beta - \varepsilon_i\|_{t+1,q} \\ &\quad \geq \|\beta\|_t + 2\|\beta\|_{1,t} - 1 + 3\|\beta - \varepsilon_i\|_{t+1,q} \\ &\quad = \|\beta\|_q + 2\|\beta\|_{1,q} - 1. \end{aligned} \quad (32)$$

(3) The assumption $x^{(\beta)}x^{v} \in K_1$ implies that $|\beta| + \beta_m + |v| \geq 3$. Then it is trivially verified that (3) holds. □

Lemma 7. *Suppose that* $x^{(\beta)}x^{v} \in K_1$, $x^{(\alpha)}x^{u} \in K$, $q \geq \max\{1, \mathrm{ht}(x^{(\beta)}x^{v})\}$, *and* $i \in J$. *The following statements hold.*

(1) *If* $x^{(\beta)}x^{v}D_m(x^{(\alpha)}x^{u}) \neq 0$, *then* $\mathfrak{F}_q(x^{(\beta)}x^{v}D_m(x^{(\alpha)}x^{u})) \geq \mathfrak{F}_q(x^{(\alpha)}x^{u}) + 1$.

(2) *If* $x^{(\alpha)}x^{u}D_m(x^{(\beta)}x^{v}) \neq 0$, *then* $\mathfrak{F}_q(x^{(\alpha)}x^{u}D_m(x^{(\beta)}x^{v})) \geq \mathfrak{F}_q(x^{(\alpha)}x^{u}) + 1$.

(3) *If* $D_i(x^{(\beta)}x^{v})D_{i'}(x^{(\alpha)}x^{u}) \neq 0$, *then* $\mathfrak{F}_q(D_i(x^{(\beta)}x^{v})D_{i'}(x^{(\alpha)}x^{u})) \geq \mathfrak{F}_q(x^{(\alpha)}x^{u}) + 1$.

Proof. (1) The assumption $x^{(\beta)}x^{v}D_m(x^{(\alpha)}x^{u}) \neq 0$ implies that $x^{(\beta)}x^{(\alpha-\varepsilon_m)} \neq 0$. By Lemma 6(1), we have

$$\mathrm{pad}(\beta + (\alpha - \varepsilon_m)) = \mathrm{pad}(\beta) + \mathrm{pad}(\alpha - \varepsilon_m). \quad (33)$$

Consequently,

$$\mathrm{pad}_0(\beta_m + (\alpha_m - 1)) = \mathrm{pad}_0(\beta_m) + \mathrm{pad}_0(\alpha_m - 1), \quad (34)$$

$$\|\beta + (\alpha - \varepsilon_m)\|_q = \|\beta\|_q + \|\alpha - \varepsilon_m\|_q, \quad (35)$$

$$\|\beta + (\alpha - \varepsilon_m)\|_{1,q} = \|\beta\|_{1,q} + \|\alpha - \varepsilon_m\|_{1,q}. \quad (36)$$

By (2) and (3) of Lemma 6, we obtain

$$\|\alpha - \varepsilon_m\|_q + \|\alpha - \varepsilon_m\|_{1,q} \geq \|\alpha\|_q + \|\alpha\|_{1,q} - 1, \quad (37)$$

$$\mathfrak{F}_q(x^{(\beta)}x^{v}) \geq 3. \quad (38)$$

Combining (34)–(38), we have

$$\begin{aligned} &\mathfrak{F}_q(x^{(\beta)}x^{v}D_m(x^{(\alpha)}x^{u})) \\ &\quad = \mathfrak{F}_q(x^{(\beta+(\alpha-\varepsilon_m))}x^{v}x^{u}) \\ &\quad = \|\beta + (\alpha - \varepsilon_m)\|_q + 2\|\beta + (\alpha - \varepsilon_m)\|_{1,q} + |u| + |v| \\ &\qquad + \mathrm{pad}_0(\beta_m + (\alpha_m - 1)) \\ &\quad = \|\beta\|_q + \|\alpha - \varepsilon_m\|_q + 2\|\beta\|_{1,q} + 2\|\alpha - \varepsilon_m\|_{1,q} + |u| + |v| \\ &\qquad + \mathrm{pad}_0(\beta_m) + \mathrm{pad}_0(\alpha_m - 1) \\ &\quad = \|\beta\|_q + 2\|\beta\|_{1,q} + |v| + \mathrm{pad}_0(\beta_m) + \|\alpha - \varepsilon_m\|_q \\ &\qquad + 2\|\alpha - \varepsilon_m\|_{1,q} + |u| + \mathrm{pad}_0(\alpha_m - 1) \\ &\quad \geq \mathfrak{F}_q(x^{(\beta)}x^{v}) + \|\alpha\|_q + 2\|\alpha\|_{1,q} - 1 + |u| + \mathrm{pad}_0(\alpha_m) - 1 \\ &\quad \geq \mathfrak{F}_q(x^{(\alpha)}x^{u}) + 1. \end{aligned} \quad (39)$$

(2) Suppose that $x^{(\alpha)}x^{u}D_m(x^{(\beta)}x^{v}) \neq 0$. Then,

$$\begin{aligned} &\mathfrak{F}_q(x^{(\alpha)}x^{u}D_m(x^{(\beta)}x^{v})) \\ &\quad = \mathfrak{F}_q(x^{(\alpha+(\beta-\varepsilon_m))}x^{v}x^{u}) \\ &\quad = \|\alpha + (\beta - \varepsilon_m)\|_q + 2\|\alpha + (\beta - \varepsilon_m)\|_{1,q} + |u| + |v| \\ &\qquad + \mathrm{pad}_0(\alpha_m + (\beta_m - 1)) \end{aligned}$$

$$
\begin{aligned}
&= \|\alpha\|_q + \|\beta-\varepsilon_m\|_q + 2\|\alpha\|_{1,q} + 2\|\beta-\varepsilon_m\|_{1,q} + |u| + |v| \\
&\quad + \mathrm{pad}_0(\alpha_m) + \mathrm{pad}_0(\beta_m - 1) \\
&= \|\alpha\|_q + 2\|\alpha\|_{1,q} + |u| + \mathrm{pad}_0(\alpha_m) + \|\beta-\varepsilon_m\|_q \\
&\quad + 2\|\beta-\varepsilon_m\|_{1,q} + |v| + \mathrm{pad}_0(\beta_m - 1) \\
&\geq \mathfrak{F}_q\left(x^{(\alpha)}x^u\right) + \|\beta\|_q + 2\|\beta\|_{1,q} - 1 + |v| + \mathrm{pad}_0(\beta_m) - 1 \\
&\geq \mathfrak{F}_q\left(x^{(\alpha)}x^u\right) + 1.
\end{aligned} \tag{40}
$$

(3) Similarly, we have

$$
\begin{aligned}
&\mathfrak{F}_q\left(D_i\left(x^{(\beta)}x^v\right)D_{i'}\left(x^{(\alpha)}x^u\right)\right) \\
&= \mathfrak{F}_q\left(x^{((\beta-\varepsilon_i)+(\alpha-\varepsilon_{i'}))}x^v x^u\right) \\
&= \|(\beta-\varepsilon_i)+(\alpha-\varepsilon_{i'})\|_q + 2\|(\beta-\varepsilon_i)+(\alpha-\varepsilon_{i'})\|_{1,q} + |v| \\
&\quad + |u| + \mathrm{pad}_0(\beta_m+\alpha_m) \\
&= \|\beta-\varepsilon_i\|_q + \|\alpha-\varepsilon_{i'}\|_q + 2\|\beta-\varepsilon_i\|_{1,q} + 2\|\alpha-\varepsilon_{i'}\|_{1,q} + |v| \\
&\quad + |u| + \mathrm{pad}_0(\beta_m) + \mathrm{pad}_0(\alpha_m) \\
&= \|\beta-\varepsilon_i\|_q + 2\|\beta-\varepsilon_i\|_{1,q} + \|\alpha-\varepsilon_{i'}\|_q + 2\|\alpha-\varepsilon_{i'}\|_{1,q} + |v| \\
&\quad + |u| + \mathrm{pad}_0(\beta_m) + \mathrm{pad}_0(\alpha_m) \\
&\geq \|\beta\|_q + 2\|\beta\|_{1,q} - 1 + \|\alpha\|_q + 2\|\alpha\|_{1,q} - 1 + |v| + |u| \\
&\quad + \mathrm{pad}_0(\beta_m) + \mathrm{pad}_0(\alpha_m) \\
&= \mathfrak{F}_q\left(x^{(\beta)}x^v\right) + \mathfrak{F}_q\left(x^{(\alpha)}x^u\right) - 2 \\
&\geq \mathfrak{F}_q\left(x^{(\alpha)}x^u\right) + 1.
\end{aligned} \tag{41}
$$

□

Lemma 8. *Suppose that $x^{(\beta)}x^v \in K_1$, $x^{(\alpha)}x^u \in K$, and $q \geq \max\{1, \mathrm{ht}(x^{(\beta)}x^v)\}$. Let $x^{(\alpha')}x^{u'}$ be a nonzero summand of $[\![x^{(\beta)}x^v, x^{(\alpha)}x^u]\!]$. Then $\mathfrak{F}_q(x^{(\alpha')}x^{u'}) \geq \mathfrak{F}_q(x^{(\alpha)}x^u) + 1$.*

Proof. A direct calculation shows that $x^{(\alpha')}x^{u'}$ fulfills the conditions of Lemma 7. □

Given that $q \in \mathbb{N}$, let

$$\ell_q = m(q+1)(p-1) + 2mq(p-1) + (p-1) + n + 1. \tag{42}$$

Clearly, the inequality $\mathfrak{F}_q(x^{(\alpha)}x^u) < \ell_q$ holds for all $x^{(\alpha)}x^u \in K$.

Lemma 9. $K_1 \subset \mathrm{nil}(K)$.

Proof. Suppose that $y = \sum_{\beta,u} c_{\alpha,v} x^{(\beta)}x^v$ is an arbitrary element of K_1, where $c_{\beta,v} \in \mathbb{F}$ and $c_{\beta,v} \neq 0$. Let $q \in \mathbb{N}$ such that $q \geq \mathrm{ht}(y)$. Let $x^{(\alpha)}x^u$ be a standard basis element of K. By using Lemma 8 repeatedly, we see that $(\mathrm{ad}\,y)^{\ell_q}(x^{(\alpha)}x^u) = 0$. □

Lemma 10. *For $i, j \in J_0$, the following statements hold.*

(1) $x^{(2\varepsilon_i)} \in \mathrm{nil}(K_{[0]} \cap K_{\bar{0}})$.

(2) *If $\sigma(i) = \sigma(j)$ and $i \neq j$, then $x_i x_j \in \mathrm{nil}(K_{[0]} \cap K_{\bar{0}})$.*

(3) *If $\sigma(i) \neq \sigma(j)$ and $i \neq j'$, then $x_i x_j \in \mathrm{nil}(K_{[0]} \cap K_{\bar{0}})$.*

(4) $x_i x_{i'} \in \mathrm{Nil}(K_{[0]} \cap K_{\bar{0}})$.

Proof. (1) Let $x^{(\alpha)}x^u$ be a standard basis element of K. A direct calculation shows that

$$
\begin{aligned}
\left(\mathrm{ad}x^{(2\varepsilon_i)}\right)^p\left(x^{(\alpha)}x^u\right) &= \left(\mathrm{ad}x^{(2\varepsilon_i)}\right)^{p-1}\left(\sigma(i)\, x_i x^{(\alpha-\varepsilon_{i'})}x^u\right) \\
&= \sigma(i)^p x_i^p x^{(\alpha-p\varepsilon_{i'})}x^u = 0.
\end{aligned} \tag{43}
$$

(2) Since $\mathrm{ad}x_i x_j = \sigma(i)(x_j D_{i'} + x_i D_{j'})$ and $x_j D_{i'} \circ x_i D_{j'} = x_i D_{j'} \circ x_j D_{i'}$, it follows from the binomial theorem that $(\mathrm{ad}x_i x_j)^p = \sigma(i)^p(x_j D_{i'} + x_i D_{j'})^p = \sigma(i)^p((x_j D_{i'})^p + (x_i D_{j'})^p) = 0$.

(3) Since $\mathrm{ad}x_i x_j = \sigma(i)(x_j D_{i'} - x_i D_{j'})$ and $x_j D_{i'} \circ x_i D_{j'} = x_i D_{j'} \circ x_j D_{i'}$, we have $(\mathrm{ad}x_i x_j)^p = \sigma(i)^p(x_j D_{i'} - x_i D_{j'})^p = \sigma(i)^p((x_j D_{i'})^p - (x_i D_{j'})^p) = 0$.

(4) It follows from (1) that $x_i x_{i'} = \sigma(i)[\![x^{(2\varepsilon_i)}, x^{(2\varepsilon_{i'})}]\!] \in \mathrm{Nil}(K_{[0]} \cap K_{\bar{0}})$. □

Lemma 11. *Suppose that $n \geq 3$. The following statements hold.*

(1) *Let i, j, k be distinct elements of Y_1, and let $a, b \in \mathbb{F}$ be such that $a^2 + b^2 = 0$. Then $y = a x_i x_j + b x_i x_k \in \mathrm{nil}(K)$.*

(2)*$x_i x_j \in \mathrm{Nil}(K_{[0]} \cap K_{\bar{0}})$ holds for all distinct $i, j \in Y_1$.*

Proof. (1) A direct calculation shows that $\mathrm{ad}\,y = a x_j D_i - a x_i D_j + b x_k D_i - b x_i D_k$. For simplicity, we denote $a x_j D_i$, $-a x_i D_j$, $b x_k D_i$, $-b x_i D_k$ by A, B, C, D, respectively. Clearly,

$$A^2 = B^2 = C^2 = D^2 = 0, \qquad AC = CA = BD = DB = 0. \tag{44}$$

Then

$$(\mathrm{ad}\,y)^2 = AB + BA + AD + DA + BC + CB + DC + CD. \tag{45}$$

By $ADA = DAD = BCB = CBC = 0$, we obtain

$$
\begin{aligned}
(\mathrm{ad}\,y)^3 &= ABA + ABC + ADC + BAB + BAD + BCD \\
&\quad + CBA + CDA + CDC + DAB + DCB + DCD.
\end{aligned} \tag{46}
$$

Note that

$$\begin{aligned} ABA &= -a^2 A, \qquad BAB = -a^2 B, \\ CDC &= -b^2 C, \qquad DCD = -b^2 D, \\ a^2 A &= CDA + ADC, \qquad a^2 B = BCD + DCB, \\ b^2 C &= ABC + CBA, \qquad b^2 D = BAD + DAB. \end{aligned} \tag{47}$$

It follows that

$$\begin{aligned} &ABA + CDA + ADC = 0, \qquad BAB + BCD + DCB = 0, \\ &CDC + ABC + CBA = 0, \qquad DCD + BAC + DAB = 0, \end{aligned} \tag{48}$$

thus proving that $(\mathrm{ad}y)^3 = 0$.

(2) Let $a \in \mathbb{F}$ such that $a^2 = -1$. Since $\mathrm{char}(\mathbb{F}) > 2$, we have $a^2 - 1 = -2 \neq 0$. Let $k \in Y_1 \setminus \{i, j\}$. Then (1) yields

$$\begin{aligned} y_1 &= a x_i x_j + x_i x_k \in \mathrm{Nil}\left(H_{[0]} \cap H_{\bar{0}}\right), \\ y_2 &= x_i x_j + a x_i x_k \in \mathrm{Nil}\left(H_{[0]} \cap H_{\bar{0}}\right). \end{aligned} \tag{49}$$

Hence $x_i x_j = -(1/2)(a y_1 - y_2) \in \mathrm{Nil}(H_{[0]} \cap H_{\bar{0}})$. □

Lemma 12. *Let $i \in J_0$ and $j \in Y_1$. Then $x_i x_j \in \mathrm{nil}(K_{[0]} \cap K_{\bar{1}})$.*

Proof. A direct calculation shows that

$$\begin{aligned} \mathrm{ad} x_i x_j &= \sigma(i)\, x_j D_{i'} - x_i D_j, \\ \left(\mathrm{ad} x_i x_j\right)^2 &= -\sigma(i)\left(x_i D_j \circ x_j D_{i'} + x_j D_{i'} \circ x_i D_j\right). \end{aligned} \tag{50}$$

Since

$$\begin{aligned} &\left(x_i D_j \circ x_j D_{i'}\right) \circ \left(x_j D_{i'} \circ x_i D_j\right) \\ &\quad = 0 = \left(x_j D_{i'} \circ x_i D_j\right) \circ \left(x_i D_j \circ x_j D_{i'}\right), \end{aligned} \tag{51}$$

we have

$$\begin{aligned} &\left(\mathrm{ad} x_i x_j\right)^{2p} \\ &\quad = -\sigma(i)\left(\left(x_i D_j \circ x_j D_{i'}\right)^p + \left(x_j D_{i'} \circ x_i D_j\right)^p\right) = 0. \end{aligned} \tag{52}$$

□

Lemma 13. (1) $\mathrm{Nil}(K_{[0]}) = \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J\}$.

(2) $\mathrm{Nil}(K_0) = \mathrm{Nil}(K_{[0]}) + K_1$.

(3) *If $n \geq 3$, then* $\mathrm{Nil}(K_{[0]} \cap K_{\bar{0}}) = \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J, \mu(i) = \mu(j)\}$.

(4) *If $n \leq 2$, then* $\mathrm{Nil}(K_{[0]} \cap K_{\bar{0}}) = \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J_0\}$.

(5) $\mathrm{Nil}(K_0 \cap K_{\bar{0}}) = \mathrm{Nil}(K_{[0]} \cap K_{\bar{0}}) + K_1 \cap K_{\bar{0}}$.

Proof. (1) Let $y = a_m x_m + \sum_{i,j \in J} a_{ij} x_i x_j$ be an arbitrary element of $\mathrm{nil}(K_{[0]})$. Suppose that $a_m \neq 0$. Since $(\mathrm{ad}y)^t(1) = (-2a_m)^t \neq 0$ and $\forall t \in \mathbb{N}$, it follows that y is not ad-nilpotent, a contradiction. Hence $a_m = 0$, and therefore $\mathrm{nil}(K_{[0]}) \subseteq \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J\}$. Noting that $\mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J\}$ is a subalgebra of K, we obtain $\mathrm{Nil}(K_{[0]}) \subseteq \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J\}$.

Conversely, Lemma 10 shows that $\mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J_0\} \subseteq \mathrm{Nil}(K_{[0]})$, and Lemma 12 implies that $\mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i \in J_0, j \in Y_1\} \subseteq \mathrm{Nil}(K_{[0]})$. Moreover, since $x_i x_j = [\![x_1 x_i, x_{1'} x_j]\!]$ for all $i, j \in Y_1$, we have $\mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in Y_1\} \subseteq \mathrm{Nil}(K_{[0]})$. Therefore $\mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J\} \subseteq \mathrm{Nil}(K_{[0]})$.

(2) It is clear that $\mathrm{Nil}(K_{[0]}) \subseteq \mathrm{Nil}(K_0)$, which combined with Lemma 9 yields $\mathrm{Nil}(K_{[0]}) + K_1 \subseteq \mathrm{Nil}(K_0)$.

On the other hand, suppose that $y = y_{[0]} + y_1$ is an arbitrary element of $\mathrm{nil}(K_0)$, where $y_{[0]} \in K_{[0]}$ and $y_1 \in K_1$. By Lemma 5, we have $y_{[0]} \in \mathrm{Nil}(K_{[0]})$, and hence $y = y_{[0]} + y_1 \in \mathrm{Nil}(K_{[0]}) + K_1$. Since $\mathrm{Nil}(K_{[0]}) + K_1$ is a subalgebra of K, it follows that $\mathrm{Nil}(K_0) \subseteq \mathrm{Nil}(K_{[0]}) + K_1$.

(3) By (1), we see that $\mathrm{Nil}(K_{[0]} \cap K_{\bar{0}}) \subseteq \mathrm{Nil}(K_{[0]}) \cap K_{\bar{0}} = \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J, \mu(i) = \mu(j)\}$. The reverse inclusion follows from Lemmas 10 and 11.

(4) Clearly the statement holds when $n = 1$. Now we consider the case $n = 2$. By (1), we can suppose that $y = a x_{m+1} x_s + \sum_{i,j \in J_0} a_{ij} x_i x_j$ is an arbitrary element of $\mathrm{nil}(K_{[0]} \cap K_{\bar{0}})$, where $a, a_{ij} \in \mathbb{F}$. If $a \neq 0$, a direct calculation shows that $(\mathrm{ad}y)^{2t}(x_s) = (-1)^t a^{2t} x_s \neq 0$ for all $t \in \mathbb{N}$, contradicting that y is ad-nilpotent. Hence $a = 0$ and $y \in \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J_0\}$, proving $\mathrm{nil}(K_{[0]} \cap K_{\bar{0}}) \subseteq \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J_0\}$. Since $\mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J_0\}$ is a subalgebra of K, it follows that $\mathrm{Nil}(K_{[0]} \cap K_{\bar{0}}) \subseteq \mathrm{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in J_0\}$. The reverse inclusion follows from Lemma 10.

(5) is completely analogous to the proof of (2). □

Let ρ be the corresponding representation with respect to $K_{[0]}$-module $K_{[-1]}$; that is, $\rho(y) = \mathrm{ad}y|_{K_{[-1]}}$, $\forall y \in K_{[0]}$. It is easy to see that ρ is faithful. For $y \in K_{[0]}$, we also denote by $\rho(y)$ the matrix of $\rho(y)$ relative to the fixed ordered $\mathbb{F}$-basis as follows:

$$\{x_1, x_2, \ldots, x_{m-1}, x_{m+1}, \ldots, x_s\}. \tag{53}$$

Denote by $\mathrm{gl}(2r, n)$ the general linear Lie superalgebra of $(2r+n) \times (2r+n)$ matrices over $\mathbb{F}$. Let e_{ij} denote the $(s-1) \times (s-1)$ matrix whose (i, j)-entry is 1 and 0 elsewhere, and $G = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$, where I_r is $r \times r$ unit matrix. Let $\mathrm{sp}(2r, \mathbb{F})$ be the Lie algebra consisting of all $2r \times 2r$ matrices A over $\mathbb{F}$ satisfying $A^T G + GA = 0$, where A^T is the transpose of A. Set $\mathscr{K} = \mathscr{L} \oplus \mathbb{F} I_{s-1}$; here

$$\begin{aligned} \mathscr{L} = \Bigg\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{gl}(2r, n) \mid A \in \mathrm{sp}(2r, \mathbb{F}), B^T G + C = 0, \\ D \text{ anti-symmetric} \Bigg\}. \end{aligned} \tag{54}$$

Lemma 14. (1) $\rho(K_{[0]}) = \mathscr{K}$.

(2) *If $y \in \mathrm{nil}(K_{[0]})$, then $\rho(y)$ is a nilpotent matrix.*

Proof. (1) For $i, j \in J$, a direct calculation shows that

$$\begin{aligned} [\![x_i x_j, x_k]\!] &= \sigma(i)(-1)^{\mu(i)(\mu(i)+\mu(j))} \delta_{i'k} x_j \\ &\quad + \sigma(j)(-1)^{\mu(j)(\mu(i)+\mu(j))+\mu(i)\mu(j)} \delta_{j'k} x_i. \end{aligned} \tag{55}$$

Therefore

$$\rho\left(x_i x_j\right) = \sigma(i)(-1)^{\mu(i)+\mu(i)\mu(j)} e_{\widehat{j}\widehat{i}'} + \sigma(j)(-1)^{\mu(j)} e_{\widehat{i}\widehat{j}'}, \quad (56)$$

where

$$\widehat{l} = \begin{cases} l & \text{if } l \in J_0, \\ l-1 & \text{if } l \in Y_1. \end{cases} \quad (57)$$

Since $[\![x_m, x_k]\!] = -x_k, \forall k \in J$, we see that

$$\rho\left(x_m\right) = -I_{s-1}. \quad (58)$$

From (56) and (58), we can easily verify (1).

(2) follows from the definition of ρ. □

If $A = (a_{ij})$ is an $n \times n$ antisymmetric matrix over $\mathbb{F}$, we write $\Gamma(A)$ to denote $\sum_{1\le i<j\le n} a_{ij}^2$. As $\operatorname{tr}(A^2) = -2\sum_{1\le i<j\le n} a_{ij}^2$, it is clear that if A is a nilpotent matrix then $\Gamma(A) = 0$.

Lemma 15. *Suppose that $n \geq 3$. Let $A = (a_{ij})$ be an antisymmetric matrix over $\mathbb{F}$ of order n. If A satisfies the following properties:*

(1) $\Gamma(A) = 0$;

(2) $\Gamma([A, B]) = 0$ *holds for every $n \times n$ antisymmetric matrix B.*

Then, $A = 0$.

Proof. Let $N = \{1, 2, \ldots, n\}$. Three cases arise as follows.

Case 1 ($n > 4$). By the property (2) of A, we see that

$$\begin{aligned} &\Gamma\left(\left[A, E_{ij} - E_{ji} + E_{kl} - E_{lk}\right]\right) - \Gamma\left(\left[A, E_{ij} - E_{ji}\right]\right) \\ &\quad - \Gamma\left(\left[A, E_{kl} - E_{lk}\right]\right) = 0 \end{aligned} \quad (59)$$

holds for all $i, j, k, l \in N$. Therefore, if $i < j < k < l$, a direct calculation shows that the left-hand side of (59) is equal to $4a_{il}a_{jk} - 4a_{ik}a_{jl}$. Hence $4a_{il}a_{jk} - 4a_{ik}a_{jl} = 0$. Similarly, if $i < k < j < l$, then $-4a_{il}a_{kj} - 4a_{ik}a_{jl} = 0$, and if $i < k < l < j$, then $-4a_{il}a_{kj} + 4a_{ik}a_{lj} = 0$. Thus, for all $i_1 < i_2 < i_3 < i_4$, we see that $a_{i_1i_3}a_{i_2i_4} = a_{i_1i_4}a_{i_2i_3} = -a_{i_1i_2}a_{i_3i_4} = -a_{i_1i_3}a_{i_2i_4}$. Then

$$a_{i_1i_2}a_{i_3i_4} = a_{i_1i_3}a_{i_2i_4} = a_{i_1i_4}a_{i_2i_3} = 0. \quad (60)$$

Denote $\eta_i = \sum_{t=1}^{n} a_{it}^2$ for $i \in N$. If $i \neq j$, a direct calculation shows that

$$0 = \Gamma\left(\left[A, E_{ij} - E_{ji}\right]\right) = \eta_i + \eta_j - 2a_{ij}^2. \quad (61)$$

Let $k \in N \setminus \{i, j\}$. Then

$$\eta_i + \eta_k - 2a_{ik}^2 = 0, \qquad \eta_j + \eta_k - 2a_{jk}^2 = 0. \quad (62)$$

From the equalities above, we see that $\eta_i = a_{ij}^2 + a_{ik}^2 - a_{jk}^2$ holds for all distinct $i, j, k \in N$. Pick $l \in N \setminus \{i, j, k\}$. Then $\eta_i = a_{ij}^2 + a_{il}^2 - a_{jl}^2$. Hence $a_{il}^2 + a_{jk}^2 = a_{ik}^2 + a_{jl}^2$ holds for all distinct $i, j, k, l \in N$. It follows that

$$a_{i_1i_2}^2 + a_{i_3i_4}^2 = a_{i_1i_3}^2 + a_{i_2i_4}^2 = a_{i_2i_3}^2 + a_{i_1i_4}^2 \quad (63)$$

holds for all $i_1 < i_2 < i_3 < i_4$.

Assume that there exists some $a_{ij} \neq 0$. For distinct $k, l \in N \setminus \{i, j\}$, we have $a_{kl} = 0$ by (60). Then we can write the following:

$$A = \sum_{t\in N} a_{it}E_{it} + \sum_{t\in N} a_{jt}E_{jt} - \sum_{t\in N} a_{it}E_{ti} - \sum_{t\in N} a_{jt}E_{tj}. \quad (64)$$

A direct calculation shows that

$$0 = \Gamma\left(\left[A, E_{kl} - E_{lk}\right]\right) = a_{il}^2 + a_{jl}^2 + a_{ik}^2 + a_{jk}^2. \quad (65)$$

Since $a_{il}^2 + a_{jk}^2 = a_{jl}^2 + a_{ik}^2$ by (63), we obtain $a_{il}^2 + a_{jk}^2 = 0$. Hence $a_{ij}^2 = a_{ij}^2 + a_{kl}^2 = a_{il}^2 + a_{jk}^2 = 0$ by (63), contradicting the assumption that $a_{ij} \neq 0$.

Case 2 ($n = 4$). Note that (60) and (63) hold; that is,

$$\begin{aligned} &a_{12}a_{34} = a_{13}a_{24} = a_{14}a_{23} = 0, \\ &a_{12}^2 + a_{34}^2 = a_{13}^2 + a_{24}^2 = a_{23}^2 + a_{14}^2. \end{aligned} \quad (66)$$

Moreover, since $0 = \Gamma(A) - \Gamma([A, E_{12} - E_{21}]) = a_{12}^2 + a_{34}^2$, we see that $A = 0$.

Case 3 ($n = 3$). Since $0 = \Gamma(A) = a_{12}^2 + a_{23}^2 + a_{13}^2$ and $\Gamma([A, E_{12} - E_{21}]) = a_{23}^2 + a_{13}^2$, it follows that $a_{12} = 0$. Similarly $a_{23} = a_{13} = 0$. □

Lemma 16. *Let y be a nonzero element of* $\operatorname{nil}(K_{[0]} \cap K_{\bar{0}})$. *Then there exists $z \in K_{[0]} \cap K_{\bar{0}}$ such that $[\![y, z]\!]$ is not ad-nilpotent.*

Proof. By Lemma 13, we can suppose that

$$y = \sum_{l\in J_0} a_l x^{(2\varepsilon_l)} + \sum_{l,t\in J_0, l<t} b_{lt}x_l x_t + \sum_{l,t\in Y_1, l<t} c_{lt}x_l x_t, \quad (67)$$

where $a_l, b_{lt}, c_{lt} \in \mathbb{F}$. Three cases arise as follows.

Case 1. There exists some $a_i \neq 0$. Let $z = x^{(2\varepsilon_{i'})}$. Then $y, z = \sigma(i)a_i x_i x_{i'} + h$, where x_i does not appear in the expression of h. Noting that $(\operatorname{ad}[\![y, z]\!])^{\ell}(x_{i'}) = a_i^{\ell} x_{i'}$ for all $\ell \in \mathbb{N}$, we conclude that $[\![y, z]\!]$ is not ad-nilpotent.

Case 2. All $a_l = 0$, and there exists $b_{ij} \neq 0$. Let $z = x_{i'}x_{j'}$. A direct calculation shows that $[\![y, z]\!] = \sigma(j)a_{ij}x_i x_{i'} + h$, where x_i does not appear in the expression of h. As $(\operatorname{ad}[\![y, z]\!])^{\ell}(x_{i'}) = \sigma(i)^{\ell}\sigma(j)^{\ell}a_{ij}^{\ell}x_{i'}$ and $\forall \ell \in \mathbb{N}$, we see that $[\![y, z]\!]$ is not ad-nilpotent.

Case 3. All $a_l = 0$ and $b_{lt} = 0$. So $y \in \operatorname{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in Y_1\}$. This can happen only if $n \geq 3$ by Lemma 13(4). Note that $\rho(y)$ is an antisymmetric nilpotent matrix. Then Lemma 15 provides an element z of $\operatorname{span}_{\mathbb{F}}\{x_i x_j \mid i, j \in Y_1\}$ such that $[\![\rho(y), \rho(z)]\!]$ is not a nilpotent matrix. Hence $[\![y, z]\!]$ is not ad-nilpotent. □

4. Proofs of Theorems

Proof of Theorem 1. We proceed in several steps.

(I) $\mathrm{Nil}(K_{\bar{0}}) = \mathrm{Nil}(K_0 \cap K_{\bar{0}})$. Suppose that y is an arbitrary element of $\mathrm{nil}(K_{\bar{0}})$. It follows from (2) and (3) of Lemma 5 that $y \in K_0$, which combined with $y \in \mathrm{nil}(K_{\bar{0}})$ yields $y \in \mathrm{nil}(K_0 \cap K_{\bar{0}})$, thus proving $\mathrm{nil}(K_{\bar{0}}) \subseteq \mathrm{nil}(K_0 \cap K_{\bar{0}})$. Hence $\mathrm{Nil}(K_{\bar{0}}) \subseteq \mathrm{Nil}(K_0 \cap K_{\bar{0}})$.

The reverse inclusion is clear.

(II) $K_0 \cap K_{\bar{0}} = \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$. We first prove $K_0 \cap K_{\bar{0}} \subseteq \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$. It follows from (I) that

$$\mathrm{Nor}_{K_{\bar{0}}}\left(\mathrm{Nil}\left(K_{\bar{0}}\right)\right) = \mathrm{Nor}_{K_{\bar{0}}}\left(\mathrm{Nil}\left(K_0 \cap K_{\bar{0}}\right)\right). \tag{68}$$

Therefore

$$\mathrm{Nil}\left(K_0 \cap K_{\bar{0}}\right) \subseteq \mathrm{Nor}_{K_{\bar{0}}}\left(\mathrm{Nil}\left(K_0 \cap K_{\bar{0}}\right)\right) = \mathrm{Nor}_{K_{\bar{0}}}\left(\mathrm{Nil}\left(K_{\bar{0}}\right)\right). \tag{69}$$

By formula (12), we see that $[\![x_m, K_{[0]}]\!] = 0$ and $[\![x_m, K_1 \cap K_{\bar{0}}]\!] \subseteq K_1 \cap K_{\bar{0}}$, proving

$$x_m \in \mathrm{Nor}_{K_{\bar{0}}}\left(\mathrm{Nil}\left(K_{\bar{0}}\right)\right). \tag{70}$$

In the case of $n \geq 3$, by (3) and (5) of Lemma 13, we obtain $K_0 \cap K_{\bar{0}} = \mathrm{Nil}(K_0 \cap K_{\bar{0}}) + \mathbb{F}x_m$. By (69) and (70), we have $K_0 \cap K_{\bar{0}} \subseteq \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$. In the case of $n = 2$, by (4) and (5) of Lemma 13, we have $K_0 \cap K_{\bar{0}} = \mathrm{Nil}(K_0 \cap K_{\bar{0}}) + \mathbb{F}x_m + \mathbb{F}x_{m+1}x_s$. Note that $x_{m+1}x_s \in \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$, which combined with (69) and (70) yields $K_0 \cap K_{\bar{0}} \subseteq \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$. In the case of $n = 1$, by (4) and (5) of Lemma 13, we have $K_0 \cap K_{\bar{0}} = \mathrm{Nil}(K_0 \cap K_{\bar{0}}) + \mathbb{F}x_m$. Hence $K_0 \cap K_{\bar{0}} \subseteq \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$ by (69) and (70).

Conversely, suppose that $y = y_{[-2]} + y_{-1} \in \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$, where $y_{[-2]} \in K_{[-2]}$, $y_{-1} \in K_{-1}$. If $y_{[-2]} \neq 0$, then $[\![y, x_1 x_m]\!] = 2y_{[-2]}x_1 + h \notin \mathrm{Nil}(K_{\bar{0}})$ by Lemma 13(5), contradicting $y \in \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$, where $h \in K_0$. Hence $y_{[-2]} = 0$ and we can write $y = y_{[-1]} + y_0$, where $y_{[-1]} = \sum_{i \in J_0} a_i x_i \in K_{[-1]}$, $a_i \in \mathbb{F}$, $y_0 \in K_0$. If there is some $a_j \neq 0$, then $[\![y, x^{(2\varepsilon_{j'})}]\!] = \sigma(j) a_j x_{j'} + h$, where $h \in K_0$. Lemma 13(5) shows that $[\![y, x^{(2\varepsilon_{j'})}]\!] \notin \mathrm{Nil}(K_{\bar{0}})$, contradicting $y \in \mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))$. Hence $y_{[-1]} = 0$ and $y \in K_0 \cap K_{\bar{0}}$, proving $\mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}})) \subseteq K_0 \cap K_{\bar{0}}$.

(III) Let $M = \{y \in \mathrm{nil}(K_{\bar{0}}) \mid [\![y, K_0 \cap K_{\bar{0}}]\!] \subseteq \mathrm{nil}(K_{\bar{0}})\}$. Then $M = K_1 \cap K_{\bar{0}}$. Suppose that $y = y_{[0]} + y_1$ is an arbitrary element of M, where $y_{[0]} \in K_{[0]} \cap K_{\bar{0}}$ and $y_1 \in K_1 \cap K_{\bar{0}}$. If $y_{[0]} \neq 0$, since $y_{[0]} \in \mathrm{nil}(K_{[0]})$ by Lemma 5(4), then Lemma 16 provides an element $z \in K_{[0]} \cap K_{\bar{0}}$ such that $[\![y_{[0]}, z]\!]$ is not ad-nilpotent. Hence $[\![y, z]\!]$ is not ad-nilpotent by Lemma 5(4), contradicting $y \in M$. Therefore $y_{[0]} = 0$ and $y \in K_1 \cap K_{\bar{0}}$, thus proving $M \subseteq K_1 \cap K_{\bar{0}}$.

On the other hand, since $[\![K_1 \cap K_{\bar{0}}, K_0 \cap K_{\bar{0}}]\!] \subseteq K_1 \cap K_{\bar{0}} \subseteq \mathrm{nil}(K_0 \cap K_{\bar{0}})$ by Lemma 9, it follows that $K_1 \cap K_{\bar{0}} \subseteq M$.

(IV) Let $Q = \{y \in K_{\bar{1}} \mid [\![y, K_{\bar{1}}]\!] \subseteq K_0 \cap K_{\bar{0}}\}$. Then $Q = K_1 \cap K_{\bar{1}}$. Suppose that $y = y_{[-1]} + y_0 \in Q$, where $y_{[-1]} = \sum_{i \in Y_1} a_i x_i \in K_{[-1]} \cap K_{\bar{1}}$ and $y_0 \in K_0 \cap K_{\bar{1}}$. If there is some $a_j \neq 0$, then $[\![y, x_j]\!] = -a_j + [\![y_0, x_j]\!] \notin K_{-1} \cap K_{\bar{0}}$, and hence $[\![y, x_j]\!] \notin K_0 \cap K_{\bar{0}}$, contradicting $y \in Q$. Therefore $y_{[-1]} = 0$, and we can write $y = y_{[0]} + y_1$, where $y_{[0]} = \sum_{i \in J_0, j \in Y_1} a_{ij} x_i x_j \in K_{[0]} \cap K_{\bar{1}}$ and $y_1 \in K_1 \cap K_{\bar{1}}$. If there exists some $a_{lt} \neq 0$, then $[\![y, x_t]\!] = -\sum_{i \in J_0} a_{it} x_i + [\![y_1, x_t]\!] \notin K_0 \cap K_{\bar{0}}$, a contradiction which yields $y_{[0]} = 0$ and $y \in K_1 \cap K_{\bar{1}}$.

The reverse inclusion follows from the fact that $K_{\bar{1}} \subset K_{-1}$.

(V) $[\![K_{\bar{1}}, K_1 \cap K_{\bar{0}}]\!] = K_0 \cap K_{\bar{1}}$. It suffices to show that $K_0 \cap K_{\bar{1}} \subseteq [\![K_{\bar{1}}, K_1 \cap K_{\bar{0}}]\!]$. Suppose that $x^{(\alpha)}x^u$ is an arbitrary basis element of $K_0 \cap K_{\bar{1}}$ with $x^u = x_i x^v$. Note that $x^{(\alpha+\varepsilon_m)}x^v \in K_1 \cap K_{\bar{0}}$. Since $x^{(\alpha)}x^u = [\![x_i, x^{(\alpha+\varepsilon_m)}x^v]\!]$, it follows that $x^{(\alpha)}x^u \in [\![K_{\bar{1}}, K_1 \cap K_{\bar{0}}]\!]$, as desired.

It follows from (II) and (V) that $K_0 = K_0 \cap K_{\bar{0}} + K_0 \cap K_{\bar{1}}$ is invariant under automorphisms of K. By (III) and (IV), we obtain that $K_1 = K_1 \cap K_{\bar{0}} + K_1 \cap K_{\bar{1}}$ is invariant. Therefore $K_{-1} = \{x \in K \mid [\![x, K_1]\!] \subseteq K_0\}$ is invariant. By the transitivity of K, we conclude that

$$K_{i+1} = \left\{x \in K_i \mid [\![x, K_{-1}]\!] \subseteq K_i\right\}, \quad \forall i \geq 0. \tag{71}$$

Hence the natural filtration of K is invariant under automorphisms of K. □

Proof of Theorem 2. Let $\varphi : K(2r+1, n) \to K(2r'+1, n')$ be an isomorphism of Lie superalgebras. Let K and K' denote $K(2r+1, n)$ and $K(2r'+1, n')$, respectively. Since $\varphi(K_{\bar{0}}) = K'_{\bar{0}}$ and $\varphi(\mathrm{Nil}(K_{\bar{0}})) = \mathrm{Nil}(K'_{\bar{0}})$, it follows that $\varphi(\mathrm{Nor}_{K_{\bar{0}}}(\mathrm{Nil}(K_{\bar{0}}))) = \mathrm{Nor}_{K'_{\bar{0}}}(\mathrm{Nil}(K'_{\bar{0}}))$. By (II) in the proof of Theorem 1, we have

$$\begin{aligned}\varphi\left(K_0 \cap K_{\bar{0}}\right) &= \varphi\left(\mathrm{Nor}_{K_{\bar{0}}}\left(\mathrm{Nil}\left(K_{\bar{0}}\right)\right)\right) \\ &= \mathrm{Nor}_{K'_{\bar{0}}}\left(\mathrm{Nil}\left({K'}_{\bar{0}}\right)\right) = K'_0 \cap K'_{\bar{0}}.\end{aligned} \tag{72}$$

Consequently

$$\begin{aligned}&\varphi\left(\left\{y \in \mathrm{nil}\left(K_{\bar{0}}\right) \mid [\![y, K_0 \cap K_{\bar{0}}]\!] \subseteq \mathrm{nil}\left(K_{\bar{0}}\right)\right\}\right) \\ &\quad = \left\{y \in \mathrm{nil}\left(K'_{\bar{0}}\right) \mid [\![y, K'_0 \cap K'_{\bar{0}}]\!] \subseteq \mathrm{nil}\left(K'_{\bar{0}}\right)\right\}.\end{aligned} \tag{73}$$

Applying (III) in the proof of Theorem 1, we see that $\varphi(K_1 \cap K_{\bar{0}}) = K'_1 \cap K'_{\bar{0}}$, which combined with (V) yields

$$\begin{aligned}\varphi\left(K_0 \cap K_{\bar{1}}\right) &= \varphi\left([\![K_{\bar{1}}, K_1 \cap K_{\bar{0}}]\!]\right) = [\![K'_{\bar{1}}, K'_1 \cap K'_{\bar{0}}]\!] \\ &= K'_0 \cap K'_{\bar{1}}.\end{aligned} \tag{74}$$

It follows from (72) and (74) that $\varphi(K_0) = K'_0$. Therefore φ induces an isomorphism of $\mathbb{Z}_2$-graded spaces $\tilde{\varphi} : K/K_0 \to K'/K'_0$. A comparison of dimensions shows that $r = r'$ and $n = n'$. The converse implication is clear. □

Proof of Theorem 3. It suffices to prove that $\phi|_{K_{[-1]}} = \psi|_{K_{[-1]}}$ implies that $\phi = \psi$. Since $\varphi(1) = \varphi([\![x_1, x_{1'}]\!]) = \psi([\![x_1, x_{1'}]\!]) = \psi(1)$, it follows that $\varphi|_{K_{[-2]}} = \psi|_{K_{[-2]}}$. We use induction on ℓ to show that

$$\varphi|_{K_{[\ell]}} = \psi|_{K_{[\ell]}}, \quad \forall \ell \geq -1. \tag{75}$$

Assume that $\ell \geq 0$ and (75) holds for $\ell - 1$. Suppose that $y \in K_{[\ell]}$ and let $z = \varphi(y) - \psi(y)$. We want to prove $z = 0$. The induction hypothesis yields that $\varphi(x_i) = \psi(x_i)$ and $\varphi([\![y, x_i]\!]) = \psi([\![y, x_i]\!])$, $i \in J$. Therefore,

$$\begin{aligned}[\![z, \varphi\left(x_i\right)]\!] &= [\![\varphi\left(y\right) - \psi\left(y\right), \varphi\left(x_i\right)]\!] \\ &= \varphi\left([\![y, x_i]\!]\right) - \psi\left([\![y, x_i]\!]\right) = 0.\end{aligned} \tag{76}$$

Since $\varphi(K_0) = K_0$ and $\varphi(K_{-1}) = K_{-1}$ by Theorem 1, φ induces an automorphism $\widetilde{\varphi}$ of the $\mathbb{Z}_2$-graded space K_{-1}/K_0. Consequently there exists a homogeneous basis $\{h_1, \ldots, h_{m-1}, h_{m+1}, \ldots, h_s\}$ of $K_{[-1]}$ such that $(x_i) \equiv h_i (\mathrm{mod}\ K_0)$. Thus there exist $g_i \in K_0$ such that $\varphi(x_i) = h_i - g_i$, $i \in J$. Therefore, (76) shows that $[\![z, h_i]\!] = [\![z, g_i]\!]$ for all $i \in J$. As $z = \varphi(y) - \psi(y) \in K_0$ by Theorem 1, it can be decomposed into $z = \sum_{j=0}^{t} z_{[j]}$, where $z_{[j]} \in K_{[j]}$. Noting that $[\![z_{[0]}, h_i]\!] \in K_{[-1]}$ and $[\![z, g_i]\!] \in K_0$, we obtain $[\![z_{[0]}, h_i]\!] = 0$ for all $i \in J$, and hence $z_{[0]} = 0$ since K is transitively graded. By induction, we conclude that $z_{[j]} = 0$, $j = 0, 1, \ldots, t$. Hence, $z = 0$, as desired. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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