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Research Article

A Note on Entire Functions That Share Two Small Functions

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This note is to show that if f is a nonconstant entire function that shares two pairs of small functions ignoring multiplicities with its first derivative f', then there exists a close linear relationship between f and f'. This result is a generalization of some results obtained by Rubel and Yang, Mues and Steinmetz, Zheng and Wang, and Qiu. Moreover, examples are provided to show that the conditions in the result are sharp.

1. Introduction and Main Result

Throughout this paper, we use standard notations in the Nevanlinna theory (see, e.g., [1–4]). Let $f(z)$ be a meromorphic function. Here and in the following the word "meromorphic" means meromorphic in the whole complex plane. We denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. The meromorphic function a is called a small function with respect to f provided that $T(r, a) = S(r, f)$.

Let f and g be two nonconstant meromorphic functions, and let a and b be two small functions with respect to f and g. If the zeros of $f - a$ and $g - b$ coincide in locations and multiplicities, then we say that f and g share the pair of small functions (a, b) CM (counting multiplicities); if we do not consider the multiplicities, then f and g are said to share the pair of small functions (a, b) IM (ignoring multiplicities). We see that f and g share the pair of small functions (a, a) CM if and only if f and g share the small function a CM, and f and g share the pair of small functions (a, a) IM if and only if f and g share the small function a IM. The same argument applies in the case when a and b are two values in the extended plane.

Moreover, we introduce the following notations. Denote the set of those points $z \in \mathbf{C}$ by $S_{(m,n)}(a_1, a_2)$ such that z is a zero of $f - a_1$ of multiplicity m and a zero of $f' - a_2$ of multiplicity n. The set $S_{(m,n)}(b_1, b_2)$ can be similarly defined. Now the notations $N_{(m,n)}(r, 1/(f - a_1))$ and $\overline{N}_{(m,n)}(r, 1/(f - a_1))$ denote the counting function and the reduced counting function of f with respect to the set $S_{(m,n)}(a_1, a_2)$, respectively. The notations $N_{(m,n)}(r, 1/(f' - a_2))$ and $\overline{N}_{(m,n)}(r, 1/(f' - a_2))$ can be similarly defined.

Many mathematicians have been interested in the value distribution of different expressions of an entire or meromorphic function and obtained a lot of fruitful and significant results. When dealing with an entire function f and its derivative f', Rubel and Yang [5] proved the following.

Theorem A. *Let f be a nonconstant entire function, and let a and b be distinct finite complex numbers. If f and f' share a and b CM, then $f \equiv f'$.*

Mues and Steinmetz [6] improved Theorem A and obtained the following.

Theorem B. *Let f be a nonconstant entire function, and let a and b be distinct finite complex numbers. If f and f' share a and b IM, then $f \equiv f'$.*

When the values a and b were replaced by two small functions related to f, Zheng and Wang [7] proved the following.

Theorem C. *Let f be a nonconstant entire function, and let a and b be distinct small functions with respect to f. If f and f' share a and b CM, then $f \equiv f'$.*

Recently, Qiu [8] proved the following result which was an improvement of Theorem C.

Theorem D. *Let f be a nonconstant entire function, and let a and b be distinct small functions with respect to f. If f and f' share a and b IM, then $f \equiv f'$.*

This paper is concerned with what can be said when the IM shared small function is replaced by the IM shared the pair of small functions in Theorem D. In fact, we prove the following result by using the method of [8], which generalizes the above theorems from the point of view of shared pairs.

Theorem 1. *Let f be a nonconstant entire function, and let a_1, a_2, b_1, and b_2 be four small functions of f such that none of them is identically equal to ∞ and $a_1 \not\equiv b_1$, $a_2 \not\equiv b_2$. If f and f' share (a_1, a_2) and (b_1, b_2) IM, then $(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \equiv 0$.*

Remark 2. Let $a_1 \equiv a_2$ and $b_1 \equiv b_2$. Then by Theorem 1 we can get Theorem D.

Remark 3. Theorem 1 shows that a nonconstant entire function sharing two pairs of small functions ignoring multiplicities with its first derivative implies that there exists a close linear relationship between them.

Example 4 (see [9]). Let $f = \beta + (\beta - \alpha)/(h - 1)$, where

$$\alpha = -\frac{1}{3}e^{-2z} - \frac{1}{2}e^{-z}, \qquad \beta = -\frac{1}{3}e^{-2z} + \frac{1}{2}e^{-z}, \qquad h = e^{-e^z}. \tag{1}$$

Set $a = \beta'$, $b = \alpha'$. Then $T(r, a) = S(r, f)$ and $T(r, b) = S(r, f)$. It is easy to verify that

$$\begin{aligned} f' - a &= e^{2z}(f - a)(f - \beta), \\ f' - b &= e^{2z}(f - b)(f - \alpha). \end{aligned} \tag{2}$$

Thus f and f' share (a, a) and (b, b) IM, but $f \not\equiv f'$. This shows that the conclusion in Theorem 1 is not valid generally for a meromorphic function f.

Example 5. Let $f = e^{2z} + z$, $a_1 = 2z$, $a_2 = 2z + 1$, $b_1 = z$, and $b_2 = z + 1$. Then f and f' share (a_1, a_2) IM but do not share (b_1, b_2) IM. Clearly, $(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \not\equiv 0$. This shows that the condition in Theorem 1 that f and f' share (a_1, a_2) and (b_1, b_2) IM cannot be weakened.

2. Some Lemmas

Lemma 1. *Let f be a nonconstant entire function, and let a_1, a_2, b_1, and b_2 be four small functions of f such that none of them is identically equal to ∞ and $a_1 \not\equiv b_1$, $a_2 \not\equiv b_2$. If f and f' share (a_1, a_2) and (b_1, b_2) IM, then $S(r, f) = S(r, f') := S(r)$.*

Proof. Note that f and f' share (a_1, a_2) and (b_1, b_2) IM. By the second fundamental theorem, we get

$$\begin{aligned} T(r, f) &\le \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f' - a_2}\right) + \overline{N}\left(r, \frac{1}{f' - b_2}\right) + S(r, f) \\ &\le 2T(r, f') + S(r, f), \\ T(r, f') &\le \overline{N}\left(r, \frac{1}{f' - a_2}\right) + \overline{N}\left(r, \frac{1}{f' - b_2}\right) + S(r, f') \\ &= \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f') \\ &\le 2T(r, f) + S(r, f) + S(r, f'), \end{aligned} \tag{3}$$

which implies from the definition of $S(r, f)$ that $S(r, f) = S(r, f')$ and $S(r, f') = S(r, f)$, respectively.

This completes the proof of Lemma 1. □

Lemma 2. *Let f be a nonconstant entire function, and let a_1, a_2, b_1, and b_2 be four small functions of f such that none of them is identically equal to ∞ and $a_1 \not\equiv b_1$, $a_2 \not\equiv b_2$. If f and f' share (a_1, a_2) and (b_1, b_2) IM, then*

$$m(r, f) = \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) + S(r), \tag{4}$$

provided that $(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \not\equiv 0$.

Proof. Note that

$$(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \not\equiv 0, \tag{5}$$

$$\begin{aligned} &(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \\ &\quad \equiv (a_2 - b_2)(f - a_1) - (a_1 - b_1)(f' - a_2), \end{aligned} \tag{6}$$

$$\begin{aligned} &(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \\ &\quad \equiv (a_2 - b_2)(f - b_1) - (a_1 - b_1)(f' - b_2). \end{aligned} \tag{7}$$

Since f and f' share (a_1, a_2) and (b_1, b_2) IM, from Lemma 1, (5)–(7), and the condition that f is entire, we have

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) \\ &\quad \le N\left(r, \frac{1}{(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1}\right) \\ &\quad \le T\left(r, (a_2 - b_2)f - (a_1 - b_1)f'\right) + S(r) \\ &\quad = m\left(r, (a_2 - b_2)f - (a_1 - b_1)f'\right) \\ &\qquad + N\left(r, (a_2 - b_2)f - (a_1 - b_1)f'\right) + S(r) \\ &\quad = m\left(r, (a_2 - b_2)f - (a_1 - b_1)f'\right) + S(r) \\ &\quad \le m\left(r, \frac{(a_2 - b_2)f - (a_1 - b_1)f'}{f}\right) + m(r, f) + S(r) \\ &\quad \le m(r, f) + S(r). \end{aligned} \tag{8}$$

On the other hand, by the second fundamental theorem, Lemma 1, and the condition that f is entire, we get

$$m(r,f)=T(r,f)\le \overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-b_1}\right)+S(r). \tag{9}$$

Now (8) and (9) imply

$$m(r,f)=\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-b_1}\right)+S(r). \tag{10}$$

This completes the proof of Lemma 2. □

Lemma 3. *Let f be a nonconstant entire function, and let a_1, a_2, b_1, and b_2 be four small functions of f such that none of them is identically equal to ∞ and $a_1\not\equiv b_1$, $a_2\not\equiv b_2$. Suppose that f and f' share (a_1,a_2) and (b_1,b_2) IM. Set*

$$\begin{aligned}\alpha&=\left(a_1'-b_1'\right)(f-a_1)-(a_1-b_1)\left(f'-a_1'\right)\\&=\left(a_1'-b_1'\right)(f-b_1)-(a_1-b_1)\left(f'-b_1'\right),\end{aligned} \tag{11}$$

$$\begin{aligned}\beta&=\left(a_2'-b_2'\right)\left(f'-a_2\right)-(a_2-b_2)\left(f''-a_2'\right)\\&=\left(a_2'-b_2'\right)\left(f'-b_2\right)-(a_2-b_2)\left(f''-b_2'\right),\end{aligned} \tag{12}$$

$$g=\frac{\alpha\left[(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\right]}{(f-a_1)(f-b_1)}, \tag{13}$$

$$h=\frac{\beta\left[(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\right]}{(f'-a_2)(f'-b_2)}, \tag{14}$$

$$\gamma_i=a_2+i(a_2-b_2),\quad (i=1,2). \tag{15}$$

If $(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\not\equiv 0$, then

(i) $T(r,g)=S(r)$,

(ii) $T(r,h)\le T(r,f)-N(r,1/(f'-\gamma_i))+S(r)$ *for* $i=1,2$.

Proof. Since f and f' share (a_1,a_2) and (b_1,b_2) IM, by Lemma 1 we know $S(r,f)=S(r,f'):=S(r)$. Noting

$$\begin{aligned}\frac{\alpha}{f-a_1}&=\frac{\left(a_1'-b_1'\right)(f-a_1)-(a_1-b_1)\left(f'-a_1'\right)}{f-a_1}\\&=a_1'-b_1'-(a_1-b_1)\frac{f'-a_1'}{f-a_1},\\ \frac{\alpha}{f-b_1}&=\frac{\left(a_1'-b_1'\right)(f-b_1)-(a_1-b_1)\left(f'-b_1'\right)}{f-b_1}\\&=a_1'-b_1'-(a_1-b_1)\frac{f'-b_1'}{f-b_1},\\ \frac{1}{(f-a_1)(f-b_1)}&=\frac{1}{a_1-b_1}\left(\frac{1}{f-a_1}-\frac{1}{f-b_1}\right),\\ \frac{\beta}{f'-a_2}&=\frac{\left(a_2'-b_2'\right)\left(f'-a_2\right)-(a_2-b_2)\left(f''-a_2'\right)}{f'-a_2}\\&=a_2'-b_2'-(a_2-b_2)\frac{f''-a_2'}{f'-a_2},\\ \frac{\beta}{f'-b_2}&=\frac{\left(a_2'-b_2'\right)\left(f'-b_2\right)-(a_2-b_2)\left(f''-b_2'\right)}{f'-b_2}\\&=a_2'-b_2'-(a_2-b_2)\frac{f''-b_2'}{f'-b_2},\\ \frac{1}{(f'-a_2)(f'-b_2)}&=\frac{1}{a_2-b_2}\left(\frac{1}{f'-a_2}-\frac{1}{f'-b_2}\right),\end{aligned} \tag{16}$$

and the lemma of the logarithmic derivative, we obtain

$$\begin{gathered}m\left(r,\frac{\alpha}{f-a_1}\right)=S(r),\qquad m\left(r,\frac{\alpha}{f-b_1}\right)=S(r),\\ m\left(r,\frac{\alpha}{(f-a_1)(f-b_1)}\right)=S(r),\end{gathered} \tag{17}$$

$$\begin{gathered}m\left(r,\frac{\beta}{f'-a_2}\right)=S(r),\qquad m\left(r,\frac{\beta}{f'-b_2}\right)=S(r),\\ m\left(r,\frac{\beta}{(f'-a_2)(f'-b_2)}\right)=S(r).\end{gathered} \tag{18}$$

Clearly, $\alpha\not\equiv 0$ and $\beta\not\equiv 0$. Otherwise from (11) and (12) we have $f=a_1+C_1(a_1-b_1)$ and $f'=a_2+C_2(a_2-b_2)$ for nonzero constants C_1,C_2, which implies that $T(r,f)=S(r)$ and $T(r,f')=S(r)$, a contradiction. Then by using a similar method we can deduce that $g\not\equiv 0$ and $h\not\equiv 0$. It is easy to see by (11) if any zero of $f-a_1(f-b_1)$ of multiplicity l is not the pole of a_1-b_1 and is not the zero of a_1-b_1, then it must be a zero of α of multiplicity $l-1$ at least. Thus from (6), (7), (11), (13), the condition that f and f' share (a_1,a_2) and (b_1,b_2) IM, and the condition that f is entire, we get

$$N(r,g)=S(r). \tag{19}$$

Likewise,

$$N(r,h)=S(r). \tag{20}$$

Now by (13) and (17) together with

$$\frac{\alpha f}{(f-a_1)(f-b_1)}=\frac{\alpha}{f-b_1}+\frac{a_1\alpha}{(f-a_1)(f-b_1)}, \tag{21}$$

it follows that

$$m(r,g)\le m\left(r,\frac{\alpha\left[(a_2-b_2)f-(a_1-b_1)f'\right]}{(f-a_1)(f-b_1)}\right)+m\left(r,\frac{\alpha(a_1b_2-a_2b_1)}{(f-a_1)(f-b_1)}\right)+\log 2$$

$$\leq m\left(r,\frac{\alpha f}{(f-a_1)(f-b_1)}\right) + m\left(r,\frac{(a_2-b_2)f-(a_1-b_1)f'}{f}\right)+S(r) \leq S(r). \tag{22}$$

Thus from this and (19) we have

$$T(r,g)=S(r), \tag{23}$$

implying (i). Next, it is easy to see that $\gamma_i \not\equiv a_2$ and $\gamma_i \not\equiv b_2$ $(i=1,2)$. For $i=1,2$, by (14), (18), $a_1 \not\equiv b_1, a_2 \not\equiv b_2$, and the condition that f is entire, we have

$$\begin{aligned}
m(r,h) &\leq m\left(r,\frac{\beta\left[(a_2-b_2)f-(a_1-b_1)f'\right]}{(f'-a_2)(f'-b_2)}\right) + m\left(r,\frac{\beta(a_1b_2-a_2b_1)}{(f'-a_2)(f'-b_2)}\right)+\log 2\\
&\leq m\left(r,\frac{\beta(f'-\gamma_i)}{(f'-a_2)(f'-b_2)} \times \frac{(a_2-b_2)f-(a_1-b_1)f'}{f'-\gamma_i}\right)+S(r)\\
&\leq m\left(r,\frac{\beta}{f'-b_2}\right)+m\left(r,\frac{\beta(a_2-\gamma_i)}{(f'-a_2)(f'-b_2)}\right) + m\left(r,\frac{(a_2-b_2)f-(a_1-b_1)f'}{f'-\gamma_i}\right)+S(r)\\
&\leq m\left(r,\frac{((a_2-b_2)/(a_1-b_1))f-f'}{f'-\gamma_i}\right)+S(r)\\
&\leq m\left(r,\frac{((a_2-b_2)/(a_1-b_1))f-\gamma_i'}{f'-\gamma_i}-1\right)+S(r)\\
&\leq m\left(r,\frac{f'-\gamma_i}{((a_2-b_2)/(a_1-b_1))f-\gamma_i}\right) + N\left(r,\frac{f'-\gamma_i}{((a_2-b_2)/(a_1-b_1))f-\gamma_i}\right) - N\left(r,\frac{((a_2-b_2)/(a_1-b_1))f-\gamma_i}{f'-\gamma_i}\right)+S(r)\\
&\leq m\left(r,\frac{1}{(a_2-b_2)/(a_1-b_1)} \times \frac{f'-\gamma_i}{f-((a_1-b_1)/(a_2-b_2))\gamma_i}\right) + N(r,f'-\gamma_i) + N\left(r,\frac{1}{((a_2-b_2)/(a_1-b_1))f-\gamma_i}\right)\\
&\quad - N\left(r,\frac{a_2-b_2}{a_1-b_1}f-\gamma_i\right) - N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r)\\
&\leq m\left(r,\frac{f'-(((a_1-b_1)/(a_2-b_2))\gamma_i)'}{f-((a_1-b_1)/(a_2-b_2))\gamma_i}\right) + m\left(r,\frac{(((a_1-b_1)/(a_2-b_2))\gamma_i)'-\gamma_i}{f-((a_1-b_1)/(a_2-b_2))\gamma_i}\right)\\
&\quad + N\left(r,\frac{1}{((a_2-b_2)/(a_1-b_1))f-\gamma_i}\right) - N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r)\\
&\leq m\left(r,\frac{1}{f-((a_1-b_1)/(a_2-b_2))\gamma_i}\right) + N\left(r,\frac{1}{((a_2-b_2)/(a_1-b_1))f-\gamma_i}\right) - N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r)\\
&\leq m\left(r,\frac{1}{((a_2-b_2)/(a_1-b_1))f-\gamma_i}\right) + N\left(r,\frac{1}{((a_2-b_2)/(a_1-b_1))f-\gamma_i}\right) - N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r)\\
&\leq T(r,f)-N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r).
\end{aligned} \tag{24}$$

Thus from (20) and (24) it follows that

$$T(r,h) \leq T(r,f)-N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r), \quad i=1,2. \tag{25}$$

This proves (ii) and completes the proof of Lemma 3. □

Lemma 4 (see [10]; cf. [11, 12]). *Let f be a nonconstant meromorphic function, and let $f^nP(f)=Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in f and the degree of $Q(f)$ is at most n. Then*

$$m(r,P(f))=S(r,f). \tag{26}$$

Lemma 5. *Let f be a nonconstant entire function, and let a_1, a_2, b_1, and b_2 be four small functions of f such that none of them is identically equal to ∞ and $a_1 \not\equiv b_1$, $a_2 \not\equiv b_2$. Suppose that f and f' share (a_1, a_2) and (b_1, b_2) IM. If*

$$m(r, f) = m(r, f') + S(r, f), \tag{27}$$

then $(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \equiv 0$.

Proof. Assume that $(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \not\equiv 0$. Let α, β, g, h, and γ_i be defined by (11)–(15), respectively. Then from the proof process of Lemma 3 we know $\alpha \not\equiv 0$, $\beta \not\equiv 0$, $g \not\equiv 0$, $h \not\equiv 0$, $\gamma_i \not\equiv a_2$, and $\gamma_i \not\equiv b_2$ $(i = 1, 2)$. Since f and f' share (a_1, a_2) and (b_1, b_2) IM, by Lemmas 1, 2, and 3 it follows that

$$m(r, f) = \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) + S(r), \tag{28}$$

$$T(r, g) = S(r), \tag{29}$$

$$T(r, h) \le T(r, f) - N\left(r, \frac{1}{f' - \gamma_i}\right) + S(r), \quad i = 1, 2. \tag{30}$$

Now from the second fundamental theorem, (27), (28), and the assumption that f is entire, we deduce

$$\begin{aligned} 2m(r, f') &= 2T(r, f') \\ &\le \overline{N}\left(r, \frac{1}{f' - a_2}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f' - b_2}\right) + \overline{N}\left(r, \frac{1}{f' - \gamma_i}\right) + S(r) \\ &\le \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) \\ &\quad + N\left(r, \frac{1}{f' - \gamma_i}\right) + S(r) \\ &= m(r, f) + N\left(r, \frac{1}{f' - \gamma_i}\right) + S(r) \\ &= m(r, f') + N\left(r, \frac{1}{f' - \gamma_i}\right) + S(r), \end{aligned} \tag{31}$$

which yields

$$N\left(r, \frac{1}{f' - \gamma_i}\right) = m(r, f') + S(r). \tag{32}$$

Again by (27), (30), (32), and the assumption that f is entire, we obtain

$$T(r, h) = S(r). \tag{33}$$

For any $z_0 \in S_{(m,n)}(a_1, a_2) \cup S_{(m,n)}(b_1, b_2)$, from (13) and (14), we can get $ng(z_0) - mh(z_0) = 0$.

If $ng - mh \equiv 0$, then by (13) and (14) we deduce

$$n\left(\frac{f' - b_1'}{f - b_1} - \frac{f' - a_1'}{f - a_1}\right) \equiv m\left(\frac{f'' - b_2'}{f' - b_2} - \frac{f'' - a_2'}{f' - a_2}\right), \tag{34}$$

which implies that

$$\left(\frac{f - b_1}{f - a_1}\right)^n \equiv c_1\left(\frac{f' - b_2}{f' - a_2}\right)^m, \tag{35}$$

where c_1 is a nonzero constant. If $n \ne m$, then from (35) and the condition that f is entire, we obtain $nm(r, f) = mm(r, f') + S(r)$, which contradicts (27). If $n = m$, then we get

$$\frac{f - b_1}{f - a_1} \equiv c_2\left(\frac{f' - b_2}{f' - a_2}\right), \tag{36}$$

where c_2 is a nonzero constant. We claim that $c_2 \ne 1$. Indeed, if $c_2 = 1$, then by (36) we deduce

$$\frac{f - b_1}{f - a_1} \equiv \frac{f' - b_2}{f' - a_2}, \tag{37}$$

which leads to $(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \equiv 0$. This contradicts the assumption. Thus $c_2 \ne 1$ and so from (36) we have

$$f\left[(1 - c_2)f' + c_2b_2 - a_2\right] = (b_1 - c_2a_1)f' + c_2a_1b_2 - a_2b_1. \tag{38}$$

This and Lemma 4 yield

$$m(r, (1 - c_2)f' + c_2b_2 - a_2) = S(r), \tag{39}$$

which gives $m(r, f') = S(r)$. From this and the condition that f is entire, it follows that $T(r, f') = S(r)$, a contradiction. Hence $ng - mh \not\equiv 0$, for any positive integers m and n.

Therefore by (29) and (33) we obtain

$$\begin{aligned} &\overline{N}_{(m,n)}\left(r, \frac{1}{f - a_1}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f - b_1}\right) \\ &\quad \le N\left(r, \frac{1}{ng - mh}\right) + S(r) \\ &\quad \le T(r, g) + T(r, h) + S(r) \\ &\quad = S(r), \end{aligned} \tag{40}$$

for any positive integers m and n. It follows from this, Lemma 1, the second fundamental theorem, and the condition that f is entire that

$$\begin{aligned}
T(r,f) &\le \overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-b_1}\right)+S(r)\\
&=\sum_{m,n}\left(\overline{N}_{(m,n)}\left(r,\frac{1}{f-a_1}\right)+\overline{N}_{(m,n)}\left(r,\frac{1}{f-b_1}\right)\right)+S(r)\\
&=\sum_{m+n\ge 6}\left(\overline{N}_{(m,n)}\left(r,\frac{1}{f-a_1}\right)+\overline{N}_{(m,n)}\left(r,\frac{1}{f-b_1}\right)\right)+S(r)\\
&\le \sum_{m+n\ge 6}\frac{1}{6}\left(N_{(m,n)}\left(r,\frac{1}{f-a_1}\right)+N_{(m,n)}\left(r,\frac{1}{f'-a_2}\right)\right)\\
&\quad+\sum_{m+n\ge 6}\frac{1}{6}\left(N_{(m,n)}\left(r,\frac{1}{f-b_1}\right)+N_{(m,n)}\left(r,\frac{1}{f'-b_2}\right)\right)+S(r)\\
&\le \frac{1}{6}\left(N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f'-a_2}\right)\right)\\
&\quad+\frac{1}{6}\left(N\left(r,\frac{1}{f-b_1}\right)+N\left(r,\frac{1}{f'-b_2}\right)\right)+S(r)\\
&\le \frac{1}{3}T(r,f)+\frac{1}{3}T(r,f')+S(r)\\
&=\frac{1}{3}T(r,f)+\frac{1}{3}m(r,f')+S(r)\\
&=\frac{1}{3}T(r,f)+\frac{1}{3}m\left(r,\frac{f'}{f}f\right)+S(r)\\
&\le \frac{1}{3}T(r,f)+\frac{1}{3}m(r,f)+S(r)\\
&=\frac{1}{3}T(r,f)+\frac{1}{3}T(r,f)+S(r)\\
&=\frac{2}{3}T(r,f)+S(r),
\end{aligned} \tag{41}$$

which implies that $T(r,f)=S(r)$, a contradiction. Thus $(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\equiv 0$.

This completes the proof of Lemma 5. □

3. Proof of Theorem 1

Suppose that $(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\not\equiv 0$. Since f and f' share (a_1,a_2) and (b_1,b_2) IM, by Lemma 1 we have $S(r,f)=S(r,f'):=S(r)$. Let α, β, g, h, and γ_i be defined by (11)–(15), respectively. Then from the proof process of Lemma 3 we know $\alpha\not\equiv 0$, $\beta\not\equiv 0$, $g\not\equiv 0$, $h\not\equiv 0$, $\gamma_i\not\equiv a_2$, and $\gamma_i\not\equiv b_2$ $(i=1,2)$. Next we rewrite (13) as

$$\begin{aligned}
&\left[g-(a_1'-b_1')(a_2-b_2)\right]f^2\\
&\quad=d_1ff'+d_2f+d_3f'+d_4f'^2+d_5,
\end{aligned} \tag{42}$$

where $d_1=(a_1-b_1)(b_1'+b_2-a_2-a_1')$, $d_2=(a_1'-b_1')(a_1b_2-a_2b_1)+(a_2-b_2)(a_1b_1'-a_1'b_1)+(a_1+b_1)g$, $d_3=(a_1-b_1)(a_2b_1+a_1'b_1-a_1b_2-a_1b_1')$, $d_4=(a_1-b_1)^2$, and $d_5=(a_1b_1'-a_1'b_1)(a_1b_2-a_2b_1)-a_1b_1g$ are all small functions with respect to f.

Now we divide into two cases.

Case 1. $g-(a_1'-b_1')(a_2-b_2)\equiv 0$; that is, $g\equiv(a_1'-b_1')(a_2-b_2)$. We again discuss the three subcases.

Subcase 1. $a_1'\not\equiv a_2$ and $b_1'\not\equiv b_2$. Since f and f' share (a_1,a_2) and (b_1,b_2) IM, the zeros of $f-a_1$ and $f-b_1$ of multiplicity larger than one are the zeros $a_1'-a_2$ and $b_1'-b_2$, respectively. It then follows that

$$\begin{aligned}
&\sum_{m\ge 2,n\ge 1}\left(\overline{N}_{(m,n)}\left(r,\frac{1}{f-a_1}\right)+\overline{N}_{(m,n)}\left(r,\frac{1}{f-b_1}\right)\right)\\
&\quad\le N\left(r,\frac{1}{a_1'-a_2}\right)+N\left(r,\frac{1}{b_1'-b_2}\right)+S(r)\\
&\quad\le T(r,a_1')+T(r,a_2)+T(r,b_1')\\
&\qquad+T(r,b_2)+S(r)\\
&\quad=S(r);
\end{aligned} \tag{43}$$

that is,

$$\sum_{m\ge 2,n\ge 1}\left(\overline{N}_{(m,n)}\left(r,\frac{1}{f-a_1}\right)+\overline{N}_{(m,n)}\left(r,\frac{1}{f-b_1}\right)\right)=S(r). \tag{44}$$

Let $z_0\in S_{(1,n)}(a_1,a_2)$. For $n\ge 2$, from (13) we get

$$\begin{aligned}
g(z_0)&=\left(a_1'(z_0)-a_2(z_0)\right)\left(a_2(z_0)-b_2(z_0)\right)\\
&=\left(a_1'(z_0)-b_1'(z_0)\right)\left(a_2(z_0)-b_2(z_0)\right),
\end{aligned} \tag{45}$$

which implies that $a_2(z_0)-b_1'(z_0)=0$ or $a_2(z_0)-b_2(z_0)=0$.

If $a_2-b_1'\equiv 0$, then by (13) we deduce

$$f'=\frac{a_1'-b_2}{a_1-b_1}f+\frac{a_1b_2-a_1'b_1}{a_1-b_1}, \tag{46}$$

which, in view of the condition that f is entire, implies that $m(r,f)=m(r,f')+S(r)$. From this and Lemma 5, it follows that $(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\equiv 0$, contradicting the assumption. Thus $a_2-b_1'\not\equiv 0$. By the conditions in Theorem 1, we know that $a_2-b_2\not\equiv 0$.

Hence

$$\sum_{n\geq 2}\overline{N}_{(1,n)}\left(r,\frac{1}{f-a_1}\right) \leq N\left(r,\frac{1}{a_2-b_1'}\right)+N\left(r,\frac{1}{a_2-b_2}\right)+S(r) \leq 2T(r,a_2)+T(r,b_2)+T\left(r,b_1'\right)+S(r)=S(r); \tag{47}$$

that is,

$$\sum_{n\geq 2}\overline{N}_{(1,n)}\left(r,\frac{1}{f-a_1}\right)=S(r). \tag{48}$$

Similarly,

$$\sum_{n\geq 2}\overline{N}_{(1,n)}\left(r,\frac{1}{f-b_1}\right)=S(r). \tag{49}$$

It then follows from (44)–(49) and the second fundamental theorem that

$$T(r,f)\leq \overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-b_1}\right)+S(r) = \overline{N}_{(1,1)}\left(r,\frac{1}{f-a_1}\right)+\overline{N}_{(1,1)}\left(r,\frac{1}{f-b_1}\right)+S(r). \tag{50}$$

For any $z_1 \in S_{(1,1)}(a_1,a_2)\cup S_{(1,1)}(b_1,b_2)$, from (13) and (14), we can get $g(z_1)-h(z_1)=0$.

If $g-h\equiv 0$, then by (13) and (14) we have

$$\frac{f'-b_1'}{f-b_1}-\frac{f'-a_1'}{f-a_1}\equiv\frac{f''-b_2'}{f'-b_2}-\frac{f''-a_2'}{f'-a_2}, \tag{51}$$

which implies that

$$\frac{f-b_1}{f-a_1}\equiv c_3\frac{f'-b_2}{f'-a_2}, \tag{52}$$

where c_3 is a nonzero constant. We claim that $c_3\neq 1$. Indeed, if $c_3=1$, then by (52) we have

$$\frac{f-b_1}{f-a_1}\equiv\frac{f'-b_2}{f'-a_2}, \tag{53}$$

which leads to $(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\equiv 0$. This contradicts the assumption. Thus $c_3\neq 1$ and so from (52) we get

$$f\left[(1-c_3)f'+c_3b_2-a_2\right]=(b_1-c_3a_1)f'+c_3a_1b_2-a_2b_1. \tag{54}$$

This and Lemma 4 yield

$$m\left(r,(1-c_3)f'+c_3b_2-a_2\right)=S(r), \tag{55}$$

which gives $m(r,f')=S(r)$. From this and the condition that f is entire, it follows that $T(r,f')=S(r)$, a contradiction. Hence $g-h\not\equiv 0$.

Therefore by (50) and Lemma 3 we obtain

$$T(r,f)\leq \overline{N}_{(1,1)}\left(r,\frac{1}{f-a_1}\right)+\overline{N}_{(1,1)}\left(r,\frac{1}{f-b_1}\right)+S(r) \leq N\left(r,\frac{1}{g-h}\right)+S(r) \leq T(r,g)+T(r,h)+S(r) \leq T(r,f)-N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r), \tag{56}$$

which implies that

$$N\left(r,\frac{1}{f'-\gamma_i}\right)=S(r),\quad i=1,2. \tag{57}$$

This is impossible by the second fundamental theorem.

Subcase 2. Either $a_1'\equiv a_2$ and $b_1'\not\equiv b_2$ or $a_1'\not\equiv a_2$ and $b_1'\equiv b_2$. Without loss of generality, we assume that $a_1'\equiv a_2$ and $b_1'\not\equiv b_2$. It is easy to see by (13) that the zeros of $f-a_1$ and $f'-a_2$ of multiplicity all larger than one are the zeros of g. Thus by Lemma 3,

$$\sum_{m\geq 2,n\geq 2}\overline{N}_{(m,n)}\left(r,\frac{1}{f-a_1}\right)\leq N\left(r,\frac{1}{g}\right)+S(r) \leq T(r,g)+S(r)=S(r); \tag{58}$$

that is,

$$\sum_{m\geq 2,n\geq 2}\overline{N}_{(m,n)}\left(r,\frac{1}{f-a_1}\right)=S(r). \tag{59}$$

By the discussion of Subcase 1, we see

$$\overline{N}\left(r,\frac{1}{f-b_1}\right)=\overline{N}_{(1,1)}\left(r,\frac{1}{f-b_1}\right)+S(r) \leq N\left(r,\frac{1}{g-h}\right)+S(r) \leq T(r,g)+T(r,h)+S(r) \leq T(r,f)-N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r); \tag{60}$$

that is,

$$\overline{N}\left(r,\frac{1}{f-b_1}\right)\leq T(r,f)-N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r),\quad i=1,2. \tag{61}$$

Note that the zeros of $f-a_1$ of multiplicity larger than one are all the zeros of $f'-a_2=f'-a_1'$. Since f and f' share (a_1,a_2) IM, it follows that

$$\overline{N}_{(1,1)}\left(r,\frac{1}{f-a_1}\right)=S(r). \tag{62}$$

Now from (59)–(62) and the second fundamental theorem, we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-b_1}\right)+S(r)\\ &\leq \overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right)+T(r,f)\\ &\quad -N\left(r,\frac{1}{f'-\gamma_i}\right)+S(r); \end{aligned} \tag{63}$$

that is,

$$N\left(r,\frac{1}{f'-\gamma_i}\right)\leq \overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right)+S(r), \quad i=1,2. \tag{64}$$

Let

$$\phi = 2\frac{f''-b_2'}{f'-b_2}-2\frac{f'-b_1'}{f-b_1}+\frac{a_1'-b_1'}{a_1-b_1}-2\frac{a_2'-b_2'}{a_2-b_2}. \tag{65}$$

It is easily seen from (65) and the lemma of the logarithmic derivative that

$$m(r,\phi)=S(r). \tag{66}$$

Note that common simple zeros of $f-b_1$ and $f'-b_2$ are not the poles of ϕ. In terms of the discussion of Subcase 1, from (65) we know $N(r,\phi)=S(r)$, which together with (66) gives that

$$T(r,\phi)=S(r). \tag{67}$$

Let $z_2 \in S_{(2,1)}(a_1,a_2)$. Then by (65) and (13) we have $\phi(z_2)=0$.

If $\phi\equiv 0$, then from (65) we derive

$$\left(f'-b_2\right)^2 = c_4\frac{(a_2-b_2)^2}{a_1-b_1}(f-b_1)^2, \tag{68}$$

where c_4 is a nonzero constant. This, in view of the condition that f is entire, implies that $m(r,f)=m(r,f')+S(r)$. From this and Lemma 5, it follows that $(a_2-b_2)f-(a_1-b_1)f'+a_1b_2-a_2b_1\equiv 0$, contradicting the assumption. Thus $\phi\not\equiv 0$.

Hence by (64) and (67) we obtain

$$\begin{aligned} N\left(r,\frac{1}{f'-\gamma_i}\right) &\leq \overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right)+S(r)\\ &\leq N\left(r,\frac{1}{\phi}\right)+S(r)\\ &\leq T(r,\phi)+S(r)\\ &\leq S(r); \end{aligned} \tag{69}$$

that is,

$$N\left(r,\frac{1}{f'-\gamma_i}\right)=S(r), \quad i=1,2. \tag{70}$$

This is also impossible by the second fundamental theorem.

Subcase 3. $a_1'\equiv a_2$ and $b_1'\equiv b_2$. By the discussion of Subcase 2, we see

$$T(r,f)\leq \overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right)+\overline{N}_{(2,1)}\left(r,\frac{1}{f-b_1}\right)+S(r). \tag{71}$$

We claim that

$$\overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right)=S(r), \tag{72}$$

$$\overline{N}_{(2,1)}\left(r,\frac{1}{f-b_1}\right)=S(r). \tag{73}$$

Let

$$\psi = 2\frac{f''-b_2'}{f'-b_2}-\frac{f'-b_1'}{f-b_1}-2\frac{a_2'-b_2'}{a_2-b_2}. \tag{74}$$

It is easily known from (74) and the lemma of the logarithmic derivative that

$$m(r,\psi)=S(r). \tag{75}$$

Note that common zeros of $f-b_1$ of multiplicity two and $f'-b_2$ of multiplicity one are not the poles of ψ. In terms of the discussion of Subcase 2, we know $N(r,\psi)=S(r)$, which together with (75) gives that

$$T(r,\psi)=S(r). \tag{76}$$

Let $z_3 \in S_{(2,1)}(a_1,a_2)$. Then by (74) and (13) we have $\psi(z_3)=0$.

If $\psi\not\equiv 0$, then from (76) we get

$$\begin{aligned} \overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right) &\leq N\left(r,\frac{1}{\psi}\right)+S(r)\\ &\leq T(r,\psi)+S(r)\leq S(r), \end{aligned} \tag{77}$$

that is,

$$\overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right)=S(r). \tag{78}$$

If $\psi\equiv 0$, then by (74) we deduce

$$\left(f'-b_2\right)^2 = c_5(a_2-b_2)^2(f-b_1), \tag{79}$$

where c_5 is a nonzero constant. This implies $a_1(z_3)-b_1(z_3)-1/c_5=0$. Since $a_1'\equiv a_2$, $b_1'\equiv b_2$, and $a_2\not\equiv b_2$, we obtain $a_1-b_1-1/c_5\not\equiv 0$. Thus

$$\begin{aligned} \overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right) &\leq N\left(r,\frac{1}{a_1-b_1-1/c_5}\right)+S(r)\\ &\leq T(r,a_1)+T(r,b_1)+S(r)\leq S(r); \end{aligned} \tag{80}$$

that is,

$$\overline{N}_{(2,1)}\left(r,\frac{1}{f-a_1}\right)=S(r). \tag{81}$$

Hence (72) follows. In the same manner as above, we can prove (73). The proof of the claim is complete. Now by (71)–(73) we get $T(r,f)=S(r)$, a contradiction.

Case 2. $g - (a_1' - b_1')(a_2 - b_2) \not\equiv 0$. Then by (42) and (i) in Lemma 3 we have

$$\begin{aligned} 2m(r,f) &\le m\left(r, \frac{1}{g-(a_1'-b_1')(a_2-b_2)}\right) \\ &\quad + m\left(r, d_1 ff' + d_2 f + d_3 f' + d_4 f'^2 + d_5\right) \\ &\le m\left(r, f\left(d_1 f' + d_2 + d_3\frac{f'}{f} + d_4 f'\frac{f'}{f}\right)\right) \\ &\quad + m(r,d_5) + S(r) \\ &\le m(r,f) + m\left(r, f'\left(d_1 + d_4\frac{f'}{f}\right)\right) + S(r) \\ &\le m(r,f) + m\left(r,f'\right) + S(r), \end{aligned} \quad (82)$$

which implies that

$$m(r,f) \le m\left(r,f'\right) + S(r). \quad (83)$$

On the other hand,

$$m\left(r,f'\right) \le m\left(r, f\frac{f'}{f}\right) \le m(r,f) + S(r). \quad (84)$$

Combining (83) with (84) yields

$$m(r,f) = m\left(r,f'\right) + S(r). \quad (85)$$

This and Lemma 5 lead to $(a_2 - b_2)f - (a_1 - b_1)f' + a_1b_2 - a_2b_1 \equiv 0$, contradicting the assumption.

This completes the proof of Theorem 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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