Research Article

Symmetry and Nonexistence of Positive Solutions for Weighted HLS System of Integral Equations on a Half Space

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We consider system of integral equations related to the weighted Hardy-Littlewood-Sobolev (HLS) inequality in a half space. By the Pohozaev type identity in integral form, we present a Liouville type theorem when the system is in both supercritical and subcritical cases under some integrability conditions. Ruling out these nonexistence results, we also discuss the positive solutions of the integral system in critical case. By the method of moving planes, we show that a pair of positive solutions to such system is rotationally symmetric about x_n-axis, which is much more general than the main result of Zhuo and Li, 2011.

1. Introduction

In [1], Jin and Li studied the weighted HLS system of nonlinear equations in R^n:

$$u(x) = \frac{1}{|x|^\alpha} \int_{R^n} \frac{1}{|x-y|^\lambda} \frac{v^q(y)}{|y|^\beta} dy,$$
$$v(x) = \frac{1}{|x|^\beta} \int_{R^n} \frac{1}{|x-y|^\lambda} \frac{u^p(y)}{|y|^\alpha} dy, \tag{1}$$

where $0 < \lambda < n$ and $1/(p+1) + 1/(q+1) = (\lambda + \alpha + \beta)/n$.

By the method of moving planes in integral forms they derived symmetry and monotonicity of positive solutions of (1) under some integrability conditions.

Theorem 1 (see [1]). *Let the pair (u, v) be a positive solution of system (1) with $u \in L^{p+1}(R^n)$, $v \in L^{q+1}(R^n)$ and $p, q \geq 1$, $pq \neq 1$, and $\alpha, \beta \geq 0$. Then u and v are radially symmetric and decreasing about some point x_0.*

Jin and Li [2] and Chen et al. [3] also discussed the regularity of solutions to (1).

Let R_+^n be the upper half Euclidean space

$$R_+^n = \{x = (x_1, \ldots, x_n) \in R^n \mid x_n > 0\}. \tag{2}$$

In this paper, we want to consider the similar integral system in the half space R_+^n as (1). More precisely, we discuss the following weighted HLS type system of nonlinear equations in R_+^n:

$$u(x) = \frac{1}{|x|^\alpha} \int_{R_+^n} G(x, y, \gamma) \frac{v^q(y)}{|y|^\beta} dy,$$
$$v(x) = \frac{1}{|x|^\beta} \int_{R_+^n} G(x, y, \gamma) \frac{u^p(y)}{|y|^\alpha} dy, \tag{3}$$

where $u, v \geq 0$, $0 < p, q < \infty$, $0 < \gamma < n$, $\alpha + \beta \geq 0$, $\alpha/n < 1/(p+1) < (n-\gamma+\alpha)/n$, and

$$G(x, y, \gamma) = \frac{1}{|x-y|^{n-\gamma}} - \frac{1}{|x^*-y|^{n-\gamma}}; \tag{4}$$

here x^* is the reflection point of x about the plane ∂R_+^n.

Similar to some integral systems or PDEs systems, the integral system (3) is usually divided into three cases according to the value of exponents (p, q). We say that system (3) is in critical case when the pair (p, q) satisfies the relation

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\gamma+\alpha+\beta}{n}. \tag{5}$$

It is in supercritical case when "<" holds; and in subcritical case when ">" holds; that is

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-\gamma+\alpha+\beta}{n}. \tag{6}$$

In the special case, where $\alpha = 0$ and $\beta = 0$, system (3) reduces to

$$u(x) = \int_{R_+^n} G(x, y, \gamma) v^q(y) \, dy,$$
$$v(x) = \int_{R_+^n} G(x, y, \gamma) u^p(y) \, dy, \tag{7}$$

and system (7) is closely related to the following system of PDEs with Navier boundary conditions:

$$(-\Delta)^{\gamma/2} u = v^q, \quad \text{in } R_+^n;$$
$$(-\Delta)^{\gamma/2} v = u^p, \quad \text{in } R_+^n;$$
$$(-\Delta)^k u = 0 \quad \text{on } \partial R_+^n; \tag{8}$$
$$(-\Delta)^k v = 0, \quad \text{on } \partial R_+^n;$$
$$k = 0, \ 1, \dots, \frac{\gamma}{2} - 1.$$

In particular, when γ is an even number, the authors ([4]) proved the equivalence between the two systems (7) and (8) under some mild growth condition.

Symmetry of solutions to integral system (8) was established by Zhuo and Li [5]. They proved that in critical case $1/(p+1) + 1/(q+1) = (n-\alpha)/n$, any pair of positive solutions of (7) with $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$ is rotationally symmetric about some line parallel to x_n-axis. Under the same integrability conditions, in [6], we obtained the nonexistence of positive solutions of (7).

The general case is that, for $\alpha \neq 0$ and $\beta \neq 0$ in (3), there are few results concerning symmetry and nonexistence for this doubled weighted system. In this paper, by the Pohozaev type identity in integral form, we present a Liouville type theorem when the system (3) is in both supercritical and subcritical cases under some integrability conditions. Based on these nonexistence results, we discuss the positive solutions of (3) in critical case. By the method of moving planes, we show that a pair of positive solutions to such system is rotationally symmetric about x_n-axis. To carry on the moving of planes, we explore global features of the integral equations and estimate certain integral norms. This is the essence of the method of moving planes in integral forms. The readers who are interested in the integral system and the applications of this method may consult [7–10] and the references therein.

The paper is organized as follows.

In Section 2, by the Pohozaev type identity in integral forms, we prove the following nonexistence results.

Theorem 2. *Suppose that $(u(x), v(x)) \in C^1(R^n)$ are nonnegative solutions of (3) with $u \in L^{p+1}(R_+^n)$, $v \in L^{q+1}(R_+^n)$.*

(i) *If p and q are both supercritical, that is,*

$$\frac{1}{p+1} < \frac{n-\gamma}{2n} + \frac{\alpha}{n}, \qquad \frac{1}{q+1} < \frac{n-\gamma}{2n} + \frac{\beta}{n}, \tag{9}$$

or

(ii) *if p and q are both subcritical, that is,*

$$\frac{1}{p+1} \in \left(\frac{n-\gamma}{2n} + \frac{\alpha}{n}, \frac{n-\gamma+\alpha}{n} \right),$$
$$\frac{1}{q+1} \in \left(\frac{n-\gamma}{2n} + \frac{\beta}{n}, \frac{n-\gamma+\beta}{n} \right), \tag{10}$$

then $u \equiv 0$ and $v \equiv 0$.

Based on these results and ruling out cases where there are no solutions, we are only interested in critical case (5). In Section 3, by means of method of moving planes in integral form, we establish rotational symmetry of solutions of (3) in critical case (5) as follows.

Theorem 3. *Assume that $u \in L^{p+1}(R_+^n)$, $v \in L^{q+1}(R_+^n)$ and p, q satisfy (5). If (u, v) is a pair of positive solutions of (3), then (u, v) is rotationally symmetric about x_n-axis.*

Remark 4. When $\alpha = \beta = 0$, Theorem 3 is coincident with the result in [5].

2. Proof of Theorem 2

In this section we will prove the nonexistence of positive solutions to the weighted HLS type system (3). These nonexistence results, known as Liouville type theorems, are useful in deriving existence, a priori estimate, regularity, and asymptotic analysis of solutions.

A celebrated result of S. I. Pohozaev is known as the Pohozaev identity. This classical result has many consequences, the most immediate one being the nonexistence of nontrivial bounded solutions to PDE. Here we apply the Pohozaev type identity in integral forms to the integral system (3) (see in [9, 11]).

For any $\rho \neq 0$, there holds

$$u(\rho x) = \frac{1}{|\rho x|^\alpha} \int_{R_+^n} \left(\frac{1}{|\rho x - y|^{n-\gamma}} - \frac{1}{|\rho x^* - y|^{n-\gamma}} \right) \frac{v^q(y)}{|y|^\beta} \, dy. \tag{11}$$

By an elementary calculation,

$$\frac{d\left(|\rho x|^{-\alpha} \right)}{d\rho}$$
$$= -\frac{\alpha}{2} |\rho x|^{-\alpha-2} \cdot (2\rho x \cdot x)$$
$$= (-\alpha\rho) |\rho x|^{-\alpha-2} |x|^2.$$

$$\frac{d\left(|\rho x - y|^{\gamma-n}\right)}{d\rho}$$

$$= \frac{\gamma - n}{2}|\rho x - y|^{\gamma-n-2}$$

$$\times \frac{d}{d\rho}\left[(\rho x_1 - y_1)^2 + \cdots + (\rho x_n - y_n)^2\right]$$

$$= (\gamma - n)|\rho x - y|^{\gamma-n-2}x \cdot (\rho x - y),$$

$$\frac{d\left(|\rho x^* - y|^{\gamma-n}\right)}{d\rho}$$

$$= \frac{\gamma - n}{2}|\rho x^* - y|^{\gamma-n-2}$$

$$\times \frac{d}{d\rho}\left[(\rho x_1 - y_1)^2 + \cdots + (\rho x_{n-1} - y_{n-1})^2\right.$$

$$\left. +(-\rho x_n - y_n)^2\right]$$

$$= (\gamma - n)|\rho x - y|^{\gamma-n-2}x^* \cdot (\rho x^* - y). \tag{12}$$

Noting $u \in C^1(R^n)$, differentiating both sides of (11) with respect to ρ and letting $\rho = 1$, we have

$$x \cdot \nabla u(x) = (-\alpha)u(x)$$

$$+ (\gamma - n)\frac{1}{|x|^\alpha}\int_{R_+^n}\left[\frac{x \cdot (x - y)}{|x - y|^{n-\gamma+2}}\right.$$

$$\left. -\frac{x^* \cdot (x^* - y)}{|x^* - y|^{n-\gamma+2}}\right]\frac{v^q(y)}{|y|^\beta}dy. \tag{13}$$

Let $B_r^+(0) = B_r(0) \cap R_+^n$ be the upper half ball in the half space in R_+^n. Multiplying left side of (13) by $u^p(x)$ and integrating on B_r^+ yields

$$\int_{B_r^+}u^p(x)(x \cdot \nabla u(x))dx$$

$$= \frac{1}{p+1}\int_{B_r^+}x \cdot \nabla\left(u^{p+1}(x)\right)dx \tag{14}$$

$$= \frac{1}{p+1}\int_{\partial B_r^+}ru^{p+1}(x)d\sigma - \frac{n}{p+1}\int_{B_r^+}u^{p+1}(x)dx.$$

Similarly, we also have

$$\int_{B_r^+}v^q(x)(x \cdot \nabla v(x))dx$$

$$= \frac{1}{q+1}\int_{\partial B_r^+}rv^{q+1}(x)d\sigma - \frac{n}{q+1}\int_{B_r^+}v^{q+1}(x)dx. \tag{15}$$

Since

$$\int_{R_+^n}u^{p+1}(x)dx < \infty, \qquad \int_{R_+^n}v^{q+1}(x)dx < \infty. \tag{16}$$

Thus, there exists a sequence $\{r_m\}$ such that

$$r_m\int_{\partial B_{r_m}^+}u^{p+1}(x)d\sigma \longrightarrow 0,$$

$$\int_{\partial B_{r_m}^+}r_mv^{q+1}(x)d\sigma \longrightarrow 0, \tag{17}$$

$$r_m \longrightarrow \infty.$$

Let $r_m \to \infty$; by (14), (15), and (17), we have

$$\int_{R_+^n}u^p(x)(x \cdot \nabla u(x))dx + \int_{R_+^n}v^q(x)(x \cdot \nabla v(x))dx$$

$$= -\frac{n}{p+1}\int_{R_+^n}u^{p+1}(x)dx - \frac{n}{q+1}\int_{R_+^n}v^{q+1}(x)dx < \infty. \tag{18}$$

On the other hand,

$$\int_{R_+^n}u^p(x)(x \cdot \nabla u(x))dx$$

$$= (\gamma - n)\int\int_{R_+^n}\left[\frac{x \cdot (x - y)}{|x - y|^{n-\gamma+2}} - \frac{x^* \cdot (x^* - y)}{|x^* - y|^{n-\gamma+2}}\right]$$

$$\times \frac{u^p(x)v^q(y)}{|x|^\alpha|y|^\beta}dx\,dy$$

$$+ (-\alpha)\int_{R_+^n}u^{p+1}(x)dx$$

$$= \frac{\gamma - n}{2}\int\int_{R_+^n}\left[\frac{x \cdot (x - y)}{|x - y|^{n-\gamma+2}} - \frac{x^* \cdot (x^* - y)}{|x^* - y|^{n-\gamma+2}}\right] \tag{19}$$

$$\times \frac{u^p(x)v^q(y)}{|x|^\alpha|y|^\beta}dx\,dy$$

$$+ \frac{\gamma - n}{2}\int\int_{R_+^n}\left[\frac{y \cdot (y - x)}{|x - y|^{n-\gamma+2}} - \frac{y^* \cdot (y^* - x)}{|y^* - x|^{n-\gamma+2}}\right]$$

$$\times \frac{u^p(y)v^q(x)}{|y|^\alpha|x|^\beta}dx\,dy$$

$$+ (-\alpha)\int_{R_+^n}u^{p+1}(x)dx.$$

There also holds

$$\int_{R_+^n}v^q(x)(x \cdot \nabla v(x))dx$$

$$= (\gamma - n)\int\int_{R_+^n}\left[\frac{x \cdot (x - y)}{|x - y|^{n-\gamma+2}} - \frac{x^* \cdot (x^* - y)}{|x^* - y|^{n-\gamma+2}}\right]$$

$$\times \frac{u^p(y)v^q(x)}{|y|^\alpha|x|^\beta}dx\,dy$$

$$+ (-\beta)\int_{R_+^n}v^{q+1}(x)dx$$

$$= \frac{\gamma - n}{2} \int \int_{R_+^n} \left[\frac{x \cdot (x - y)}{|x - y|^{n-\gamma+2}} - \frac{x^* \cdot (x^* - y)}{|x^* - y|^{n-\gamma+2}} \right]$$
$$\times \frac{u^p(y) v^q(x)}{|y|^\alpha |x|^\beta}$$
$$+ \frac{\gamma - n}{2} \int \int_{R_+^n} \left[\frac{y \cdot (y - x)}{|x - y|^{n-\gamma+2}} - \frac{y^* \cdot (y^* - x)}{|y^* - x|^{n-\gamma+2}} \right]$$
$$\times \frac{u^p(x) v^q(y)}{|x|^\alpha |y|^\beta} dx \, dy$$
$$+ (-\beta) \int_{R_+^n} v^{q+1}(x) \, dx.$$
$$(20)$$

Using

$$x \cdot (x - y) + y \cdot (y - x) = |x - y|^2,$$
$$x^* \cdot (x^* - y) + y^* \cdot (y^* - x) = |x^* - y|^2.$$
$$(21)$$

Combining the fact $|x^* - y| = |y^* - x|$, (19), and (20), we have

$$\int_{R_+^n} u^p(x)(x \cdot \nabla u(x)) \, dx + \int_{R_+^n} v^q(x)(x \cdot \nabla v(x)) \, dx$$
$$= \left(\frac{\gamma - n}{2} - \alpha \right) \int_{R_+^n} u^{p+1}(x) \, dx$$
$$+ \left(\frac{\gamma - n}{2} - \beta \right) \int_{R_+^n} v^{q+1}(x) \, dx.$$
$$(22)$$

By (18) and (22), we have

$$\left(\frac{\gamma - n}{2} - \alpha + \frac{n}{p+1} \right) \int_{R_+^n} u^{p+1}(x) \, dx$$
$$+ \left(\frac{\gamma - n}{2} - \beta + \frac{n}{q+1} \right) \int_{R_+^n} v^{q+1}(x) \, dx = 0.$$
$$(23)$$

Hence, if

$$\frac{\gamma - n}{2} - \alpha + \frac{n}{p+1} > 0, \qquad \frac{\gamma - n}{2} - \beta + \frac{n}{q+1} > 0 \quad (24)$$

or

$$\frac{\gamma - n}{2} - \alpha + \frac{n}{p+1} < 0, \qquad \frac{\gamma - n}{2} - \beta + \frac{n}{q+1} < 0, \quad (25)$$

hold, it follows that $u \equiv 0$ and $v \equiv 0$.

This completes the proof of Theorem 2.

Remark 5. In [11], the authors consider another weighted HLS type integral system

$$u(x) = \int_{R_+^n} G(x, y, \gamma) |y|^{-s} v^q(y) \, dy,$$
$$v(x) = \int_{R_+^n} G(x, y, \gamma) |y|^{-t} u^p(y) \, dy,$$
$$(26)$$
$$\forall x \in R_+^n$$

and showed the Liouville type theorem as follows.

Theorem 6 (see [11]). *Suppose that* $u(x), v(x) \in C^1(R^n)$ *are positive solutions of* (26) *when p and q are both subcritical; that is* $1/(p+1) > (n-\gamma)/2(n-t)$ *and* $1/(q+1) > (n-\gamma)/2(n-s)$. *If* $\int_{R_+^n} (u^{p+1}/|x|^t) dx < \infty$, $\int_{R_+^n} (v^{q+1}/|x|^s) dx < \infty$ *and* $\gamma - s > 1$, $\gamma - t > 1$, *then* $u \equiv 0$ *and* $v \equiv 0$.

When $s = t = 0$ in system (26) or $\alpha = \beta = 0$ in system (3), the two systems reduce to the simple integral system (7). In this special case, we can find that Theorem 6 is coincident with case (ii) in Theorem 2.

3. Proof of Theorem 3

In this section, we will consider rotational symmetry of weighted HLS type system (3) in critical case (5).

Firstly, we need the following weighted HLS inequality.

Lemma 7 (see [12]). *Let* $1 < l, m < \infty$, $0 < \gamma < n$, $\tau + \beta \geq 0$, $1/l + 1/m + (\gamma + \tau + \beta)/n = 2$, *and* $1 - 1/m - \gamma/n < \tau/n < 1 - 1/m$. *Then*

$$\left| \int \int_{R^n} \frac{f(x) g(y)}{|x|^\tau |x - y|^\gamma |y|^\beta} dx \, dy \right| \leq C \|f\|_m \|g\|_l. \quad (27)$$

One can also write the weighted HLS inequality in another form. Let

$$Tg(x) = \int_{R^n} \frac{g(y)}{|x|^\tau |x - y|^\gamma |y|^\beta} dy. \quad (28)$$

Then

$$\|Tg(x)\|_{L^\mu} = \operatorname*{Sup}_{\|f\|_m = 1} \; < Tg(x), \qquad f(x) > \leq \|g\|_{L^l}, \quad (29)$$

where $1/l + (\gamma + \tau + \beta)/n = 1 + 1/\mu$, $1/\mu + 1/m = 1$.

For a given real number λ, define

$$\Sigma_\lambda = \{x = (x_1, x_2, \ldots, x_n) \in R_+^n \mid x_1 < \lambda\},$$
$$T_\lambda = \{x \in R_+^n \mid x_1 = \lambda\}. \quad (30)$$

Let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_{n-1}, x_n)$ be the reflection of the point $x = (x_1, x_2, \ldots, x_n)$ about the plane T_λ. Set

$$u_\lambda(x) = u(x^\lambda), \qquad v_\lambda(x) = v(x^\lambda). \quad (31)$$

Lemma 8 (see [8, 13]). *For* $x, y \in \Sigma_\lambda$, $x \neq y$, *one has*

$$G(x, y, \gamma) \geq G(x^\lambda, y, \gamma). \quad (32)$$

Lemma 9. *Let (u, v) be any pair of positive solutions of* (3) *in critical case* (5); *for any $x \in \Sigma_\lambda$ and $|x| > |x^\lambda|$, one has*

$$u(x) - u_\lambda(x)$$

$$\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma) \right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy,$$

$$v(x) - v_\lambda(x)$$

$$\leq \frac{1}{|x|^\beta} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma) \right] \frac{u^p(y) - u_\lambda^p(y)}{|y|^\alpha} dy.$$

(33)

Proof. Through the calculation, we have

$$u(x) = \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} G(x, y, \gamma) \frac{v^q(y)}{|y|^\beta} dy$$

$$+ \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} G(x^\lambda, y, \gamma) \frac{v_\lambda^q(y)}{|y^\lambda|^\beta} dy,$$

$$u_\lambda(x) = \frac{1}{|x^\lambda|^\alpha} \int_{\Sigma_\lambda} G(x^\lambda, y, \gamma) \frac{v^q(y)}{|y|^\beta} dy$$

$$+ \frac{1}{|x^\lambda|^\alpha} \int_{\Sigma_\lambda} G(x, y, \gamma) \frac{u_\lambda^p(y)}{|y^\lambda|^\beta} dy.$$

(34)

By the assumption $|x| > |x^\lambda|$, we have

$$u(x) - u_\lambda(x)$$

$$\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma) \right]$$

$$\times \left(\frac{v^q(y)}{|y|^\beta} - \frac{v_\lambda^q(y)}{|y^\lambda|^\beta} \right) dy$$

$$\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma) \right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy.$$

(35)

Similarly, we have

$$v(x) - v_\lambda(x)$$

$$\leq \frac{1}{|x|^\beta} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma) \right]$$

$$\times \frac{u^p(y) - u_\lambda^p(y)}{|y|^\alpha} dy.$$

(36)

□

Proof of Theorem 3. Step 1. We will show that for sufficiently negative λ,

$$u_\lambda(x) \geq u(x), \quad v_\lambda(x) \geq v(x), \quad \text{a.e. } \forall x \in \Sigma_\lambda. \quad (37)$$

Define

$$\Sigma_\lambda^u = \{ x \in \Sigma_\lambda, u(x) > u_\lambda(x) \},$$

$$\Sigma_\lambda^v = \{ x \in \Sigma_\lambda, v(x) > v_\lambda(x) \}.$$

(38)

We prove that, for sufficiently negative λ, both Σ_λ^u and Σ_λ^v must be empty and thus (37) holds.

In fact, by Lemma 9 and the mean value theorem, we have, for $x \in \Sigma_\lambda^u$,

$$0 < u(x) - u_\lambda(x)$$

$$\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma) \right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy$$

$$\leq \int_{\Sigma_\lambda^v} \frac{1}{|x|^\alpha} \left[G(x, y, \gamma) - G(x^\lambda, y, \gamma) \right] \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy$$

$$\leq \frac{1}{|x|^\alpha} \int_{\Sigma_\lambda^v} G(x, y, \gamma) \frac{v^q(y) - v_\lambda^q(y)}{|y|^\beta} dy$$

$$\leq \frac{q}{|x|^\alpha} \int_{\Sigma_\lambda^v} \frac{1}{|x - y|^{n-\gamma}} \psi_\lambda^{q-1}(y) \frac{v(y) - v_\lambda(y)}{|y|^\beta} dy$$

$$\leq q \int_{\Sigma_\lambda^v} \frac{1}{|x|^\alpha |x - y|^{n-\gamma} |y|^\beta} v^{q-1}(y) [v(y) - v_\lambda(y)] dy,$$

(39)

where $\psi_\lambda(y)$ is valued between $v(y)$ and $v_\lambda(y)$; therefore, on Σ_λ^v, we have

$$0 \leq v_\lambda(y) \leq \psi_\lambda(y) \leq v(y). \quad (40)$$

By Lemma 7 and the Hölder inequality, we have

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \leq C \|v^{q-1}(v_\lambda - v)\|_{L^{T_1}(\Sigma_\lambda^v)}$$

$$\leq C \|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)},$$

(41)

$$\|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)} \leq C \|u^{p-1}(u_\lambda - u)\|_{L^{T_2}(\Sigma_\lambda^u)}$$

$$\leq C \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)},$$

(42)

where $T_1 = n(p+1)/(n + (\gamma - \alpha - \beta)(p+1))$ and $T_2 = n(q+1)/(n + (\gamma - \alpha - \beta)(q+1))$. It easy to show that $T_1, T_2 > 1$. Combining (41) and (42), we arrive

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}$$

$$\leq C \|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}.$$

(43)

The conditions $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$ make us able to choose sufficiently negative λ, so that

$$C \|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \leq \frac{1}{2}. \quad (44)$$

Now inequality (43) implies

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} = 0, \qquad (45)$$

and therefore Σ_λ^u must be measure zero. Similarly, one can show that Σ_λ^v is measure zero. Therefore (37) holds.

Step 2. Inequality (37) provides a starting point to move the plane $T_\lambda = \{x \in R_+^n \mid x_1 = \lambda\}$. Now we start from the neighborhood of $x_1 = -\infty$ and move the plane to the right as long as (37) holds to the limiting position. More precisely, define

$$\lambda_0 = \sup \ \{\lambda \mid u(x) \le u_\mu(x),$$
$$v(x) \le v_\mu(x), \mu \le \lambda, \forall x \in \Sigma_\mu\}. \qquad (46)$$

We will prove that $\lambda_0 = 0$. On the contrary, we suppose $\lambda_0 < 0$. We show that $u(x)$ and $v(x)$ are symmetric about the plane T_{λ_0}; that is

$$u_{\lambda_0}(x) \equiv u(x), \quad v_{\lambda_0}(x) \equiv v(x), \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0}. \quad (47)$$

Otherwise, on Σ_{λ_0},

$$u(x) \le u_{\lambda_0}(x), \qquad v(x) \le v_{\lambda_0}(x),$$
$$\text{but } u(x) \not\equiv u_{\lambda_0}(x) \quad \text{or} \quad v(x) \not\equiv v_{\lambda_0}(x). \qquad (48)$$

We show that the plane can be moved further to the right. More precisely, there exists an $\epsilon > 0$ such that, for $\forall \lambda \in [\lambda_o, \lambda_o + \epsilon)$,

$$u(x) \le u_\lambda(x), \quad v(x) \le v_\lambda(x), \quad \text{a.e. } \forall x \in \Sigma_\lambda. \qquad (49)$$

Without loss of generality, we assume

$$v(x) \not\equiv v_{\lambda_0}(x), \quad \text{on } \Sigma_{\lambda_0}. \qquad (50)$$

by Lemma 9, we have in fact $u(x) < u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Let

$$\overline{\Sigma_{\lambda_0}^u} = \left\{x \in \Sigma_{\lambda_0} \mid u(x) \ge u_{\lambda_0}(x)\right\},$$
$$\overline{\Sigma_{\lambda_0}^v} = \left\{x \in \Sigma_{\lambda_0} \mid v(x) \ge v_{\lambda_0}(x)\right\}. \qquad (51)$$

Then obviously $\overline{\Sigma_{\lambda_0}^u}$ has measure zero and $\lim_{\lambda \to \lambda_0} \Sigma_\lambda^u \subset \overline{\Sigma_{\lambda_0}^u}$. The same argument above is also true for the other solution v of (3). From (41) and (42), we deduce

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}$$
$$\le C\|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \qquad (52)$$

Again the conditions that $u \in L^{p+1}(R_+^n)$ and $v \in L^{q+1}(R_+^n)$ ensure that one can choose ϵ sufficiently small, so that, for all λ in $[\lambda_o, \lambda_o + \epsilon)$,

$$C\|v\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \le \frac{1}{2}. \qquad (53)$$

The method to verify this inequality is standard and the proofs of the rest are similar to the proof in paper [6, 11, 14].

Now by (52) and (53), we have $\|u_\lambda - u\|_{L^p(\Sigma_\lambda^u)} = 0$, and therefore Σ_λ^u must be measure zero. Similarly, Σ_λ^v must also be measure zero. Hence, for these values of $\lambda > \lambda_o$, we have

$$u_\lambda(x) \ge u(x), \quad v_\lambda(x) \ge v(x), \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0}. \qquad (54)$$

This (47) must hold and therefore both $u(x)$ and $v(x)$ are symmetric about the plane T_{λ_0}.

Now we show that the plane cannot stop before hitting the origin. Otherwise, assume that the plane stops at $x_1 = \lambda_0 < 0$. By the fact that $|y| > |y^{\lambda_0}|$, we have

$$u(x) - u_{\lambda_0}(x)$$
$$\le \frac{1}{|x|^\alpha} \int_{\Sigma_{\lambda_0}} \left[G(x, y, \gamma) - G(x^{\lambda_0}, y, \gamma)\right]$$
$$\times \left[\frac{v^q(y)}{|y|^\beta} - \frac{v_{\lambda_0}^q(y)}{|y^{\lambda_0}|^\beta}\right] dy$$
$$< \frac{1}{|x|^\alpha} \int_{\Sigma_{\lambda_0}} \left[G(x, y, \gamma) - G(x^{\lambda_0}, y, \gamma)\right]$$
$$\times \left[\frac{v^q(y) - v_{\lambda_0}^q(y)}{|y|^\beta}\right] dy = 0. \qquad (55)$$

This contradicts with (47).

As the direction of x_1 can be chosen arbitrarily, we derive that $(u(x), v(x))$ is rotationally symmetric about x_n-axis. This completes the proof of Theorem 3. $\qquad \square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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