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*Research Article*

# Existence and Uniqueness of Positive Periodic Solutions for a Delayed Predator-Prey Model with Dispersion and Impulses

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An impulsive Lotka-Volterra type predator-prey model with prey dispersal in two-patch environments and time delays is investigated, where we assume the model of patches with a barrier only as far as the prey population is concerned, whereas the predator population has no barriers between patches. By applying the continuation theorem of coincidence degree theory and by means of a suitable Lyapunov functional, a set of easily verifiable sufficient conditions are obtained to guarantee the existence, uniqueness, and global stability of positive periodic solutions of the system. Some known results subject to the underlying systems without impulses are improved and generalized. As an application, we also give two examples to illustrate the feasibility of our main results.

## 1. Introduction

The aim of this paper is to investigate the existence and uniqueness of the positive periodic solution of the following impulsive Lotka-Volterra type predator-prey model with prey dispersal in two-patch environments and time delays:

$$\begin{aligned} x_1'(t) &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t) - a_{13}(t)x_3(t)\right] \\ &\quad + D_1(t)\left(x_2(t) - x_1(t)\right), \\ x_2'(t) &= x_2(t)\left[r_2(t) - a_{22}(t)x_2(t) - a_{23}(t)x_3(t)\right] \\ &\quad + D_2(t)\left(x_1(t) - x_2(t)\right), \\ x_3'(t) &= x_3(t)\left[-r_3(t) + a_{31}(t)x_1\left(t-\tau_1(t)\right) + a_{32}(t)x_2\right. \\ &\quad \left.\times\left(t-\tau_1(t)\right) - a_{33}(t)x_3\left(t-\tau_2(t)\right)\right], \\ &\qquad t \neq t_k, \\ \Delta x_i(t) &= x_i\left(t^+\right) - x_i(t) = c_{ik}x_i(t), \\ &\quad i = 1,2,3, \quad k = 1,2,\ldots, \quad t = t_k, \end{aligned} \tag{1}$$

with the following initial conditions:

$$\begin{gathered} x_i(\theta) = \psi_i(\theta), \\ \theta \in [-\tau, 0], \quad \psi_i(0) > 0, \quad \psi_i \in C\left([-\tau, 0), R^+\right), \\ \tau = \max_{t\in[0,\omega]}\left\{\tau_1(t), \tau_2(t)\right\}, \quad i = 1,2,3, \end{gathered} \tag{2}$$

where $x_i(t)$ represents the prey population in the ith patch ($i = 1, 2$) and $x_3(t)$ represents the predator population for both patches. $r_i(t)$ is the intrinsic growth rate of the prey in the ith patch ($i = 1, 2$) and $a_{ii}(t)$ ($i = 1, 2$) are the density-dependent coefficients of the prey at the ith patch. $a_{13}(t)$ and $a_{23}(t)$ are the capturing rates of the predator in patches 1 and 2, respectively, and $a_{31}(t)/a_{13}(t)$ and $a_{32}(t)/a_{23}(t)$ are the conversion rates of nutrients into the reproduction of the predator. $r_3(t)$ is the death rate of the predator and $D_i(t)$ denotes the dispersal rate of the prey in the ith patch ($i = 1, 2$). $\tau_1(t)$ is the delay due to gestation; that is, mature adult predators can only contribute to the production of predator biomass. In addition, we have included the term $a_{33}(t)x_3(t - \tau_2(t))$ in the dynamics of the predator to incorporate the negative feedback of predator crowding, where $c_{ik}x_i(t_k)$ $(i = 1, 2, 3)$ represent the population $x_i(t)$ at $t_k$ regular harvest pulse.

As was pointed out by Xu and Chen [1], dispersal between patches often occurs in ecological environments, and more realistic models should include the dispersal process. During the last decade, many scholars had done excellent works on the predator-prey system with dispersal; see [2–16] and the references cited therein. In [5], Cui proposed the following two species predator-prey system with prey dispersal:

$$\begin{aligned} x_1'(t) &= x_1(a_1(t) - b_1(t)x_1 - c(t)y) + D(t)(x_2 - x_1), \\ x_2'(t) &= x_2(a_2(t) - b_2(t)x_2) + D(t)(x_1 - x_2), \\ y'(t) &= y(t)(-d(t) + e(t)x_1 - q(t)y - \delta(t)y(t-\tau)), \end{aligned} \tag{3}$$

where $x_1(t)$ and $y(t)$ represent the population density of prey species $x$ and predator species $y$ in patch 1 and $x_2(t)$ is the density of prey species $x$ in patch 2. Predator species $y$ is confined to patch 1, while the prey species $x$ can diffuse between two patches. $D(t)$ is strictly positive functions that can be viewed as the dispersal rate or inverse barrier strength. By giving a thoroughly analysis on the right hand side of the system (3), Cui obtained a sufficient and necessary condition to guarantee the predator and prey species to be permanent.

It is unlike system (3), where the predator species is confined on patch 1. In [10], the authors proposed a model of patches with a barrier only as far as the prey population is concerned, whereas the predator population has no barriers between patches; that is, they considered the following predator-prey system in two-patch environment:

$$\begin{aligned} x_1' &= x_1 g_1(x_1) - y p_1(x_1) + \varepsilon(x_2 - x_1), \\ x_2' &= x_2 g_2(x_2) - y p_2(x_2) + \varepsilon(x_1 - x_2), \\ y' &= y(t)[-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2)], \\ x_i(0) &> 0, \quad y(0) > 0, \quad i = 1, 2, \end{aligned} \tag{4}$$

where $x_i(t)$ represents the prey population in the ith patch, $i = 1, 2$, at time $t \geq 0$. $y(t)$ stands for the total predator population for both patches. The predator population is assumed to have no barriers between patches. $g_i(x_i)$ is the specific growth rate for the prey population in the absence of predation when it is restricted to the ith patch. $p_i(x_i)$ is the predator functional response of the predator population on the prey in the ith patch. $\varepsilon$ is a positive constant that can be viewed as the dispersal rate or inverse barrier strength. $s(y)$ is the density-dependent death rate of the predator in the absence of prey. $c_i > 0$ is the conversion ratio of prey into predator. Conditions have been established in [10] for the existence, uniform persistence, and local and global stability of positive steady states of system (4).

The model (4), however, as was pointed out by Yang [11], is not perfect. Therefore, Xu et al. [12] had considered the following delayed periodic Lotka-Volterra type predator-prey system with prey dispersal in two-patch environments:

$$\begin{aligned} x_1'(t) &= x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{13}(t)y(t)] \\ &\quad + D_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) &= x_2(t)[r_2(t) - a_{22}(t)x_2(t) - a_{23}(t)y(t)] \\ &\quad + D_2(t)(x_1(t) - x_2(t)), \\ y'(t) &= y(t)[-r_3(t) + a_{31}(t)x_1(t-\tau_1) \\ &\quad + a_{32}(t)x_2(t-\tau_1) - a_{33}(t)y(t-\tau_2)], \\ x_i(0) &> 0, \quad y(0) > 0, \quad i = 1, 2, \end{aligned} \tag{5}$$

with initial conditions:

$$\begin{gathered} x_i(\theta) = \phi_i(\theta), \qquad y(\theta) = \psi(\theta), \\ \theta \in [-\tau, 0], \quad \phi_i(0) > 0, \quad \psi(0) > 0, \\ \phi_i, \psi \in C([-\tau, 0), R^+), \\ \tau = \max\{\tau_1, \tau_2\}, \quad i = 1, 2, \end{gathered} \tag{6}$$

by using Gaines and Mawhins continuation theorem of coincidence degree theory and by means of a suitable Lyapunov functional, they obtained a set of easily verifiable sufficient conditions to guarantee the existence, uniqueness, and global stability of positive periodic solutions of the system (5).

On the other hand, impulsive differential equations [17–19] arise frequently in the modeling of many physical systems whose states are subjects to sudden change at certain moments, for example, in population biology, the diffusion of chemicals, the spread of heat, the radiation of electromagnetic waves, the maintenance of a species through instantaneous stocking, and harvesting. There has been an increasing interest in the investigation for such equations during the past few years. There are many researchers who introduced impulsive differential equations in population dynamics [20–28]. However, to the best of the authors' knowledge, to this day, no scholars had done works on the existence, uniqueness, and global stability of positive periodic solution of (1). Based on the idea of [10–15], we propose and study the system (1) in this paper.

For the sake of generality and convenience, we always make the following fundamental assumptions:

$(A_1)$ $a_{ij}(t), r_i(t)$ $(i, j = 1, 2, 3)$, $D_1(t), \tau_1(t), \tau_2(t)$, and $D_2(t)$ are all positive periodic continuous functions with period $\omega > 0$, and $\tau_i'(t) < 1$ $(i = 1, 2)$;

$(A_2)$ $\{t_k\}_{k \in N}$ satisfies $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k \to \infty} t_k = +\infty$, $c_{ik}$ $(i = 1, 2)$ are constants with $1 + c_{ik} > 0$ and there exists a positive integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $c_{i(k+q)} = c_{ik}$. Without loss of generality, we can assume that $t_k \neq 0$ and $[0, \omega] \cap \{t_k\} = t_1, t_2, \ldots, t_m$, then $q = m$.

In what follows, we will use the notation.

Throughout this paper, we make the following notation and assumptions.

Let $\omega > 0$ be a constant and

$C_\omega = \{x \mid x \in C(R,R),\ x(t+\omega) = x(t)\}$, with the norm defined by $|x|_0 = \max_{t\in[0,\omega]}|x(t)|$;

$C_\omega^1 = \{x \mid x \in C^1(R,R),\ x(t+\omega) = x(t)\}$, with the norm defined by $\|x\| = \max_{t\in[0,\omega]}\{|x|_0, |x'|_0\}$;

$PC = \{x \mid x : R \to R^+,\ \lim_{s\to t} x(s) = x(t)$, if $t \neq t_k$, $\lim_{t\to t_k^-} x(t) = x(t_k)$, $\lim_{t\to t_k^+} x(t)$ exists, $k \in Z^+\}$;

$PC^1 = \{x \mid x : R \to R^+,\ x' \in PC\}$;

$PC_\omega = \{x \mid x \in PC,\ x(t+\omega) = x(t)\}$, with the norm defined by $|x|_0 = \max_{t\in[0,\omega]}|x(t)|$;

$PC_\omega^1 = \{x \mid x \in PC^1,\ x(t+\omega) = x(t)\}$, with the norm defined by $\|x\| = \max_{t\in[0,\omega]}\{|x|_0, |x'|_0\}$.

Then those spaces are all Banach spaces. We also denote

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)\,dt, \qquad f^L = \min_{t\in[0,\omega]} f(t),$$
$$f^M = \max_{t\in[0,\omega]} f(t), \qquad c_j = \sum_{k=1}^{m} \ln\left(1 + c_{jk}\right), \quad j = 1,2,3. \tag{7}$$

The aim of this paper is to obtain a set of easily verifiable sufficient conditions to guarantee the existence, uniqueness, and global stability of positive periodic solutions of the system (1) by further developing the analysis technique of [10–15]. The organization of this paper is as follows. In the next section, first, the necessary knowledge and lemmas are provided. Second, by using continuation theorem developed by Gaines and Mawhin [29], we establish the existence of at least one periodic solution of system (1). In Section 3, the uniqueness and global attractivity of periodic solution of system (1) are presented. Finally, we give two examples to show our results.

## 2. Existence of Positive Periodic Solutions

In this section, by using the continuation theorem which was proposed in [29] by Gaines and Mawhin, we will establish the existence conditions of at least one positive periodic solution to system (1). In doing so, we will introduce the following definitions and lemmas.

Let $X$, $Z$ be a real Banach space, let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and let $N : X \to Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, it follows that $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P} : (I-P)X \to \operatorname{Im} L$ is invertible; we denote the inverse of that map by $K_p$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Let $PC_\omega$ denote the space of $\omega$-periodic functions $\Psi : J \to R$ which are continuous for $t \neq t_k$, are continuous from the left for $t \in R$, and have discontinuities of the first kind at point $t = t_k$. We also denote $PC_\omega^1 = \{\Psi \in PC_\omega : \Psi' \in PC_\omega\}$.

*Definition 1* (see [18]). The set $F \in PC_\omega$ is said to be quasiequicontinuous in $[0,\omega]$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in F$, $k \in N^+$, $t_1, t_2 \in (t_{k-1}, t_k) \cap [0,\omega]$, and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.

**Lemma 2** (Gaines and Mawhin [29]). *Let $X$ and $Z$ be two Banach spaces and let $L : \operatorname{Dom} L \subset X \to Z$ be a Fredholm operator with index zero. $\Omega \subset X$ is an open bounded set, and $N : \overline{\Omega} \to Z$ is $L$-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*

(b) $QNx \neq 0$ *for each $x \in \partial\Omega \cap \ker L$;*

(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

*Then, the equation $Lx = Nx$ has at least one solution lying in* $\operatorname{Dom} L \cap \overline{\Omega}$.

**Lemma 3** (see [30]). *Assume that $f(t)$, $g(t)$ are continuous nonnegative functions defined on the interval $[\alpha,\beta]$. Then there exists $\xi \in [\alpha,\beta]$ such that $\int_\alpha^\beta f(t)g(t)dt = f(\xi)\int_\alpha^\beta g(t)dt$.*

**Lemma 4** (see [20, 27, 28]). *Assume that $\Phi \in PC_\omega^1$, $[0,\omega] \cap \{t_k\} = t_1, t_2, \ldots, t_q$. Then the following inequality holds:*

$$\sup_{s\in[0,\omega]} \Phi(s) - \inf_{s\in[0,\omega]} \Phi(s) \leq \frac{1}{2}\left[\int_0^\omega \left|\Phi'(s)\right|\big)\,ds + \sum_{k=1}^{q}\left|\Delta\Phi(t_k)\right|\right]. \tag{8}$$

**Lemma 5.** *The region $R_+^3 = \{(x_1,x_2,x_3) : x_1(0) > 0,\ x_2(0) > 0,\ x_3(0) > 0\}$ is the positive invariable region of the system* (1).

*Proof.* In view of biological population, we obtain $x_1 > 0$, $x_2 > 0$, $x_3 > 0$. By the system (1), we have

$$x_1(t) = x_1(0)\, e^{\int_0^t [r_1(s) - a_{11}(s)x_1(s) - a_{13}(s)x_3(s) + D_1(s)(x_2(s)/x_1(s) - 1)]ds}, \quad t \in [0, t_1],$$

$$\begin{aligned}
&x_1(t) \\
&= x_1(t_k)\, e^{\int_0^t [r_1(s)-a_{11}(s)x_1(s)-a_{13}(s)x_3(s)+D_1(s)(x_2(s)/x_1(s)-1)]ds}, \\
&\qquad t \in (t_k, t_{k+1}], \\
&x_1(t_k^+) = (1+c_{1k})\, x_1(t_k) > 0, \quad k \in N; \\
&x_2(t) \\
&= x_2(0)\, e^{\int_0^t [r_2(s)-a_{22}(s)x_2(s)-a_{23}(s)x_3(s)+D_2(s)(x_1(s)/x_2(s)-1)]ds}, \\
&\qquad t \in [0, t_1], \\
&x_2(t) \\
&= x_2(t_k)\, e^{\int_0^t [r_2(s)-a_{22}(s)x_2(s)-a_{23}(s)x_3(s)+D_2(s)(x_1(s)/x_2(s)-1)]ds}, \\
&\qquad t \in (t_k, t_{k+1}], \\
&x_2(t_k^+) = (1+c_{2k})\, x_2(t_k) > 0, \quad k \in N; \\
&x_3(t) \\
&= x_3(0) \\
&\quad \times e^{\int_0^t [-r_3(s)+a_{31}(s)x_1(s-\tau_1(t))+a_{32}(s)x_2(s-\tau_1(t))-a_{33}(s)x_3(s-\tau_2(t))]ds}, \\
&\qquad t \in [0, t_1], \\
&x_3(t) \\
&= x_3(t_k) \\
&\quad \times e^{\int_0^t [-r_3(s)+a_{31}(s)x_1(s-\tau_1(t))+a_{32}(s)x_2(s-\tau_1(t))-a_{33}(s)x_3(s-\tau_2(t))]ds}, \\
&\qquad t \in (t_k, t_{k+1}], \\
&x_3(t_k^+) = (1+c_{3k})\, x_3(t_k) > 0, \quad k \in N.
\end{aligned} \tag{9}$$

Therefore, the conclusion is true. □

**Lemma 6** (see [27, 28, 31]). *Suppose $\sigma \in C^1_\omega$ and $\sigma'(t) < 1$, $t \in [0,\omega]$. Then the function $t-\sigma(t)$ has a unique inverse $\mu(t)$ satisfying $\mu \in C(R,R)$ with $\mu(a+\omega) = \mu(a)+\omega$, $\forall a \in R$. If $g \in PC_\omega$, $\tau'(t) < 1$, $t \in [0,\omega]$; then $g(\mu(t)) \in PC_\omega$.*

*Proof.* Since $\sigma'(t) < 1$, $t \in [0,\omega]$ and $t-\sigma(t)$ is continuous on $R$, it follows that $t-\sigma(t)$ has a unique inverse function $\mu(t) \in C(R,R)$ on $R$. Hence, it suffices to show that $\mu(a+\omega) = \mu(a)+\omega$, $\forall a \in R$. For any $a \in R$, by the condition $\sigma'(t) < 1$, one can find that, for the equation $t-\sigma(t) = a$, exists a unique solution $t_0$ and, for the equation $t-\sigma(t) = a+\omega$, exists a unique solution $t_1$; that is $t_0 - \sigma(t_0) = a$ and $t_1 - \sigma(t_1) = a+\omega$, that is, $\mu(a) = t_0 = \sigma(t_0)+a$ and $\mu(a+\omega) = t_1$. As

$$\begin{aligned}
a+\omega+\sigma(t_0) - \sigma(a+\omega+\sigma(t_0)) &= a+\omega+\sigma(t_0) \\
&\quad - \sigma(a+\sigma(t_0)) \\
&= a+\omega+\sigma(t_0)-\sigma(t_0) \\
&= a+\omega,
\end{aligned} \tag{10}$$

it follows that $t_1 = a+\omega+\sigma(t_0)$. Since $\mu(a+\omega) = t_1$, thus, we have $\mu(a+\omega) = t_1 = a+\omega+\sigma(t_0)$ and $\mu(a+\omega) = t_1 = \mu(a)+\omega$. We can easily obtain that if $g \in PC_\omega$, $\tau'(t) < 1$, $t \in [0,\omega]$, then $g(\mu(t+\omega)) = g(\mu(t)+\omega) = g(\mu(t))$, $t \in R$, where $\mu(t)$ is the unique inverse function of $t-\tau(t)$, which together with $\mu \in C(R,R)$ implies that $g(\mu(t)) \in PC_\omega$. The proof of Lemma 6 is completed. □

We denote by $\mu_i(t)$ the inverse of $t-\tau_i(t)$, $i = 1, 2$.

**Theorem 7.** *In addition to $(A_1)$-$(A_2)$, assume the following conditions hold:*

$(A_3)$ $\overline{(r_1 - D_1)} - a_{13}^M (B_1^M + B_2^M) A / \overline{a_{33}} + c_1/\omega > 0$,

$(A_4)$ $a_{22}^M B_1^L (\overline{(r_1 - D_1)}\omega + c_1) + a_{11}^M B_2^L (\overline{(r_2 - D_2)}\omega + c_2) - a_{11}^M a_{22}^M (\overline{r_3}\omega + c_3) > 0$.

*Then, system* (1) *has at least one positive $\omega$-periodic solution, where*

$$\begin{aligned}
A &= \max\left\{ \frac{(r_1 - D_1)^M + D_1^M}{a_{11}^L}, \frac{(r_2 - D_2)^M + D_2^M}{a_{22}^L} \right\}, \\
B_1(t) &= \frac{a_{31}(\mu_1(t))}{1-\tau_1'(\mu_1(t))}, \qquad B_2(t) = \frac{a_{32}(\mu_1(t))}{1-\tau_1'(\mu_1(t))}.
\end{aligned} \tag{11}$$

*Proof.* We carry out the change of variable $u_i(t) = \ln x_i(t)$, $i = 1, 2, 3$; then (1) can be transformed to

$$\begin{aligned}
u_1'(t) &= r_1(t) - D_1(t) - a_{11}(t)\, e^{u_1(t)} \\
&\quad - a_{13}(t)\, e^{u_3(t)} + D_1(t)\, e^{u_2(t)-u_1(t)}, \\
u_2'(t) &= r_2(t) - D_2(t) - a_{22}(t)\, e^{u_2(t)} \\
&\quad - a_{23}(t)\, e^{u_3(t)} + D_2(t)\, e^{u_1(t)-u_2(t)}, \\
u_3'(t) &= -r_3(t) + a_{31}(t)\, e^{u_1(t-\tau_1(t))} \\
&\quad + a_{32}(t)\, e^{u_2(t-\tau_1(t))} \\
&\quad - a_{33}(t)\, e^{u_3(t-\tau_2(t))}, \quad t \ne t_k, \\
\Delta u_1(t) &= \ln(1+c_{1k}), \\
\Delta u_2(t) &= \ln(1+c_{2k}), \\
\Delta u_3(t) &= \ln(1+c_{3k}), \quad t = t_k.
\end{aligned} \tag{12}$$

It is easy to see that if system (12) has one $\omega$-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^T$, then $(x_1^*(t), x_2^*(t), x_3^*(t))^T = (e^{u_1^*(t)}, e^{u_2^*(t)}, e^{u_3^*(t)})^T$ is a positive $\omega$-periodic solution of system (1). Therefore, it suffices to prove system (12) has a $\omega$-periodic solution. Let

$$X = \{x = (u_1, u_2, u_3)^T \mid u_i \in PC_\omega,\ i = 1, 2, 3\},\quad Z = X \times R^{3m}, \tag{13}$$

and define

$$\|u\|_X = \sum_{i=1}^{3} \sup_{t\in[0,\omega]} |u_i(t)|, \quad u = (u_1, u_2, u_3)^T \in X,\quad \|z\|_Z = \|u\|_X + \|v\|, \quad (u, v) \in Z, \tag{14}$$

where $\|\cdot\|$ is the Euclidean norm of $R^{3m}$. Then $X$ and $Z$ are Banach spaces.

Let

$$\operatorname{Dom} L = \{u = (u_1, u_2, u_3)^T \mid u_i \in PC_\omega^1,\ i = 1, 2, 3\},$$

$$L : \operatorname{Dom} L \subset X \longrightarrow Z, \quad Lu = (u', \Delta u(t_1), \ldots, \Delta u(t_m)), \tag{15}$$

and $N : X \to Z$ with

$$Nu = \left( \begin{bmatrix} r_1(t) - D_1(t) - a_{11}(t) e^{u_1(t)} - a_{13}(t) e^{u_3(t)} + D_1(t) e^{u_2(t)-u_1(t)} \\ r_2(t) - D_2(t) - a_{22}(t) e^{u_2(t)} - a_{23}(t) e^{u_3(t)} + D_2(t) e^{u_1(t)-u_2(t)} \\ -r_3(t) + a_{31}(t) e^{u_1(t-\tau_1(t))} + a_{32}(t) e^{u_2(t-\tau_1(t))} - a_{33}(t) e^{u_3(t-\tau_2(t))} \end{bmatrix}, \begin{bmatrix} \ln(1+c_{11}) \\ \ln(1+c_{21}) \\ \ln(1+c_{31}) \end{bmatrix}, \begin{bmatrix} \ln(1+c_{12}) \\ \ln(1+c_{22}) \\ \ln(1+c_{32}) \end{bmatrix}, \ldots, \begin{bmatrix} \ln(1+c_{1m}) \\ \ln(1+c_{2m}) \\ \ln(1+c_{3m}) \end{bmatrix} \right), \quad u \in X. \tag{16}$$

It is not difficult to show that

$$\operatorname{Ker} L = \{u \in X \mid u = \psi \in R^3\},\quad \operatorname{Im} L = \left\{ z = (\psi, c_1, \ldots, c_m) \in Z \mid \int_0^\omega \psi(s)\,ds + \sum_{k=1}^{m} c_k = 0 \right\}, \tag{17}$$

and $\dim\operatorname{Ker} L = 3 = \operatorname{codim}\operatorname{Im} L$. So, $\operatorname{Im} L$ is closed in $Z$ and $L$ is a Fredholm mapping of index zero. Take

$$Pu = \frac{1}{\omega}\int_0^\omega u(t)\,dt, \quad u \in X;\quad Qz = Q(\psi, c_1, \ldots, c_m) = \left( \frac{1}{\omega}\left[ \int_0^\omega \psi(s)\,ds + \sum_{k=1}^{m} c_k \right], (0, 0, \ldots, 0)_{3\times m} \right). \tag{18}$$

It is trivial to show that $P$, $Q$ are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, and hence,the generalized inverse $K_P$ exists. In the following, we first devote ourselves to deriving the explicit expression of $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$. Taking $z = (\psi, c_1, \ldots, c_m) \in \operatorname{Im} L$, then exists an $u \in \operatorname{Dom} L \subset X$ such that

$$u'(t) = \psi(t), \quad t \neq t_k,\quad \Delta u(t) = c_k, \quad t = t_k,\ k = 1, 2, \ldots, m. \tag{19}$$

Then direct integration produces

$$u(t) = \int_0^t \psi(s)\,ds + \sum_{t>t_k} c_k + u(0) \tag{20}$$

that $u(t) \in \operatorname{Ker} P$; that is $\int_0^\omega u(s)ds = 0$, which, together with (20), implies

$$\int_0^\omega \int_0^t \psi(s)\,ds\,dt + \int_0^\omega \sum_{t>t_k} c_k dt + \omega u(0) = 0. \tag{21}$$

Then,

$$u(t) = \int_0^t \psi(s)\,ds + \sum_{t>t_k} c_k - \frac{1}{\omega}\int_0^{\omega}\int_0^t \psi(s)\,ds\,dt - \sum_{k=1}^{m} c_k + \frac{1}{\omega}\sum_{k=1}^{m} c_k t_k; \tag{22}$$

that is

$$K_P z = \int_0^t \psi(s)\,ds + \sum_{t>t_k} c_k - \frac{1}{\omega}\int_0^{\omega}\int_0^t \psi(s)\,ds\,dt - \sum_{k=1}^{m} c_k + \frac{1}{\omega}\sum_{k=1}^{m} c_k t_k. \tag{23}$$

Thus, for $u \in X$

$$QNu = \left(\begin{pmatrix} \frac{1}{\omega}\int_0^{\omega}\left[r_1(t) - D_1(t) - a_{11}(t)e^{u_1(t)} - a_{13}(t)e^{u_3(t)} + D_1(t)e^{u_2(t)-u_1(t)}\right]dt + \frac{1}{\omega}\sum_{k=1}^{m}\ln(1+c_{1k}) \\ \frac{1}{\omega}\int_0^{\omega}\left[r_2(t) - D_2(t) - a_{22}(t)e^{u_2(t)} - a_{23}(t)e^{u_3(t)} + D_2(t)e^{u_1(t)-u_2(t)}\right]dt + \frac{1}{\omega}\sum_{k=1}^{m}\ln(1+c_{2k}) \\ \frac{1}{\omega}\int_0^{\omega}\left[-r_3(t) + a_{31}(t)e^{u_1(t-\tau_1(t))} + a_{32}(t)e^{u_2(t-\tau_1(t))} - a_{33}(t)e^{u_3(t-\tau_2(t))}\right]dt + \frac{1}{\omega}\sum_{k=1}^{m}\ln(1+c_{3k}) \end{pmatrix}, 0, \ldots, 0\right), \tag{24}$$

$$K_P(I-Q)Nu = \begin{pmatrix} \int_0^t\left[r_1(s) - D_1(s) - a_{11}(s)e^{u_1(s)} - a_{13}(s)e^{u_3(s)} + D_1(s)e^{u_2(s)-u_1(s)}\right]ds + \sum_{t>t_k}\ln(1+c_{1k}) \\ \int_0^t\left[r_2(s) - D_2(s) - a_{22}(s)e^{u_2(s)} - a_{23}(s)e^{u_3(s)} + D_2(s)e^{u_1(s)-u_2(s)}\right]ds + \sum_{t>t_k}\ln(1+c_{2k}) \\ \int_0^t\left[-r_3(s) + a_{31}(s)e^{u_1(s-\tau_1(s))} + a_{32}(s)e^{u_2(s-\tau_1(s))} - a_{33}(s)e^{u_3(s-\tau_2(s))}\right]ds + \sum_{t>t_k}\ln(1+c_{3k}) \end{pmatrix}$$

$$- \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega}\int_0^t\left[r_1(s) - D_1(s) - a_{11}(s)e^{u_1(s)} - a_{13}(s)e^{u_3(s)} + D_1(s)e^{u_2(s)-u_1(s)}\right]ds\,dt + \sum_{k=1}^{m}\ln(1+c_{1k}) - \frac{1}{\omega}\sum_{k=1}^{m}\ln(1+c_{1k})t_k \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t\left[r_2(s) - D_2(s) - a_{22}(s)e^{u_2(s)} - a_{23}(s)e^{u_3(s)} + D_2(s)e^{u_1(s)-u_2(s)}\right]ds\,dt + \sum_{k=1}^{m}\ln(1+c_{2k}) - \frac{1}{\omega}\sum_{k=1}^{m}\ln(1+c_{2k})t_k \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t\left[-r_3(s) + a_{31}(s)e^{u_1(s-\tau_1(s))} + a_{32}(s)e^{u_2(s-\tau_1(s))} - a_{33}(s)e^{u_3(s-\tau_2(s))}\right]ds\,dt + \sum_{k=1}^{m}\ln(1+c_{3k}) - \frac{1}{\omega}\sum_{k=1}^{m}\ln(1+c_{3k})t_k \end{pmatrix}$$

$$-\begin{pmatrix} \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[r_1(s)-D_1(s)-a_{11}(s)e^{u_1(s)}-a_{13}(s)e^{u_3(s)}\right. \\ \left.+D_1(s)e^{u_2(s)-u_1(s)}\right]ds+\left(\frac{t}{\omega}-\frac{1}{2}\right)\sum_{k=1}^{m}\ln(1+c_{1k}) \\ \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[r_2(s)-D_2(s)-a_{22}(s)e^{u_2(s)}-a_{23}(s)e^{u_3(s)}\right. \\ \left.+D_2(s)e^{u_1(s)-u_2(s)}\right]ds+\left(\frac{t}{\omega}-\frac{1}{2}\right)\sum_{k=1}^{m}\ln(1+c_{2k}) \\ \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[-r_3(s)+a_{31}(s)e^{u_1(s-\tau_1(s))}+a_{32}(s)e^{u_2(s-\tau_1(s))}\right. \\ \left.-a_{33}(s)e^{u_3(s-\tau_2(s))}\right]ds+\left(\frac{t}{\omega}-\frac{1}{2}\right)\sum_{k=1}^{m}\ln(1+c_{3k}) \end{pmatrix}. \tag{25}$$

Clearly, $QN$ and $K_P(I-Q)N$ are continuous. By applying Ascoli-Arzela theorem, one can easily show that $QN(\overline{\Omega}), K_P(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is obviously bounded. Thus, $N$ is $L$-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. Now, we reach the position to search for an appropriate open bounded set $\Omega \subset X$ for the application of Lemma 2. Considering the operate equation $Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$\begin{aligned} u_1'(t) &= \lambda\left[r_1(t)-D_1(t)-a_{11}(t)e^{u_1(t)}-a_{13}(t)e^{u_3(t)}\right. \\ &\qquad \left.+D_1(t)e^{u_2(t)-u_1(t)}\right], \\ u_2'(t) &= \lambda\left[r_2(t)-D_2(t)-a_{22}(t)e^{u_2(t)}\right. \\ &\qquad \left.-a_{23}(t)e^{u_3(t)}+D_2(t)e^{u_1(t)-u_2(t)}\right], \\ u_3'(t) &= \lambda\left[-r_3(t)+a_{31}(t)e^{u_1(t-\tau_1(t))}\right. \\ &\qquad \left.+a_{32}(t)e^{u_2(t-\tau_1(t))}-a_{33}(t)e^{u_3(t-\tau_2(t))}\right], \\ \Delta u_i(t) &= \lambda\ln(1+c_{ik}), \quad i=1,2,3,\ k\in N. \end{aligned} \tag{26}$$

Since $u(t) = (u_1(t), u_2(t), u_3(t))^T$ are $\omega$-periodic functions, we need only to prove the result in the interval $[0,\omega]$. Integrating (26) over the interval $[0,\omega]$ leads to

$$\begin{aligned} &\int_0^{\omega}\left[r_1(t)-D_1(t)-a_{11}(t)e^{u_1(t)}-a_{13}(t)e^{u_3(t)}\right. \\ &\qquad \left.+D_1(t)e^{u_2(t)-u_1(t)}\right]dt = -c_1, \\ &\int_0^{\omega}\left[r_2(t)-D_2(t)-a_{22}(t)e^{u_2(t)}-a_{23}(t)e^{u_3(t)}\right. \\ &\qquad \left.+D_2(t)e^{u_1(t)-u_2(t)}\right]dt = -c_2, \\ &\int_0^{\omega}\left[-r_3(t)+a_{31}(t)e^{u_1(t-\tau_1(t))}+a_{32}(t)e^{u_2(t-\tau_1(t))}\right. \\ &\qquad \left.-a_{33}(t)e^{u_3(t-\tau_2(t))}\right]dt = -c_3. \end{aligned} \tag{27}$$

Hence, we have

$$\begin{aligned} &\int_0^{\omega}\left[a_{11}(t)e^{u_1(t)}+a_{13}(t)e^{u_3(t)}\right]dt+\int_0^{\omega}D_1(t)\,dt \\ &\qquad = \int_0^{\omega}D_1(t)e^{u_2(t)-u_1(t)}dt+\int_0^{\omega}r_1(t)\,dt+c_1, \\ &\int_0^{\omega}\left[a_{22}(t)e^{u_2(t)}+a_{23}(t)e^{u_3(t)}\right]dt+\int_0^{\omega}D_2(t)\,dt \\ &\qquad = \int_0^{\omega}D_2(t)e^{u_1(t)-u_2(t)}dt+\int_0^{\omega}r_2(t)\,dt+c_2, \\ &\int_0^{\omega}\left[a_{31}(t)e^{u_1(t-\tau_1(t))}+a_{32}(t)e^{u_2(t-\tau_1(t))}\right]dt \\ &\qquad = \int_0^{\omega}a_{33}(t)e^{u_3(t-\tau_2(t))}dt+\overline{r_3}\omega+c_3. \end{aligned} \tag{28}$$

It follows from (26)–(28) that

$$\begin{aligned}\int_0^\omega \left|u_1'(t)\right| dt &< \int_0^\omega \left[r_1(t) + D_1(t) e^{u_2(t)-u_1(t)}\right] dt \\ &\quad + \int_0^\omega \left[a_{11}(t) e^{u_1(t)} + a_{13}(t) e^{u_3(t)} \right. \\ &\quad \left. + D_1(t)\right] dt \\ &= 2\int_0^\omega a_{11}(t) e^{u_1(t)} dt \\ &\quad + 2\int_0^\omega a_{13}(t) e^{u_3(t)} dt + 2\overline{D_1}\omega - c_1, \\ \int_0^\omega \left|u_2'(t)\right| dt &< \int_0^\omega \left[r_2(t) + D_2(t) e^{u_1(t)-u_2(t)}\right] dt \\ &\quad + \int_0^\omega \left[a_{22}(t) e^{u_2(t)} + a_{23}(t) e^{u_3(t)} \right. \\ &\quad \left. + D_2(t)\right] dt \\ &= 2\int_0^\omega a_{22}(t) e^{u_2(t)} dt \\ &\quad + 2\int_0^\omega a_{23}(t) e^{u_3(t)} dt + 2\overline{D_2}\omega - c_2, \\ \int_0^\omega \left|u_3'(t)\right| dt &< \int_0^\omega r_3(t)\, dt \\ &\quad + \int_0^\omega \left[a_{31}(t) e^{u_1(t-\tau_1(t))} \right. \\ &\quad + a_{32}(t) e^{u_2(t-\tau_1(t))} \\ &\quad \left. + a_{33}(t) e^{u_3(t-\tau_2(t))}\right] dt \\ &= 2\int_0^\omega a_{31}(t) e^{u_1(t-\tau_1(t))} dt \\ &\quad + 2\int_0^\omega a_{32}(t) e^{u_2(t-\tau_1(t))} dt + c_3 \\ &\le 2\int_0^\omega a_{31}(t) e^{u_1(t-\tau_1(t))} dt \\ &\quad + 2\int_0^\omega a_{32}(t) e^{u_2(t-\tau_1(t))} dt.\end{aligned} \tag{29}$$

Multiplying the first equation of (26) by $e^{u_1(t)}$ and integrating over $[0, \omega]$ we have

$$\begin{aligned}-\sum_{k=1}^{m} c_{1k} e^{u_1(t_k)} + \int_0^\omega a_{11}(t) e^{2u_1(t)} dt \\ \le (r_1 - D_1)^M \int_0^\omega e^{u_1(t)} dt + D_1^M \int_0^\omega e^{u_2(t)} dt.\end{aligned} \tag{30}$$

Since $-1 < c_{1k} \le 0$, we obtain

$$\int_0^\omega a_1(t) e^{2u_1(t)} dt \le (r_1 - D_1)^M \int_0^\omega e^{u_1(t)} dt + D_1^M \int_0^\omega e^{u_2(t)} dt, \tag{31}$$

which yields

$$a_{11}^L \int_0^\omega e^{2u_1(t)} dt \le (r_1 - D_1)^M \int_0^\omega e^{u_1(t)} dt + D_1^M \int_0^\omega e^{u_2(t)} dt. \tag{32}$$

Similarly, multiplying the second equation of (26) by $e^{u_2(t)}$ and integrating over $[0, \omega]$ gives

$$a_{22}^L \int_0^\omega e^{2u_2(t)} dt \le (r_1 - D_1)^M \int_0^\omega e^{u_2(t)} dt + D_2^M \int_0^\omega e^{u_1(t)} dt. \tag{33}$$

By using the inequalities

$$\left(\int_0^\omega e^{u_i(t)} dt\right)^2 \le \omega \int_0^\omega e^{2u_i(t)} dt, \quad i = 1, 2, \tag{34}$$

it follows from (32)–(34) that

$$\begin{aligned}a_{11}^L\left(\int_0^\omega e^{2u_1(t)} dt\right)^2 &< \omega(r_1 - D_1)^M \int_0^\omega e^{u_1(t)} dt \\ &\quad + \omega D_1^M \int_0^\omega e^{u_2(t)} dt, \\ a_{22}^L\left(\int_0^\omega e^{2u_2(t)} dt\right)^2 &< \omega(r_2 - D_2)^M \int_0^\omega e^{u_2(t)} dt \\ &\quad + \omega D_2^M \int_0^\omega e^{u_1(t)} dt.\end{aligned} \tag{35}$$

If $\int_0^\omega e^{u_1(t)} dt \le \int_0^\omega e^{u_2(t)} dt$, then it follows from the second equation of (35) that

$$\begin{aligned}a_{22}^L\left(\int_0^\omega e^{2u_2(t)} dt\right)^2 &< \omega(r_2 - D_2)^M \int_0^\omega e^{u_2(t)} dt \\ &\quad + \omega D_2^M \int_0^\omega e^{u_2(t)} dt,\end{aligned} \tag{36}$$

which implies

$$\int_0^\omega e^{u_1(t)} dt \le \int_0^\omega e^{u_2(t)} dt < \frac{\omega(r_2 - D_2)^M + \omega D_2^M}{a_{22}^L}. \tag{37}$$

If $\int_0^\omega e^{u_2(t)} dt \le \int_0^\omega e^{u_1(t)} dt$, similarly, we obtain

$$\int_0^\omega e^{u_2(t)} dt \le \int_0^\omega e^{u_1(t)} dt < \frac{\omega(r_1 - D_1)^M + \omega D_1^M}{a_{11}^L}. \tag{38}$$

Set

$$A = \max\left\{\frac{(r_1 - D_1)^M + D_1^M}{a_{11}^L}, \frac{(r_2 - D_2)^M + D_2^M}{a_{22}^L}\right\}. \tag{39}$$

Then it follows from (37)–(39) that

$$\int_0^{\omega} e^{u_i(t)} dt \leq \omega A, \quad i = 1, 2. \tag{40}$$

Note that $u(t) = (u_1(t), u_2(t), u_3 t)) \in X$; then there exists $\xi_i, \eta_i \in [0, \omega]$ $(i = 1, 2, 3)$ such that

$$u_i(\xi_i) = \inf_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \sup_{t\in[0,\omega]} u_i(t), \quad i = 1, 2, 3. \tag{41}$$

Then it follows from (40) and (41), that

$$u_i(\xi_i) \leq \ln A, \quad i = 1, 2. \tag{42}$$

Since $\tau_i'(t) < 1$, we can let $s = t - \tau_i(t)$, that is, $t = \mu_i(s)$ $(i = 1, 2)$; then

$$\int_0^{\omega} a_{31}(t) e^{u_1(t-\tau_1(t))} dt = \int_{-\tau_1(0)}^{\omega-\tau_1(\omega)} \frac{a_{31}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_1(s)} ds. \tag{43}$$

According to Lemma 6, we know $(a_{31}(\mu_1(s))/(1 - \tau_1'(\mu_1(s))))e^{u_1(s)} \in C_{\omega}$. Thus,

$$\int_{-\tau_1(0)}^{\omega-\tau_1(\omega)} \frac{a_{31}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_1(s)} ds = \int_0^{\omega} \frac{a_{31}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_1(s)} ds. \tag{44}$$

Similarly, we have

$$\begin{aligned} \int_0^{\omega} a_{32}(t) e^{u_2(t-\tau_1(t))} dt &= \int_0^{\omega} \frac{a_{32}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_2(s)} ds, \\ \int_0^{\omega} a_{33}(t) e^{u_3(t-\tau_1(t))} dt &= \int_0^{\omega} \frac{a_{33}(\mu_2(s))}{1 - \tau_2'(\mu_2(s))} e^{u_3(s)} ds. \end{aligned} \tag{45}$$

On the other hand, by Lemma 6, we can see that $\mu_1(\omega) = \mu_1(0) + \omega$, so we can derive

$$\begin{aligned} \int_0^{\omega} \frac{a_{31}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} ds &= \int_{\mu_1(0)}^{\mu_1(\omega)} \frac{a_{31}(t)(1 - \tau'{}_1(t))}{1 - \tau_1'(t)} dt \\ &= \int_0^{\omega} a_{31}(t) dt = \overline{a_{31}}\omega, \\ \int_0^{\omega} \frac{a_{32}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} ds &= \int_{\mu_1(0)}^{\mu_1(\omega)} \frac{a_{32}(t)(1 - \tau_1'(t))}{1 - \tau_1'(t)} dt \\ &= \int_0^{\omega} a_{32}(t) dt = \overline{a_{31}}\omega, \\ \int_0^{\omega} \frac{a_{33}(\mu_2(s))}{1 - \tau_2'(\mu_2(s))} ds &= \int_{\mu_2(0)}^{\mu_2(\omega)} \frac{a_{33}(t)(1 - \tau'{}_1(t))}{1 - \tau_1'(t)} dt \\ &= \int_0^{\omega} a_{33}(t) dt = \overline{a_{33}}\omega; \end{aligned} \tag{46}$$

therefore, we can derive from (27) and (46) that

$$\begin{aligned} \overline{a_{33}}\omega e^{u_3(\xi_3)} &\leq e^{u_3(\xi_3)} \int_0^{\omega} \frac{a_{33}(\mu_2(s))}{1 - \tau_2'(\mu_2(s))} ds \\ &\leq \int_0^{\omega} \frac{a_{33}(\mu_2(s))}{1 - \tau_2'(\mu_2(s))} e^{u_3(s)} ds \\ &\leq \int_0^{\omega} a_{33}(t) e^{u_3(t-\tau_2(t))} dt \\ &= \int_0^{\omega} a_{31}(t) e^{u_1(t-\tau_1(t))} dt \\ &\quad + \int_0^{\omega} a_{32}(t) e^{u_2(t-\tau_1(t))} dt - \overline{r_3}\omega + c_3 \\ &= \int_0^{\omega} \frac{a_{31}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_1(s)} ds \\ &\quad + \int_0^{\omega} \frac{a_{32}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_2(s)} ds - \overline{r_3}\omega + c_3 \\ &\leq \int_0^{\omega} \frac{a_{31}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_1(s)} ds \\ &\quad + \int_0^{\omega} \frac{a_{32}(\mu_1(s))}{1 - \tau_2'(\mu_1(s))} e^{u_2(s)} ds \\ &= \int_0^{\omega} B_1(s) e^{u_1(s)} ds + \int_0^{\omega} B_2(s) e^{u_2(s)} ds \\ &\leq (B_1^M + B_2^M) A\omega, \end{aligned} \tag{47}$$

which implies

$$\overline{a_{33}} e^{u_3(\xi_3)} \leq (B_1^M + B_2^M) A; \tag{48}$$

that is

$$u_3(\xi_3) \leq \ln \frac{(B_1^M + B_2^M) A}{\overline{a_{33}}}. \tag{49}$$

It follows from (29), (42), and (49) that

$$\begin{aligned} \int_0^{\omega} |u_1'(t)| dt &< 2a_{11}^M A\omega + 2a_{13}^M \frac{(B_1^M + B_2^M) A}{\overline{a_{33}}} \\ &\quad + 2\overline{D_1}\omega - c_1 := A_1, \\ \int_0^{\omega} |u_2'(t)| dt &< 2a_{22}^M A\omega + 2a_{23}^M \frac{(B_1^M + B_2^M) A}{\overline{a_{33}}} \\ &\quad + 2\overline{D_2}\omega - c_2 := A_2, \\ \int_0^{\omega} |u_3'(t)| dt &< 2(B_1^M + B_2^M) A\omega := A_3. \end{aligned} \tag{50}$$

From (42), (50), and Lemma 4, it follows that, for $t \in [0, \omega]$,

$$\begin{aligned} u_i(t) &\le u_i(\xi_i) + \frac{1}{2}(A_i + |c_i|) \\ &\le \ln A + \frac{1}{2}(A_i - c_i), \quad i = 1, 2, \\ u_3(t) &\le u_3(\xi_3) + \frac{1}{2}(A_3 + |c_3|) \\ &\le \ln \frac{(B_1^M + B_2^M) A}{\overline{a_{33}}} + \frac{1}{2}(A_3 - c_3). \end{aligned} \tag{51}$$

It follows from (28) and (48) that

$$\begin{aligned} e^{u_1(\eta_1)} \int_0^\omega a_{11}(t)\,dt &\ge \int_0^\omega a_{11}(t) e^{u_1(t)} dt \\ &\ge \overline{(r_1 - D_1)}\omega - \int_0^\omega a_{13}(t) e^{u_3(t)} dt + c_1 \\ &\ge \overline{(r_1 - D_1)}\omega - \frac{a_{13}^M (B_1^M + B_2^M) A\omega}{\overline{a_{33}}} + c_1, \end{aligned} \tag{52}$$

which deduces

$$\begin{aligned} u_1(\eta_1) &\ge \ln \frac{\overline{(r_1 - D_1)} - a_{13}^M (B_1^M + B_2^M) A / \overline{a_{33}} + c_1/\omega}{\overline{a_{11}}} \\ &:= \ln C_1. \end{aligned} \tag{53}$$

This, together with (53) and Lemma 4, leads to

$$u_1(t) \ge u_1(\eta_1) - \frac{1}{2}(A_1 - c_1) \ge \ln C_1 - \frac{1}{2}(A_1 - c_1). \tag{54}$$

Let $R_1 := \max\{|\ln A| + (1/2)(A_1 - c_1)|, \ln C_1| + (1/2)(A_1 - c_1)\}$. It follows from (51) and (54) that

$$\sup_{t \in [0,\omega]} |u_1(t)| \le R_1. \tag{55}$$

From (28), (41) and (48) we have

$$\begin{aligned} \overline{a_{22}}\omega e^{u_2(\eta_2)} &\ge \overline{(r_2 - D_2)}\omega + \overline{D_2}\omega e^{u_1(\xi_1) - u_2(\eta_2)} \\ &\quad - \int_0^\omega a_{23}(t) e^{u_3(t)} dt + c_2 \\ &\ge \overline{(r_2 - D_2)}\omega + \overline{D_2}\omega e^{-R_1} \cdot e^{-u_2(\eta_2)} \\ &\quad - \frac{a_{23}^M (B_1^M + B_2^M) A\omega}{\overline{a_{33}}} + c_2, \end{aligned} \tag{56}$$

which deduces

$$e^{u_2(\eta_2)} \ge \frac{\alpha + \sqrt{\alpha^2 + 4\overline{a_{22}}\,\overline{D_2} e^{-R_1}}}{\overline{a_{22}}} := C_2, \tag{57}$$

where $\alpha = \overline{(r_2 - D_2)} - a_{23}^M (B_1^M + B_2^M) A / \overline{a_{33}} + c_2/\omega$, implies

$$u_2(\eta_2) \ge \ln C_2. \tag{58}$$

This, together with (41) and Lemma 4, leads to

$$u_2(t) \ge u_2(\eta_2) - \frac{1}{2}(A_2 - c_2) \ge \ln C_2 - \frac{1}{2}(A_2 - c_2). \tag{59}$$

Set

$$R_2 := \max\left\{ |\ln A| + \frac{1}{2}(A_2 - c_2), |\ln C_2| + \frac{1}{2}(A_2 - c_2) \right\}. \tag{60}$$

It follows from (51) and (59) that

$$\sup_{t \in [0,\omega]} |u_2(t)| \le R_2. \tag{61}$$

Noting that

$$\begin{aligned} \overline{(r_1 - D_1)}\omega + c_1 &\le a_{11}^M \int_0^\omega e^{u_1(t)} dt + a_{13}^M \int_0^\omega e^{u_3(t)} dt, \\ \overline{(r_2 - D_2)}\omega + c_2 &\le a_{22}^M \int_0^\omega e^{u_2(t)} dt + a_{23}^M \int_0^\omega e^{u_3(t)} dt, \end{aligned} \tag{62}$$

it follows from (28) and (46) that

$$\begin{aligned} \overline{a_{33}}\omega e^{u_3(\eta_3)} &\ge e^{u_3(\xi_3)} \int_0^\omega \frac{a_{33}(\mu_2(s))}{1 - \tau_2'(\mu_2(s))} ds \\ &\ge \int_0^\omega \frac{a_{33}(\mu_2(s))}{1 - \tau_2'(\mu_2(s))} e^{u_3(s)} ds \\ &= \int_0^\omega a_{33}(t) e^{u_3(t - \tau_2(t))} dt \\ &= \int_0^\omega a_{31}(t) e^{u_1(t - \tau_1(t))} dt \\ &\quad + \int_0^\omega a_{32}(t) e^{u_2(t - \tau_1(t))} dt - \overline{r_3}\omega + c_3 \\ &= \int_0^\omega \frac{a_{31}(\mu_1(s))}{1 - \tau_1'(\mu_1(s))} e^{u_1(s)} ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\omega} \frac{a_{32}(\mu_1(s))}{1-\tau_1'(\mu_1(s))} e^{u_2(s)} ds - \overline{r_3}\omega + c_3 \\
&= \int_0^{\omega} B_1(s) e^{u_1(s)} ds \\
&\quad + \int_0^{\omega} B_2(s) e^{u_2(s)} ds - \overline{r_3}\omega + c_3 \\
&\geq B_1^L \frac{\overline{(r_1-D_1)}\omega + c_1 - a_{13}^M \int_0^{\omega} e^{u_3(t)} dt}{a_{11}^M} \\
&\quad + B_2^L \frac{\overline{(r_2-D_2)}\omega + c_2 - a_{23}^M \int_0^{\omega} e^{u_3(t)} dt}{a_{22}^M} \\
&\quad - \overline{r_3}\omega + c_3 \\
&\geq B_1^L \frac{\overline{(r_1-D_1)}\omega + c_1 - a_{13}^M \omega e^{u_3(\eta_3)}}{a_{11}^M} \\
&\quad + B_2^L \frac{\overline{(r_2-D_2)}\omega + c_2 - a_{23}^M \omega e^{u_3(\eta_3)}}{a_{22}^M} \\
&\quad - \overline{r_3}\omega - c_3,
\end{aligned} \tag{63}
$$

which yields

$$
\begin{aligned}
e^{u_3(\eta_3)} \geq & \left(a_{22}^M B_1^L \left(\overline{(r_1-D_1)}\omega + c_1\right)\right. \\
& \left. + a_{11}^M B_2^L \left(\overline{(r_2-D_2)}\omega + c_2\right) - a_{11}^M a_{22}^M \left(\overline{r_3}\omega + c_3\right)\right) \\
& \times \left(\left[a_{11}^M a_{22}^M \overline{a_{33}} + a_{13}^M a_{22}^M B_1^L + a_{11}^M a_{23}^M B_2^L\right]\omega\right)^{-1} \\
:= & C_3.
\end{aligned} \tag{64}
$$

This, together with (41) and Lemma 4, leads to

$$
u_3(t) \geq u_3(\eta_2) - \frac{1}{2}(A_3 - c_3) \geq \ln C_3 - \frac{1}{2}(A_3 - c_3). \tag{65}
$$

Set

$$
\begin{aligned}
R_3 := \max \left\{ \left| \ln \frac{\left(a_{31}^M + a_{32}^M\right) A\omega}{a_{33}^L} \right| + \frac{1}{2}(A_3 - c_3), \right. \\
\left. \left|\ln C_3\right| + \frac{1}{2}(A_3 - c_3) \right\}.
\end{aligned} \tag{66}
$$

It follows from (51) and (65) that

$$
\sup_{t\in[0,\omega]} \left|u_3(t)\right| \leq R_3. \tag{67}
$$

Thus, we obtain

$$
\left\|u_i(t)\right\| \leq R_i, \quad i = 1, 2, 3. \tag{68}
$$

Clearly, $R_i$ $(i = 1, 2, 3)$ are independent of $\lambda$.

In order to use the invariance property of homotopy, we need to consider the following algebraic equations:

$$
\begin{aligned}
&\frac{c_1}{\omega} + \overline{(r_1-D_1)} - \overline{a_{11}} e^{u_1} + \mu\left(-\overline{a_{13}} e^{u_3} + \overline{D_1} e^{u_2-u_1}\right) = 0, \\
&\frac{c_2}{\omega} + \overline{(r_2-D_2)} - \overline{a_{22}} e^{u_2} + \mu\left(-\overline{a_{23}} e^{u_3} + \overline{D_2} e^{u_1-u_2}\right) = 0, \\
&\overline{a_{31}} e^{u_1} + \overline{a_{32}} e^{u_2} - \overline{a_{33}} e^{u_3} + \mu\left(\frac{c_3}{\omega} + \overline{r_3}\right) = 0,
\end{aligned} \tag{69}
$$

for $(u_1, u_2, u_3)^T \in R^3$, where $\mu \in [0,1]$. Carrying out similar arguments as above, one can easily show that any solution $(u_1^*, u_2^*, u_3^*)$ of (69) with $\mu \in [0,1]$ also satisfies

$$
\left\|u_i^*(t)\right\| \leq R_i, \quad i = 1, 2, 3. \tag{70}
$$

Choose $R^* > \sum_{i=1}^{3} R_i$ and define $\Omega = \{u(t) = (u_1(t), u_2(t), u_3(t))^T \in X : \|u\| < R^*, u(t_k^+) \in \Omega, k = 1, 2, \ldots, m\}$; it is clear that $\Omega$ satisfies the condition (a) of Lemma 2. Let $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$; then $u$ is a constant vector in $R^3$ with $\|u\| = R^*$. Then

$$
\begin{aligned}
&QNu \\
&= \left( \begin{pmatrix} \frac{c_1}{\omega} + \overline{(r_1-D_1)} - \overline{a_{11}} e^{u_1} - \overline{a_{13}} e^{u_3} + \overline{D_1} e^{u_2-u_1} \\ \frac{c_2}{\omega} + \overline{(r_2-D_2)} - \overline{a_{22}} e^{u_2} - \overline{a_{23}} e^{u_3} + \overline{D_2} e^{u_1-u_2} \\ \frac{c_3}{\omega} - \overline{r_3} + \overline{a_{31}} e^{u_1} + \overline{a_{32}} e^{u_2} - \overline{a_{33}} e^{u_3} \end{pmatrix}, \right. \\
&\qquad \left. \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}_{k=1}^{m} \right) \neq 0.
\end{aligned} \tag{71}
$$

That is, the condition (b) of Lemma 2 holds. Finally, for the convenience of computing the Brouwer degree, we consider a homotopy

$$
\begin{aligned}
B_\mu\left((u_1, u_2, u_3)^T\right) = \mu QN\left((u_1, u_2, u_3)^T\right) \\
+ (1-\mu)\phi\left((u_1, u_2, u_3)^T\right), \\
\mu \in [0,1],
\end{aligned} \tag{72}
$$

where

$$
\phi\left((u_1, u_2, u_3)^T\right) = \begin{bmatrix} \frac{c_1}{\omega} + \overline{(r_1-D_1)} - \overline{a_{11}} e^{u_1} \\ \frac{c_2}{\omega} + \overline{(r_2-D_2)} - \overline{a_{22}} e^{u_2} \\ \overline{a_{31}} e^{u_1} + \overline{a_{32}} e^{u_2} - \overline{a_{33}} e^{u_3} \end{bmatrix}. \tag{73}
$$

By (69) and (70), it follows that $B_\mu((u_1,u_2,u_3)^T)\neq 0$ for $u\in\partial\Omega\cap\operatorname{Ker}L$, $\mu\in[0,1]$. In addition, it is clear that the algebraic equation $\phi((u_1,u_2,u_3)^T)=0$ has a unique solution in $R^3$. Choose the isomorphism $J$ to be the identity mapping; by a direct computation and the invariance property of homotopy, one has

$$\begin{aligned}\deg\{JQNu,\operatorname{Ker}L\cap\partial\Omega,0\}&=\deg\{QNu,\operatorname{Ker}L\cap\partial\Omega,0\}\\&=\deg\{\phi,\operatorname{Ker}L\cap\partial\Omega,0\}\\&=-1\neq 0.\end{aligned}\tag{74}$$

By now we have proved that all the requirements in Lemma 2 are satisfied. Hence system (12) has at least one $\omega$-periodic solution, say $(u_1^*,u_2^*,u_3^*)^T$. Set $x_1^*(t)=e^{u_1^*(t)}$, $x_2^*(t)=e^{u_2^*(t)}$, $x_3^*(t)=e^{u_3^*(t)}$; then $(x_1^*(t),x_2^*(t),x_3^*(t))^T$ has at least one positive $\omega$-periodic solution of system (1). The proof of Theorem 7 is complete. □

*Remark 8.* If $c_{ik}=0$ $(i=1,2,3,\ k=1,2,\ldots,m)$, then (1) is translated to (5). In this case, the conditions $(A_3)$, $(A_4)$ are the same as $(H_2)$, $(H_4)$ of Theorem 2.1 in [12], but we see that $(H_3)$ of Theorem 2.1 in [12] is not needed here. Hence our result improves and generalizes the corresponding result of [12].

*Remark 9.* If $\tau_i(t)=0$ $(i=1,2,3)$, then (1) is translated to (1.2) in [14]. In this case, the conditions $(A_1)$–$(A_4)$ are the same as $(C_1)$–$(C_4)$ in [14]. Hence our result generalizes the corresponding result of [14].

## 3. Uniqueness and Global Stability

We now proceed to the discussion on the uniqueness and global stability of the $\omega$-periodic solution $x^*(t)$ in Theorem 14. It is immediate that if $x^*(t)$ is globally asymptotically stable then $x^*(t)$ is unique in fact. Under the hypotheses $(A_1)$, $(A_2)$, we consider the nonimpulsive delay differential equation

$$\begin{aligned}y_1'(t)&=y_1(t)\left[r_1(t)-A_{11}(t)y_1(t)-A_{13}(t)y_3(t)\right]\\&\quad+D_1(t)\left(B_1(t)y_2(t)-y_1(t)\right),\\y_2'(t)&=y_2(t)\left[r_2(t)-A_{22}(t)y_2(t)-A_{23}(t)y_3(t)\right]\\&\quad+D_2(t)\left(B_2(t)y_1(t)-y_2(t)\right),\\y_3'(t)&=y_3(t)\left[-r_3(t)+A_{31}(t)y_1(t-\tau_1(t))\right.\\&\quad+A_{32}(t)y_2(t-\tau_1(t))\\&\quad\left.-A_{33}(t)y_3(t-\tau_2(t))\right],\end{aligned}\tag{75}$$

with the initial conditions

$$\begin{aligned}&y_i(t)=\phi_i(\theta),\quad y_3(t)=\varphi(\theta),\quad \phi_i(0)>0,\quad \varphi(0)>0,\\&\phi_i,\varphi\in C([-\tau,0],R_+),\quad \theta\in[-\tau,0],\\&\tau=\max_{t\in[0,\omega]}\{\tau_1(t),\tau_2(t)\},\quad i=1,2,\end{aligned}\tag{76}$$

where

$$\begin{aligned}A_{ii}(t)&=a_{ii}(t)\prod_{0<t_k<t}(1+c_{ik})\quad(i=1,2),\\A_{13}(t)&=a_{13}(t)\prod_{0<t_k<t}(1+c_{3k}),\\A_{23}(t)&=a_{23}(t)\prod_{0<t_k<t}(1+c_{3k}),\\A_{31}(t)&=a_{31}(t)\prod_{0<t_k<t-\tau_1(t)}(1+c_{1k}),\\A_{32}(t)&=a_{32}(t)\prod_{0<t_k<t-\tau_1(t)}(1+c_{2k}),\\A_{33}(t)&=a_{33}(t)\prod_{0<t_k<t-\tau_2(t)}(1+c_{3k}),\\B_1(t)&=\prod_{0<t_k<t}(1+c_{1k})^{-1}(1+c_{2k}),\\B_2(t)&=\prod_{0<t_k<t}(1+c_{2k})^{-1}(1+c_{1k}).\end{aligned}\tag{77}$$

The following lemmas will be used in the proofs of our results. The proof of the first lemma is similar to that of Theorem 1 in [23].

**Lemma 10.** *Suppose that $(A_1)$, $(A_2)$ hold; then*

(i) *if $y(t)=(y_1(t),y_2(t),y_3(t))^T$ is a solution of* (75) *on $[-\tau,+\infty)$, then $x_i(t)=\prod_{0<t_k<t}(1+c_{ik})y_i(t)$ $(i=1,2,3)$ is a solution of* (1) *on $[-\tau,+\infty)$;*

(ii) *if $x(t)=(x_1(t),x_2(t),x_3(t))^T$ is a solution of* (1) *on $[-\tau,+\infty)$, then $y_i(t)=\prod_{0<t_k<t}(1+c_{ik})^{-1}x_i(t)$ $(i=1,2,3)$ is a solution of* (75) *on $[-\tau,+\infty)$.*

*Proof.* (i) It is easy to see that $x_i(t) = \prod_{0<t_k<t}(1 + c_{ik})y_i(t)$ is absolutely continuous on every interval $(t_k, t_{k+1}]$; $t \neq t_k$, $k = 1, 2, \ldots$,

$$\begin{aligned} x_1'(t) - x_1(t)\left[r_1(t) - a_{11}(t)x_1(t) - a_{13}(t)x_3(t)\right] \\ + D_1(t)\left(x_2(t) - x_1(t)\right) \\ = \prod_{0<t_k<t}\left(1 + c_{1k}\right)y_1'(t) \\ - \prod_{0<t_k<t}\left(1 + c_{1k}\right)y_1(t) \\ \times \left[r_1(t) - a_{11}(t)\prod_{0<t_k<t}\left(1 + c_{1k}\right)y_1(t)\right. \\ \left. - a_{13}(t)\prod_{0<t_k<t}\left(1 + c_{3k}\right)y_3(t)\right] \\ + D_1(t)\left(\prod_{0<t_k<t}\left(1 + c_{2k}\right)y_2(t)\right. \\ \left. - \prod_{0<t_k<t}\left(1 + c_{1k}\right)y_1(t)\right) \\ = \prod_{0<t_k<t}\left(1 + c_{1k}\right) \\ \times \left\{y_1'(t) - y_1(t)\right. \\ \times \left[r_1(t) - A_{11}(t)y_1(t) - A_{13}(t)y_3(t)\right] \\ \left. - D_1(t)\left(B_1(t)y_2(t) - y_1(t)\right)\right\} = 0. \end{aligned} \tag{78}$$

On the other hand, for any $t = t_k$, $k = 1, 2, \ldots$,

$$\begin{aligned} x_1\left(t_k^+\right) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}\left(1 + c_{1k}\right)y_1(t) \\ &= \prod_{0<t_j\le t_k}\left(1 + c_{1k}\right)y_1\left(t_k\right), \\ x_1\left(t_k\right) &= \prod_{0<t_j<t_k}\left(1 + c_{1k}\right)y_1\left(t_k\right). \end{aligned} \tag{79}$$

Thus

$$\Delta x_1\left(t_k^+\right) = \left(1 + c_{1k}\right)y_1\left(t_k\right), \tag{80}$$

which implies that $x_1(t)$ is a solution of (1); similarly, we can prove that $x_2(t)$, $x_3(t)$ are also solutions of (i). Therefore, $x_i(t)$ $(i = 1, 2, 3)$ are solutions of (1) on $[-\tau, +\infty)$.

(ii) Since $x_i(t) = \prod_{0<t_k<t}(1 + c_{ik})y_i(t)$ is absolutely continuous on every interval $(t_k, t_{k+1}]$; $t \neq t_k$, $k = 1, 2, \ldots$, and in view of (79), it follows that, for any $k = 1, 2, \ldots$,

$$\begin{aligned} y_1\left(t_k^+\right) &= \prod_{0<t_j\le t_k}\left(1 + c_{1k}\right)^{-1}x_1\left(t_k^+\right) \\ &= \prod_{0<t_j<t_k}\left(1 + c_{1k}\right)^{-1}x_1\left(t_k\right) = y_1\left(t_k\right), \\ y_1\left(t_k^-\right) &= \prod_{0<t_j<t_k}\left(1 + c_{1k}\right)^{-1}x_1\left(t_k^-\right) \\ &= \prod_{0<t_j\le t_k^-}\left(1 + c_{1k}\right)^{-1}x_1\left(t_k^-\right) = y_1\left(t_k\right), \end{aligned} \tag{81}$$

which implies that $y_1(t)$ is continuous on $[-\tau, +\infty)$. It is easy to prove that $y_1(t)$ is absolutely continuous on $[-\tau, +\infty)$. Similarly, we can prove that $y_2(t)$, $y_3(t)$ are absolutely continuous on $[-\tau, +\infty)$. Similar to the proof of $(i)$, we can check that $y_i(t) = \prod_{0<t_k<t}(1 + c_{ik})^{-1}x_i(t)$ $(i = 1, 2, 3)$ are solutions of (75) on $[-\tau, +\infty)$. The proof of Lemma 10 is completed. □

**Lemma 11.** *Let $y(t) = (y_1(t), y_2(t), y_3(t))^T$ denote any positive solution of system* (75) *with initial conditions* (76). *Then there exists a $T_2 > 0$ such that $0 < y_i(t) \le M_i$ $(i = 1, 2, 3)$, for $t \ge T_2$, where*

$$\begin{aligned} M_1 = M_2 > M^* &= \max\left\{\frac{r_1^M + \widetilde{D}_1^M}{A_{11}^L}, \frac{r_2^M + \widetilde{D}_2^M}{A_{22}^L}\right\}, \\ M_3 &= \frac{\left(A_{31}^M + A_{32}^M\right)M_1e^{(A_{31}^M + A_{32}^M)M_1\tau}}{A_{33}^L}, \\ \widetilde{D}_1(t) = B_2(t)D_1(t), &\qquad \widetilde{D}_2(t) = B_1(t)D_2(t). \end{aligned} \tag{82}$$

*Proof.* Let $V_1(t) = \max\{y_1(t), y_2(t)\}$. Calculating the upper-right derivative of $V_1(t)$ along the positive solution of system (75), we have the following:

$(C_1)$ if $y_1(t) \ge y_2(t)$ in some intervals, then

$$\begin{aligned} D^+V_1(t) &= y_1'(t) \\ &= y_1(t)\left[r_1(t) - A_{11}(t)y_1(t) - A_{13}(t)y_3(t)\right] \\ &\quad + D_1(t)\left(B_2(t)y_2(t) - y_1(t)\right) \\ &\le y_1(t)\left[r_1^M - A_{11}^Ly_1(t)\right] + \widetilde{D}_1(t)y_2(t) \\ &\le y_1(t)\left[r_1^M + \widetilde{D}_1^M - A_{11}^Ly_1(t)\right], \end{aligned} \tag{83}$$

$(C_2)$ if $y_2(t) > y_1(t)$ in other intervals, similarly, we have

$$\begin{aligned} D^+V_1(t) &= y_2'(t) \\ &\le y_2(t)\left[r_2^M - A_{22}^Ly_2(t)\right] + \widetilde{D}_2(t)y_1(t) \\ &\le y_2(t)\left[r_2^M + \widetilde{D}_2^M - A_{22}^Ly_2(t)\right]. \end{aligned} \tag{84}$$

It follows from $(C_1)$ and $(C_2)$ that

$$D^+V_1(t) \leq y_i(t)\left[r_i^M + \widetilde{D}_i^M - A_{ii}^L y_i(t)\right], \quad i = 1,2. \tag{85}$$

By (85) we can derive the following.

(A) If $\max\{y_1(0), y_2(0)\} \leq M_1$, then $\max\{y_1(t), y_2(t)\} \leq M_1, t \geq 0$.

(B) If $\max\{y_1(0), y_2(0)\} > M_1$, let $-\delta = \max\{M_1(r_i^M + \widetilde{D}_i^M - A_{ii}^L M_1),\ i = 1,2\}$ ($\delta > 0$ by the condition (82)). We consider the following two cases:

(a) $V_1(0) = y_1(0) > M_1\ (y_1(0) \geq y_2(0))$;
(b) $V_1(0) = y_2(0) > M_1\ (y_1(0) < y_2(0))$.

If (a) holds, then there exists $\epsilon > 0$ such that if $t \in [0, \epsilon)$, then $V_1(t) = y_1(t) > M_1$, and we have

$$D^+V_1(t) = y_1'(t) < -\delta < 0. \tag{86}$$

If (b) holds, then there exists $\epsilon > 0$ such that if $t \in [0, \epsilon)$, then $V_1(t) = y_2(t) > M_1$, and we also have

$$D^+V_1(t) = y_2'(t) < -\delta < 0. \tag{87}$$

From what has been discussed above, we can conclude that if $V(0) > M_1$, then $V(t)$ is strictly monotone decreasing with speed at least $\delta$. Therefore there exists a $T_1 > 0$ such that if $t \geq T_1$, then

$$V_1(t) = \max\{y_1(t), y_2(t)\} \leq M_1. \tag{88}$$

From the third equation of system (75) and (88) we can deduce that, for $t > T_1 + \tau$,

$$\begin{aligned} y_3'(t) &\leq y_3(t)\left[\left(A_{31}^M + A_{32}^M\right)M_1 - A_{33}^L y_3(t - \tau_2(t))\right] \\ &\leq y_3(t)\Big[\left(A_{31}^M + A_{32}^M\right)M_1 \\ &\quad - A_{33}^L e^{-(A_{31}^M + A_{32}^M)M_1\tau} y_3(t)\Big]. \end{aligned} \tag{89}$$

A standard comparison argument shows that

$$\limsup_{t\to+\infty} y_3(t) \leq \frac{\left(A_{31}^M + A_{32}^M\right)M_1 e^{(A_{31}^M + A_{32}^M)M_1\tau}}{A_{33}^L} := M_3. \tag{90}$$

Thus, there exists a $T_2 \geq T_1 + \tau$ such that

$$y_3(t) \leq M_3, \quad \text{for } t > T_2. \tag{91}$$

The proof of Lemma 11 is completed. □

**Lemma 12.** *Let $y(t) = (y_1(t), y_2(t), y_3(t))^T$ denote any positive solution of system* (75) *with initial conditions* (76). *Then there exists a $T > 0$ such that $y_i(t) \geq m_i$ ($i = 1,2,3$), for $t \geq T$, where*

$$\begin{aligned} 0 < m_1 = m_2 < m_1^* &= \min\left\{\frac{r_1^L - D_1^M - A_{13}^M M_3}{A_{11}^M}, \frac{r_2^L - D_2^M - A_{23}^M M_3}{A_{22}^M}\right\}, \\ 0 < m_3 < m_3^* &:= \frac{\left(A_{31}^L + A_{32}^L\right)m_1}{A_{33}^M} e^{[(A_{31}^L + A_{32}^L)m_1 - r_3^M - A_{33}^M M_3]\tau}, \end{aligned} \tag{92}$$

*and $M_3$ are defined in Lemma 11.*

*Proof.* Let $V_2(t) = \min\{y_1(t), y_2(t)\}$. Calculating the lower-right derivative of $V_2(t)$ along the positive solution of system (75), similar to the discussion for inequality (85), for any $t > T_1$, where $T_1$ is defined in Lemma 11, we easily obtain:

$(C_3)$ if $y_1(t) \leq y_2(t)$, in some intervals, then

$$D_+V_2(t) = y_1'(t) \geq y_1(t)\left[r_1^L - D_1^M - A_{13}^M M_3 - A_{11}^M y_1(t)\right]. \tag{93}$$

$(C_4)$ if $y_2(t) \leq y_1(t)$, in other intervals, similarly, we have

$$D_+V_2(t) = y_2'(t) \geq y_2(t)\left[r_2^L - D_2^M - A_{23}^M M_3 - A_{22}^M y_2(t)\right]. \tag{94}$$

From $(C_3)$ and $(C_4)$, we can reduce the following.

(C) If $V_2(T_2) = \min\{y_1(T_2), y_2(T_2)\} \geq m_1$, then $\min\{y_1(t), y_2(t)\} \geq m_1, t \geq T_2$.

(D) If $V_2(T_2) = \min\{y_1(T_2), y_2(T_2)\} < m_1$, and let $\sigma = \min\{y_1(T_2)(r_1^L - D_1^M - A_{11}^M m_1 - A_{13}^M M_3), y_2(T_2)(r_2^L - D_2^M - A_{22}^M m_1 - A_{23}^M M_3)\}$. There are three cases:

(c) $V_2(T_2) = y_1(T_2) < m_1\ (y_1(T_2) < y_2(T_2))$;
(d) $V_2(T_2) = y_2(T_2) < m_1\ (y_2(T_2) < y_1(T_2))$;
(e) $V_2(T_2) = y_1(T_2) = y_2(T_2) < m_1$.

If (c) holds, then there exists $\epsilon > 0$ such that if $t \in [T_2, T_2 + \epsilon)$, we have $V_2(t) = y_1(t)$ and $D_+V_2(t) = y_1'(t) > \sigma > 0$.

If (d) holds, similar to (c), there exists $[T_2, T_2 + \epsilon)$ such that if $t \in [T_2, T_2 + \epsilon)$, we have $V_2(t) = y_2(t)$ and $D_+V_2(t) = y_2'(t) > \sigma > 0$.

If (e) holds, in the same way also there exists $[T_2, T_2 + \epsilon)$ such that if $t \in [T_2, T_2 + \epsilon)$, we have $V_2(t) = y_i(t)$ and $D_+V_2(t) = y_i'(t) > \sigma > 0, i = 1,2$.

From (c)–(e), we know that if $V_2(T_2) < m_1$, $V(t)$ will strictly monotonically increase with speed $\sigma$. So there exists $T_3 > T_2$ such that if $t \geq T_3$, we have $V_2(t) = \min\{y_1(t), y_2(t)\} \geq m_1$.

From the third equation of system (75), for any $t > T_3 + \tau$, we know that

$$\begin{aligned} y_3'(t) &\geq y_3(t) \\ &\quad \times \left[-r_3^M + \left(A_{31}^L + A_{32}^L\right)m_1 - A_{33}^M y_3(t - \tau_2(t))\right] \end{aligned} \tag{95}$$

and using the fact that

$$y_3(t-\tau_2(t)) \le e^{-[(A_{31}^L+A_{32}^L)m_1-r_3^M-A_{33}^M M_3]\tau} y_3(t), \quad \text{for } t>\tau, \tag{96}$$

therefore, for $t > T_3 + \tau$, we get

$$y_3'(t) \ge y_3(t) \times \left[ \left(A_{31}^L + A_{32}^L\right) m_1 - A_{33}^M e^{-[(A_{31}^L+A_{32}^L)m_1-r_3^M-A_{33}^M M_3]\tau_2} y_3(t) \right]. \tag{97}$$

A standard comparison argument shows that

$$\liminf_{t\to+\infty} y_3(t) \ge \frac{\left(A_{31}^L + A_{32}^L\right) m_1}{A_{33}^M} e^{[(A_{31}^L+A_{32}^L)m_1-r_3^M-A_{33}^M M_3]\tau} := m_3. \tag{98}$$

Thus, there exists a $T \ge T_3 + \tau$ such that

$$y_3(t) \ge m_3, \quad \text{for } t > T. \tag{99}$$

The proof of Lemma 12 is completed. □

**Lemma 13** (see Barbălat's Lemma [32]). *Let $f(t)$ be a nonnegative function defined on $[0,+\infty)$ such that $f(t)$ is integrable and uniformly continuous on $[0,+\infty)$; then $\lim_{t\to+\infty} f(t) = 0$.*

We now formulate the uniqueness and global stability of the positive $\omega$-periodic solution $x^*(t)$ of system (1). It is immediate that if $x^*(t)$ is globally asymptotically stable then $x^*(t)$ is in fact unique.

**Theorem 14.** *In addition to $(A_1)$ and $(A_2)$, assume further that*

$(A_5)$ $\lim_{t\to+\infty} \inf C_i(t) > 0$, *then, system (1) has a unique positive $\omega$-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^T$ which is globally asymptotically stable, where*

$$\begin{aligned} C_1(t) &= A_{11}(t) - \frac{B_1^M D_2^M}{m_2} - \frac{1}{1-\tau_1'(\mu_1(t))} A_{31}(t+\tau_1(t)) \\ &\quad - \frac{1}{1-\tau_1'(\mu_1(t))} A_{31}(t+\tau_1(t)) \times \int_{t+\tau_1(t)}^{t+\tau_1(t)+\tau_2(t)} A_{33}(s)\,ds; \\ C_2(t) &= A_{22}(t) - \frac{B_2^M D_1^M}{m_1} - \frac{1}{1-\tau_1'(\mu_1(t))} \times A_{32}(t+\tau_1(t)) - \frac{1}{1-\tau_1'(\mu_1(t))} \\ &\quad \times A_{32}(t+\tau_1(t)) \int_{t+\tau_1(t)}^{t+\tau_1(t)+\tau_2(t)} A_{33}(s)\,ds; \\ C_3(t) &= A_{33}(t) - A_{13}(t) - A_{23}(t) \\ &\quad - [r_3(t) + A_{31}(t)M_1 + A_{32}(t)M_2 + A_{33}(t)M_3] \\ &\quad \times \int_t^{t+\tau_2(t)} A_{33}(s)\,ds - \frac{1}{1-\tau_1'(\mu_1(t))} \\ &\quad \times A_{33}(t+\tau_2(t)) \int_{t+\tau_2(t)}^{t+2\tau_2(t)} A_{33}(s)\,ds. \end{aligned} \tag{100}$$

*Proof.* Let $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^T$ be a positive $\omega$-periodic solution of (1); then $y^*(t) = (y_1^*(t), y_2^*(t), y_3^*(t))^T$, ($y_i^*(t) = \prod_{0<t_k<t}(1+c_{ik})^{-1} x_i^*(t)$) is the positive $\omega$-periodic solution of system (75), and let $y_i(t) = (y_1(t), y_2(t), y_3(t))^T$ be any positive solution of system (75) with the initial conditions (76). It follows from Lemmas 11 and 12 that there exist positive constants $T$, $M_i$, and $m_i$, such that, for all $t \ge T$,

$$m_i \le y_i^*(t) \le M_i, \qquad m_i \le y_i(t) \le M_i, \quad i = 1,2,3. \tag{101}$$

We define

$$V_1(t) = \left|\ln y_1^*(t) - \ln y_1(t)\right| + \left|\ln y_2^*(t) - \ln y_2(t)\right|. \tag{102}$$

Calculating the upper right derivative of $V_1(t)$ along solutions of (75), it follows that

$$\begin{aligned}
D^+V_1(t) &= \sum_{i=1}^{2}\left(\frac{\dot{y}_i^*(t)}{y_i^*(t)} - \frac{\dot{y}_i(t)}{y_i(t)}\right)\operatorname{sgn}\left(y_i^*(t) - y_i(t)\right)\\
&\le \operatorname{sgn}\left(y_1^*(t) - y_1(t)\right)\\
&\quad\times\Bigg\{ - A_{11}(t)\left(y_1^*(t) - y_1(t)\right)\\
&\quad - A_{13}(t)\left(y_3^*(t) - y_3(t)\right)\\
&\quad + B_2(t)D_1(t)\left(\frac{y_2^*(t)}{y_1^*(t)} - \frac{y_2(t)}{y_1(t)}\right)\Bigg\}\\
&\quad + \operatorname{sgn}\left(y_2^*(t) - y_2(t)\right)\\
&\quad\times\Bigg\{ - A_{22}(t)\left(y_2^*(t) - y_2(t)\right)\\
&\quad - A_{23}(t)\left(y_3^*(t) - y_3(t)\right)\\
&\quad + B_1(t)D_2(t)\left(\frac{y_1^*(t)}{y_2^*(t)} - \frac{y_1(t)}{y_2(t)}\right)\Bigg\}\\
&\le -A_1(t)\left|y_1^*(t) - y_1(t)\right| + \left(A_{13}(t) + A_{23}(t)\right)\\
&\quad\times\left|y_3^*(t) - y_3(t)\right| - A_2(t)\left|y_2^*(t) - y_2(t)\right|\\
&\quad + \overline{D}_1(t) + \overline{D}_2(t),
\end{aligned} \tag{103}$$

where

$$\begin{aligned}
&\overline{D}_1(t)\\
&= \begin{cases} B_1(t)D_1(t)\left(\dfrac{y_2^*(t)}{y_1^*(t)} - \dfrac{y_2(t)}{y_1(t)}\right), & \text{if } y_1^*(t) > y_1(t),\\ B_1(t)D_1(t)\left(\dfrac{y_2(t)}{y_1(t)} - \dfrac{y_2^*(t)}{y_1^*(t)}\right), & \text{if } y_1^*(t) \le y_1(t),\end{cases}\\
&\overline{D}_2(t)\\
&= \begin{cases} B_2(t)D_2(t)\left(\dfrac{y_1^*(t)}{y_2^*(t)} - \dfrac{y_1(t)}{y_2(t)}\right), & \text{if } y_2^*(t) > y_2(t),\\ B_2(t)D_2(t)\left(\dfrac{y_1(t)}{y_2(t)} - \dfrac{y_1^*(t)}{y_2^*(t)}\right), & \text{if } y_2^*(t) \le y_2(t).\end{cases}
\end{aligned} \tag{104}$$

We estimate $\overline{D}_1(t)$ under the following two cases:

(i) if $y_1^*(t) > y_1(t)$, then

$$\overline{D}_1(t) \le \frac{B_1^M D_1^M}{y_1^*(t)}\left(y_2^*(t) - y_2(t)\right) \le \frac{B_1^M D_1^M}{m_1}\left|y_2^*(t) - y_2(t)\right|; \tag{105}$$

(ii) if $y_1^*(t) \le y_1(t)$, then

$$\overline{D}_1(t) \le \frac{B_1^M D_1^M}{y_1(t)}\left(y_2(t) - y_2^*(t)\right) \le \frac{B_1^M D_1^M}{m_1}\left|y_2^*(t) - y_2(t)\right|. \tag{106}$$

Combining the conclusions in (i)-(ii), we obtain

$$\overline{D}_1(t) \le \frac{B_1^M D_1^M}{m_1}\left|y_2^*(t) - y_2(t)\right|. \tag{107}$$

A similar argument as in the discussion above shows that

$$\overline{D}_2(t) \le \frac{B_2^M D_2^M}{m_2}\left|y_1^*(t) - y_1(t)\right|. \tag{108}$$

It follows from (103), (107), and (108) that

$$\begin{aligned}
D^+V_1(t) &\le -\left[A_1(t) - \frac{B_2^M D_2^M}{m_2}\right]\left|y_1^*(t) - y_1(t)\right|\\
&\quad - \left[A_2(t) - \frac{B_1^M D_1^M}{m_1}\right]\left|y_2^*(t) - y_2(t)\right|\\
&\quad + \left(A_{13}(t) + A_{23}(t)\right)\left|y_3^*(t) - y_3(t)\right|.
\end{aligned} \tag{109}$$

We define $V_{21}(t) = |\ln y_3^*(t) - \ln y_3(t)|$. Calculating the upper right derivative of $V_{21}(t)$ along solutions of (75), it follows that

$$\begin{aligned}
D^+V_{21}(t) &= \left(\frac{\dot{y}_3^*(t)}{y_3^*(t)} - \frac{\dot{y}_3(t)}{y_3(t)}\right)\operatorname{sgn}\left(y_3^*(t) - y_3(t)\right)\\
&= \operatorname{sgn}\left(y_3^*(t) - y_3(t)\right)\\
&\quad\times\{-A_{33}(t)\left(y_3^*(t-\tau_2(t)) - y_3(t-\tau_2(t))\right)\\
&\quad + A_{31}(t)\left(y_1^*(t-\tau_1(t)) - y_1(t-\tau_1(t))\right)\\
&\quad + A_{32}(t)\left(y_2^*(t-\tau_1(t)) - y_2(t-\tau_1(t))\right)\}\\
&= \operatorname{sgn}\left(y_3^*(t) - y_3(t)\right)\\
&\quad\times\Bigg\{ - A_{33}(t)\left(y_3^*(t) - y_3(t)\right)\\
&\quad + A_{31}(t)\left(y_1^*(t-\tau_1(t)) - y_1(t-\tau_1(t))\right)\\
&\quad + A_{32}(t)\left(y_2^*(t-\tau_1(t)) - y_2(t-\tau_1(t))\right)\\
&\quad + A_{33}(t)\int_{t-\tau_2(t)}^{t}\left(\dot{y}_3^*(u) - \dot{y}_3(u)\right)du\Bigg\}\\
&\le -A_{33}(t)\left|y_3^*(t) - y_3(t)\right| + A_{31}(t)\\
&\quad\times\left|y_1^*(t-\tau_1(t)) - y_1(t-\tau_1(t))\right|\\
&\quad + A_{32}(t)\left|y_2^*(t-\tau_1(t)) - y_2(t-\tau_1(t))\right|\\
&\quad + A_{33}(t)\left|\int_{t-\tau_2(t)}^{t}\left(\dot{y}_3^*(u) - \dot{y}_3(u)\right)du\right|.
\end{aligned} \tag{110}$$

By substituting (75) into (110), we obtain

$$\begin{aligned}
D^{+}V_{21}(t) = & -A_{33}(t)\left|y_3^*(t)-y_3(t)\right| \\
& + A_{31}(t)\left|y_1^*(t-\tau_1(t))-y_1(t-\tau_1(t))\right| \\
& + A_{32}(t)\left|y_2^*(t-\tau_1(t))-y_2(t-\tau_1(t))\right| \\
& + A_{33}(t)\left|\int_{t-\tau_2(t)}^{t}\left[\left(-r_3(u) + A_{31}(u)y_1(u-\tau_1(t)) + A_{32}(u)y_2(u-\tau_1(t)) - A_{33}(u)y_3(u-\tau_2(t))\right)\right.\right. \\
& \times\left(y_3^*(u)-y_3(u)\right) + A_{31}(u)y_3^*(u)\times\left(y_1^*(u-\tau_1(t)) - y_1(u-\tau_1(t))\right) \\
& + A_{32}(u)y_3^*(u)\times\left(y_2^*(u-\tau_1(t)) - y_2(u-\tau_1(t))\right) \\
& \left.\left.- A_{33}(u)y_3^*(u)\times\left(y_3^*(u-\tau_2(t)) - y_3(u-\tau_2(t))\right)\right]du\right| \\
\leq & -A_{33}(t)\left|y_3^*(t)-y_3(t)\right| \\
& + A_{31}(t)\left|y_1^*(t-\tau_1(t))-y_1(t-\tau_1(t))\right| \\
& + A_{32}(t)\left|y_2^*(t-\tau_1(t))-y_2(t-\tau_1(t))\right| \\
& + A_{33}(t)\int_{t-\tau_2(t)}^{t}\left[\left(r_3(u)+A_{31}(u)M_1 + A_{32}(u)M_2 + A_{33}(u)M_3\right)\right. \\
& \times\left|y_3^*(u)-y_3(u)\right| + A_{31}(u)M_3 \times\left|y_1^*(u-\tau_1(t)) - y_1(u-\tau_1(t))\right| \\
& + A_{32}(u)M_3\times\left|y_2^*(u-\tau_1(t)) - y_2(u-\tau_1(t))\right| \\
& \left.+ A_{33}(u)M_3\times\left|y_3^*(u-\tau_2(t)) - y_3(u-\tau_2(t))\right|\right]du.
\end{aligned} \tag{111}$$

Define

$$\begin{aligned}
V_{22}(t) = & \int_{t-\tau_1(t)}^{t}\frac{1}{1-\tau_1'(\mu_1(s))}A_{31}(s+\tau_1(t)) \times\left|y_1^*(s)-y_1(s)\right|ds \\
& + \int_{t-\tau_1(t)}^{t}\frac{1}{1-\tau_1'(\mu_1(s))}A_{32}(s+\tau_1(t)) \times\left|y_2^*(s)-y_2(s)\right|ds \\
& + \int_{t}^{t+\tau_2(t)}\int_{s-\tau_2(t)}^{t}A_{33}(s) \times\left\{\left(r_3(u)+A_{31}(u)M_1 + A_{32}(u)M_2 + A_{33}(u)M_3\right)\right. \\
& \times\left|y_3^*(u)-y_3(u)\right| + A_{31}(u)M_3 \times\left|y_1^*(u-\tau_1(t)) - y_1(u-\tau_1(t))\right| \\
& + A_{32}(u)M_3 \times\left|y_2^*(u-\tau_1(t)) - y_2(u-\tau_1(t))\right| \\
& \left.+ A_{33}(u)M_3 \times\left|y_3^*(u-\tau_2(t)) - y_3(u-\tau_2(t))\right|\right\}du\,ds.
\end{aligned} \tag{112}$$

It follows from (111) and (112) that, for any $t \geq T+\tau$,

$$\begin{aligned}
D^{+}\left(V_{21}(t)+V_{22}(t)\right) \leq & -A_{33}(t)\left|y_3^*(t)-y_3(t)\right| \\
& + \frac{1}{1-\tau_1'(\mu_1(t))}A_{31}(t+\tau_1(t)) \times\left|y_1^*(t)-y_1(t)\right| + \frac{1}{1-\tau_1'(\mu_1(t))} \\
& \times A_{32}(t+\tau_1(t))\left|y_2^*(t)-y_2(t)\right| \\
& + \int_{t}^{t+\tau_2(t)}A_{33}(s)\,ds \\
& \times\left[\left(r_3(t)+A_{31}(t)M_1 + A_{32}(t)M_2 + A_{33}(t)M_3\right)\left|y_3^*(t)-y_3(t)\right|\right.
\end{aligned}$$

$$+ A_{31}(t) M_3 \times |y_1^*(t-\tau_1(t)) - y_1(t-\tau_1(t))| + A_{32}(t) M_3 |y_2^*(t-\tau_1(t)) - y_2(t-\tau_1(t))| + A_{33}(t) M_3 |y_3^*(t-\tau_2(t)) - y_3(t-\tau_2(t))| ]. \tag{113}$$

We also define

$$\begin{aligned} V_{23}(t) = {} & M_3 \int_{t-\tau_1(t)}^{t} \int_{u+\tau_1(t)}^{u+\tau_1(t)+\tau_2(t)} \left[ A_{33}(s) \times \frac{1}{1-\tau_1'(\mu_1(u))} \times A_{31}(u+\tau_1(t)) \times |y_1^*(u) - y_1(u)| + A_{33}(s) \frac{1}{1-\tau_1'(\mu_1(u))} \times A_{32}(u+\tau_1(t)) \times |y_2^*(u) - y_2(u)| \right] ds\, du \\ & + M_3 \int_{t-\tau_2(t)}^{t} \int_{u+\tau_2(t)}^{u+2\tau_2(t)} A_{33}(s) \frac{1}{1-\tau_2'(\mu_2(u))} \times A_{33}(u+\tau_2(t)) \times |y_3^*(u) - y_3(u)| \, ds\, du, \\ V_2(t) = {} & V_{21}(t) + V_{22}(t) + V_{23}(t). \end{aligned} \tag{114}$$

It follows from (112)–(114) that, for any $t \ge T + \tau_2$,

$$\begin{aligned} D^+ V_2(t) \le {} & -A_{33}(t) |y_3^*(t) - y_3(t)| + \frac{1}{1-\tau_1'(\mu_1(t))} \times A_{31}(t+\tau_1(t)) |y_1^*(t) - y_1(t)| \\ & + \frac{1}{1-\tau_1'(\mu_1(t))} A_{32}(t+\tau_1(t)) \times |y_2^*(t) - y_2(t)| \\ & + \left\{ \int_t^{t+\tau_2(t)} A_{33}(s)\, ds \left[ r_3(t) + A_{31}(t) M_1 + A_{32}(t) M_2 + A_{33}(t) M_3 \right] \times |y_3^*(t) - y_3(t)| \right. \\ & + \frac{1}{1-\tau'(\mu_1(t))} A_{31}(t+\tau_1(t)) M_3 \times \int_{t+\tau_1(t)}^{t+\tau_1(t)+\tau_2(t)} A_{33}(s)\, ds\, |y_1^*(t) - y_1(t)| \\ & + \frac{1}{1-\tau_1'(\mu_1(t))} A_{32}(t+\tau_1(t)) M_3 \times \int_{t+\tau_1(t)}^{t+\tau_1(t)+\tau_2(t)} A_{33}(s)\, ds\, |y_2^*(t) - y_2(t)| \\ & \left. + \frac{1}{1-\tau_2'(\mu_2(t))} A_{33}(t+\tau_2(t)) M_3 \times \int_{t+\tau_2(t)}^{t+2\tau_2(t)} A_{33}(s)\, ds\, |y_3^*(t) - y_3(t)| \right\}. \end{aligned} \tag{115}$$

We now define a Lyapunov functional $V(t)$ as

$$V(t) = V_1(t) + V_2(t). \tag{116}$$

Then it follows from (109), (115), and (116) that, for any $t \ge T + \tau_2$,

$$D^+ V(t) \le -\sum_{i=1}^{3} C_i(t) |y_i^*(t) - y_i(t)|, \tag{117}$$

where $C_i(t)$ $(i = 1, 2, 3)$ are defined in (100). By hypothesis, there exist positive constants $\alpha_i$, $i = 1, 2, 3$ and $T^* \ge T + \tau$ such that if $t \ge T^*$

$$B_i(t) \ge \alpha_i > 0, \quad i = 1, 2, 3. \tag{118}$$

Integrating both sides of (117) on interval $[T^*, t]$, we have

$$V(t) + \sum_{i=1}^{3} \int_{T^*}^{t} C_i(s) |y_i^*(s) - y_i(s)| \, ds \le V(T^*). \tag{119}$$

It follows from (118) and (119) that

$$V(t) + \alpha_i \sum_{i=1}^{3} \int_{T^*}^{t} |y_i^*(s) - y_i(s)| \, ds \le V(T^*), \quad \text{for } t \ge T^*. \tag{120}$$

Therefore, $V(t)$ is bounded on $[T^*, +\infty)$ and also $\int_{T^*}^{t} |y_i^*(s) - y_i(s)| ds < +\infty$ $(i = 1, 2, 3)$. Since $\dot{y}_i^*(t)$ and $\dot{y}_i(t)$ $(i = 1, 2, 3)$ are bounded for $t \ge T^*$, therefore, $|y_i^*(s) - y_i(s)|$ $(i = 1, 2, 3)$

are uniformly continuous on $[T^*, +\infty)$. By Lemma 13, we have

$$\lim_{t\to+\infty}\left|y_i^*(s)-y_i(s)\right| = \lim_{t\to+\infty}\left[\prod_{0<t_k<t}(1+c_{ik})^{-1} \times \left|x_i^*(s)-x_i(s)\right|\right] = 0, \quad i = 1,2,3. \tag{121}$$

Therefore

$$\lim_{t\to+\infty}\left|x_i^*(s)-x_i(s)\right| = 0, \quad i = 1,2,3. \tag{122}$$

By Theorems 7.4 and 8.2 in [33], we know that the periodic positive solution $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^T$ is uniformly asymptotically stable. The proof of Theorem 14 is completed. □

## 4. Some Examples

The following illustrative examples will demonstrate the effectiveness of our results.

*Example 1.* We consider the following delayed periodic Lotka-Volterra predator-prey system with prey dispersal and impulse:

$$\begin{aligned}
x_1'(t) &= x_1(t)\left[7+\sin t-8x_1(t)-2x_3(t)\right] \\
&\quad + (2+\sin t)\left(x_2(t)-x_1(t)\right), \\
x_2'(t) &= x_2(t)\left[5-\cos t-3x_2(t)-4x_3(t)\right] \\
&\quad + (3-\cos t)\left(x_1(t)-x_2(t)\right), \\
x_3'(t) &= x_3(t)\left[-2+\sin t+3x_1\left(t-\frac{\sin t}{5}\right)\right. \\
&\quad \left. +3x_2\left(t-\frac{\sin t}{5}\right)-6x_3\left(t-\frac{\cos t}{10}\right)\right], \\
&\qquad t \neq t_k, \\
\Delta x_i(t) &= x_i(t^+)-x_i(t) = c_{ik}x_i(t), \\
&\quad i = 1,2,3, \quad k = 1,2,\ldots, \quad t = t_k.
\end{aligned} \tag{123}$$

We fix the parameters $c_{ik} = e^{-1/2}-1$, $i = 1,2,3$, $t_{k+2} = t_k + 2\pi$, $[0,2\pi] \cap \{t_k\} = \{t_1, t_2\}$. It is easy to see that $\overline{(r_1-D_1)}\omega = 10\pi$, $\overline{(r_2-D_2)}\omega = 4\pi$, $\overline{r_3}\omega = 4\pi$, $a_{11} = 8$, $a_{13} = 2$, $a_{22} = 3$, $a_{23} = 4$, $a_{31} = a_{32} = 3$, $a_{33} = 6$, $D_1^M = 3$, $D_2^M = 4$, $c_i = -1$, $i = 1,2,3$, $A = 2$. Thus we have

$$\begin{aligned}
&\overline{(r_1-D_1)}\omega - \frac{\left[a_{13}^M\left(a_{31}^M+a_{32}^M\right)A\right]\omega}{a_{33}^L}+c_1 \\
&\quad = 2\pi-1>0, \\
&a_{22}^M a_{31}^L\left(\overline{(r_1-D_1)}\omega+c_1\right)+a_{11}^M a_{32}^L\left(\overline{(r_2-D_2)}\omega+c_2\right) \\
&\quad - a_{11}^M a_{22}^M\left(\overline{r_3}\omega+c_3\right) = 9(10\pi-1)>0.
\end{aligned} \tag{124}$$

According to Theorem 7, we see that system (123) has at least one positive $2\pi$-periodic solution.

*Example 2.* We consider another delayed periodic Lotka-Volterra diffusive predator-prey model with impulse:

$$\begin{aligned}
x_1'(t) &= x_1(t)\left[9-\cos t-(7+2\sin t)x_1(t)\right. \\
&\quad \left.-(2-\cos t)x_3(t)\right] \\
&\quad + (1-\cos t)\left(x_2(t)-x_1(t)\right), \\
x_2'(t) &= x_2(t)\left[6+\sin t-(4-\cos t)x_2(t)\right. \\
&\quad \left.-(3+\sin t)x_3(t)\right] \\
&\quad + (1+\sin t)\left(x_1(t)-x_2(t)\right), \\
x_3'(t) &= x_3(t)\left[-5-\cos t+(2+\sin t)x_1\left(t-\frac{\cos t}{10}\right)\right. \\
&\quad + (5-\cos t)x_2\left(t-\frac{\cos t}{10}\right) \\
&\quad \left.-(10-\sin t)x_3\left(t-\frac{\sin t}{100}\right)\right], \\
&\qquad t \neq t_k, \\
\Delta x_i(t) &= x_i(t^+)-x_i(t) = c_{ik}x_i(t), \\
&\quad i = 1,2,3, \quad k = 1,2,\ldots, \quad t = t_k.
\end{aligned} \tag{125}$$

We fix the parameters $c_{ik} = e^{-3\pi/2}-1$, $i = 1,2,3$, $t_{k+2} = t_k + 2\pi$,$[0,2\pi] \cap \{t_k\} = \{t_1, t_2\}$. It is easy to see that $\overline{(r_1-D_1)}\omega = 16\pi$, $\overline{(r_2-D_2)}\omega = 10\pi$, $\overline{r_3}\omega = 10\pi$, $a_{11}^M = 9$, $a_{11}^L = 5$, $a_{13}^M = 3$, $a_{13}^L = 1$, $a_{22}^M = 5$, $a_{22}^L = 3$, $a_{23}^M = 4$, $a_{23}^L = 2$, $a_{31}^M = 3$, $a_{31}^L = 1$,

$a_{32}^M = 6, a_{32}^L = 4, a_{33}^M = 11, a_{33}^L = 9, D_1^M = 2, D_2^M = 2, c_i = -3\pi$, $i = 1, 2, 3$, $A = 2$. Thus we have

$$\begin{aligned}
\overline{(r_1 - D_1)}\omega &- \frac{\left[a_{13}^M\left(a_{31}^M + a_{32}^M\right)A\right]\omega}{a_{33}^L} + c_1 \\
&= 16\pi - \frac{3*(3+6)*2*2\pi}{9} \\
&\quad - 3\pi = \pi > 0, \\
a_{22}^M a_{31}^L &\left(\overline{(r_1 - D_1)}\omega + c_1\right) + a_{11}^M a_{32}^L\left(\overline{(r_2 - D_2)}\omega + c_2\right) \\
&- a_{11}^M a_{22}^M\left(\overline{r_3}\omega + c_3\right) \\
&= 5*1*(16\pi - 3\pi) \\
&\quad + 9*4*(10\pi - 3\pi) - 9*5*(10\pi - 3\pi) = 2\pi > 0.
\end{aligned} \tag{126}$$

According to Theorem 7, we see that model (125) has at least one positive $2\pi$-periodic solution.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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