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Research Article

Determinants of the RFMLR Circulant Matrices with Perrin, Padovan, Tribonacci, and the Generalized Lucas Numbers

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The row first-minus-last right (RFMLR) circulant matrix and row last-minus-first left (RLMFL) circulant matrices are two special pattern matrices. By using the inverse factorization of polynomial, we give the exact formulae of determinants of the two pattern matrices involving Perrin, Padovan, Tribonacci, and the generalized Lucas sequences in terms of finite many terms of these sequences.

1. Introduction

Several special matrices arise frequently in many fields including image processing, communications, medicine, and signal encoding [1]. The application of a block-circulant matrix approach for singular value decomposition rendered the analysis independent of tracer arrival time to improve the results in [2]. Yin et al. introduced fast algorithms for reconstructing signals from incomplete Toeplitz and circulant measurements and showed that Toeplitz and circulant matrices not only were as effective as random matrices for signal encoding but also permitted much faster decoding in [3]. Wu et al. proposed a technique that was made time-shift insensitive by the use of a block-circulant matrix for deconvolution with (oSVD) and without (cSVD) minimization of oscillation of the derived residue function in [4].

The circulant matrices [5, 6], a fruitful subject of research, have in recent years been extended in many directions. The $f(x)$-circulant matrices are another natural extension of this well-studied class and can be found in [7–10]. The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes [7]. The properties and structures of the $x^n + x - 1$-circulant matrices, which are called the row first-minus-last right (RFMLR) circulant matrices, are better than those of the general $f(x)$-circulant matrices, so it is significant that we give our attention to them. We first introduce the definitions of the row first-minus-last right (RFMLR) circulant matrices and row last-minus-first left (RLMFL) circulant matrices. As regards their more properties, please refer to [11, 12].

Definition 1. A row first-minus-last right (RFMLR) circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$, denoted by RFMLRcircfr$(a_1, a_2, \ldots, a_n)$, is meant to be a square matrix of the form

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 - a_n & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_3 & a_4 - a_3 & \cdots & a_2 \\ a_2 & a_3 - a_2 & \cdots & a_1 - a_n \end{pmatrix}. \tag{1}$$

We define matrix $\mathbf{\Theta}_{(1,-1)}$ as the basic RFMLR circulant matrix; that is,

$$\mathbf{\Theta}_{(1,-1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix} = \text{RFMLRcircfr}(0, 1, 0, \ldots, 0). \tag{2}$$

Definition 2. A row last-minus-first left (RLMFL) circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$, denoted by RLMFLcircfr$(a_1, a_2, \ldots, a_n)$, is meant to be a square matrix of the form

$$\mathbf{B} = \begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ a_2 & \cdots & a_n - a_1 & a_1 \\ \vdots & \iddots & \vdots & \vdots \\ a_{n-1} & \cdots & a_{n-3} - a_{n-2} & a_{n-2} \\ a_n - a_1 & \cdots & a_{n-2} - a_{n-1} & a_{n-1} \end{pmatrix}. \quad (3)$$

Let $\mathbf{A} = \text{RFMLRcircfr}(a_n, a_{n-1}, \ldots, a_1)$ and $\mathbf{B} = \text{RLMFLcircfr}(a_1, a_2, \ldots, a_n)$. By explicit computation, we find

$$\mathbf{B} = \mathbf{A}\hat{\mathbf{I}}, \quad (4)$$

where $\hat{\mathbf{I}}$ is the backward identity matrix of the form

$$\hat{\mathbf{I}} = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}_{n\times n}. \quad (5)$$

There are many interests in properties and generalization of some special matrices with famous numbers. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [13]. Lin gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices [14]. Lind presented the determinants of circulant and skew circulant involving Fibonacci numbers in [15]. Shen et al. [16] discussed the determinant of circulant matrix involving Fibonacci and Lucas numbers. Akbulak and Bozkurt [17] gave the norms of Toeplitz involving Fibonacci and Lucas numbers.

The determinant problems of the row first-minus-last right (RFMLR) circulant matrices and row last-minus-first left (RLMFL) circulant matrices involving the Perrin, Padovan, Tribonacci, and the generalized Lucas sequences are considered in this paper. The exact formulae of determinants are presented by using some terms of these sequences. The techniques used herein are based on the inverse factorization of polynomial.

The Perrin and Padovan sequences $\{R_n\}$ and $\{\mathbb{P}_n\}$ [18–20] are defined by a third-order recurrence:

$$R_n = R_{n-2} + R_{n-3}, \quad n \ge 3, \quad (6)$$

$$\mathbb{P}_n = \mathbb{P}_{n-2} + \mathbb{P}_{n-3}, \quad n \ge 3, \quad (7)$$

with the initial conditions $R_0 = 3$, $R_1 = 0$, and $R_2 = 2$, and $\mathbb{P}_0 = 1$, $\mathbb{P}_1 = 1$, and $\mathbb{P}_2 = 1$.

The Tribonacci and the generalized Lucas sequences $\{T_n\}$ and $\{\mathbb{L}_n\}$ [20, 21] are defined by a third-order recurrence:

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3}, \quad n \ge 3, \\ \mathbb{L}_n &= \mathbb{L}_{n-1} + \mathbb{L}_{n-2} + \mathbb{L}_{n-3}, \quad n \ge 3, \end{aligned} \quad (8)$$

with the initial conditions $T_0 = 0$, $T_1 = 1$, and $T_2 = 1$ and $\mathbb{L}_0 = 3$, $\mathbb{L}_1 = 1$, and $\mathbb{L}_2 = 3$.

The first few members of these sequences are given as follows:

$$\begin{array}{cccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ R_n & 3 & 0 & 2 & 3 & 2 & 5 & 5 \\ \mathbb{P}_n & 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ T_n & 0 & 1 & 1 & 2 & 4 & 7 & 13 \\ \mathbb{L}_n & 3 & 1 & 3 & 7 & 11 & 21 & 39. \end{array} \quad (9)$$

Recurrences (6) and (7) involve the characteristic equation $x^3 - x - 1 = 0$. If its roots are denoted by r_1, r_2, r_3, then the following equalities can be derived:

$$\begin{aligned} r_1 + r_2 + r_3 &= 0, \\ r_1 r_2 + r_1 r_3 + r_2 r_3 &= -1, \\ r_1 r_2 r_3 &= 1. \end{aligned} \quad (10)$$

Moreover, the Binet form for the Perrin sequence is

$$R_n = r_1^n + r_2^n + r_3^n, \quad (11)$$

and the Binet form for Padovan sequence is

$$\mathbb{P}_n = a_1 r_1^n + a_2 r_2^n + a_3 r_3^n, \quad (12)$$

where

$$a_i = \prod_{\substack{j=1 \\ j \ne i}}^{3} \frac{r_j - 1}{r_i - r_j}, \quad i = 1, 2, 3. \quad (13)$$

Recurrences (8) as well imply the characteristic equation $x^3 - x^2 - x - 1 = 0$. If its roots are denoted by t_1, t_2, t_3, then we have

$$\begin{aligned} t_1 + t_2 + t_3 &= 1, \\ t_1 t_2 + t_1 t_3 + t_2 t_3 &= -1, \\ t_1 t_2 t_3 &= 1. \end{aligned} \quad (14)$$

Furthermore, an exact expression for the nth Tribonacci number can be given explicitly by

$$\begin{aligned} T_n &= \frac{t_1^{n+1}}{(t_1 - t_2)(t_1 - t_3)} + \frac{t_2^{n+1}}{(t_2 - t_1)(t_2 - t_3)} \\ &\quad + \frac{t_3^{n+1}}{(t_3 - t_1)(t_3 - t_2)} \\ &= \frac{t_1^n}{-t_1^2 + 4t_1 - 1} + \frac{t_2^n}{-t_2^2 + 4t_2 - 1} + \frac{t_3^n}{-t_3^2 + 4t_3 - 1}. \end{aligned} \quad (15)$$

This can be written in a slightly more concise form (the Binet form) as

$$T_n = b_1 t_1^n + b_2 t_2^n + b_3 t_3^n, \quad (16)$$

where b_i is the ith root of the polynomial $44y^3 - 2y - 1$. And the Binet form for the generalized Lucas sequence is

$$\mathbb{L}_n = t_1^n + t_2^n + t_3^n. \quad (17)$$

2. Main Results

By Proposition 5.1 in [7] and properties of RFMLR circulant matrices [12], we deduce the following lemma.

Lemma 3. *Let* $\mathbf{A} = RFMLRcircfr(a_1, a_2, \ldots, a_n)$ *and* ε_i $(i = 1, 2, \ldots, n)$ *be the roots of the characteristic equation of* $\mathbf{\Theta}_{(1,-1)}$. *Then the eigenvalues of* $\mathbf{A}$ *are given by*

$$\lambda_i = f(\varepsilon_i) = \sum_{j=1}^{n} a_j \varepsilon_i^{j-1}, \quad i = 1, 2, \ldots, n, \tag{18}$$

and the determinant of $\mathbf{A}$ *is given by*

$$\det \mathbf{A} = \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} \sum_{j=1}^{n} a_j \varepsilon_i^{j-1}. \tag{19}$$

Lemma 4. *Suppose* ε_i $(i = 1, 2, \ldots, n)$ *are the roots of the characteristic equation of* $\mathbf{\Theta}_{(1,-1)}$. *If* $a = 0$, *then*

$$\begin{aligned}\prod_{i=1}^{n} \left(a\varepsilon_i^3 + b\varepsilon_i^2 + c\varepsilon_i + d\right) &= \prod_{i=1}^{n} \left(b\varepsilon_i^2 + c\varepsilon_i + d\right) \\ &= d^n + b^{n-1}(b + c + d) \\ &\quad + d\left(\delta_1^{n-1} + \delta_2^{n-1}\right) - \left(\delta_1^n + \delta_2^n\right),\end{aligned} \tag{20}$$

where $a, b, c \in \mathbb{R}$ *and*

$$\begin{aligned}\delta_1 &= \frac{-c + \sqrt{c^2 - 4bd}}{2}, \\ \delta_2 &= \frac{-c - \sqrt{c^2 - 4bd}}{2}.\end{aligned} \tag{21}$$

If $a \neq 0$, *then*

$$\begin{aligned}&\prod_{i=1}^{n} \left(a\varepsilon_i^3 + b\varepsilon_i^2 + c\varepsilon_i + d\right) \\ &\quad = \frac{(-a)^n}{2} \left(-\Delta_n^2 + \Delta_{2n} + 2\Delta_{n+1} + 2\Delta_n\right) \\ &\qquad + \frac{(-a)^{n-1}}{2} d\left(\Delta_{n-1}^2 - \Delta_{2(n-1)} + 2\Delta_{n-1}\right) \\ &\qquad + (-a)^{n-1}\left(-b\Delta_n + a + b + c + d\right) + d^n,\end{aligned} \tag{22}$$

where $\Delta_n = \delta_1^n + \delta_2^n + \delta_3^n$, *and* δ_1, δ_2, δ_3 *are the roots of the equation* $ax^3 + bx^2 + cx + d = 0$.

Proof. Since ε_i $(i = 1, 2, \ldots, n)$ are the roots of the characteristic equation of $\mathbf{\Theta}_{(1,-1)}$, $g(x) = x^n + x - 1$ can be factored as

$$x^n + x - 1 = \prod_{i=1}^{n} (x - \varepsilon_i). \tag{23}$$

Let $\delta_1, \delta_2, \delta_3$ be the roots of the equation $ax^3 + bx^2 + cx + d = 0$.

If $a = 0$, please see [12] for details of the proof.

If $a \neq 0$, then

$$\begin{aligned}&\prod_{i=1}^{n} \left(a\varepsilon_i^3 + b\varepsilon_i^2 + c\varepsilon_i + d\right) \\ &\quad = a^n \prod_{i=1}^{n} \left(\varepsilon_i^3 + \frac{b}{a}\varepsilon_i^2 + \frac{c}{a}\varepsilon_i + \frac{d}{a}\right) \\ &\quad = a^n \prod_{i=1}^{n} (\varepsilon_i - \delta_1)(\varepsilon_i - \delta_2)(\varepsilon_i - \delta_3) \\ &\quad = (-a)^n \prod_{i=1}^{n} (\delta_1 - \varepsilon_i) \prod_{i=1}^{n} (\delta_2 - \varepsilon_i) \prod_{i=1}^{n} (\delta_3 - \varepsilon_i) \\ &\quad = (-a)^n \left(\delta_1^n + \delta_1 - 1\right)\left(\delta_2^n + \delta_2 - 1\right)\left(\delta_3^n + \delta_3 - 1\right) \\ &\quad = (-a)^n \left\{(\delta_1\delta_2\delta_3)^n + \delta_1\delta_2\delta_3\left[(\delta_1\delta_2)^{n-1} + (\delta_1\delta_3)^{n-1} \right.\right. \\ &\qquad\qquad \left. + (\delta_2\delta_3)^{n-1}\right] \\ &\qquad - \left[(\delta_1\delta_2)^n + (\delta_1\delta_3)^n + (\delta_2\delta_3)^n\right] \\ &\qquad + \delta_1\delta_2\delta_3\left(\delta_1^{n-1} + \delta_2^{n-1} + \delta_3^{n-1}\right) \\ &\qquad - \left[\delta_1^n(\delta_2 + \delta_3) + \delta_2^n(\delta_1 + \delta_3) \right. \\ &\qquad\quad \left. + \delta_3^n(\delta_1 + \delta_2)\right] + \left(\delta_1^n + \delta_2^n + \delta_3^n\right) \\ &\qquad + \delta_1\delta_2\delta_3 - (\delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3) \\ &\qquad \left. + (\delta_1 + \delta_2 + \delta_3) - 1\right\}.\end{aligned} \tag{24}$$

Let $\Delta_n = \delta_1^n + \delta_2^n + \delta_3^n$. We derive $(\delta_1\delta_2)^n + (\delta_1\delta_3)^n + (\delta_2\delta_3)^n = (\Delta_n^2 - \Delta_{2n})/2$ from $(\delta_1^n + \delta_2^n + \delta_3^n)^2 = \delta_1^{2n} + \delta_2^{2n} + \delta_3^{2n} + 2[(\delta_1\delta_2)^n + (\delta_1\delta_3)^n + (\delta_2\delta_3)^n]$. Taking the relation of roots and coefficients

$$\begin{aligned}\delta_1 + \delta_2 + \delta_3 &= -\frac{b}{a} \\ \delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3 &= \frac{c}{a} \\ \delta_1\delta_2\delta_3 &= -\frac{d}{a}\end{aligned} \tag{25}$$

into account, we deduce that

$$\begin{aligned}&\prod_{i=1}^{n} \left(a\varepsilon_i^3 + b\varepsilon_i^2 + c\varepsilon_i + d\right) \\ &\quad = \frac{(-a)^n}{2} \left(-\Delta_n^2 + \Delta_{2n} + 2\Delta_{n+1} + 2\Delta_n\right) \\ &\qquad + \frac{(-a)^{n-1}}{2} d\left(\Delta_{n-1}^2 - \Delta_{2(n-1)} + 2\Delta_{n-1}\right) \\ &\qquad + (-a)^{n-1}\left(-b\Delta_n + a + b + c + d\right) + d^n.\end{aligned} \tag{26}$$

The proof is completed. □

We present the exact formulae of determinants of the RFMLR and RLMFL circulant matrices involving four kinds of famous numbers and the detailed process.

2.1. Determinants of the RFMLR and RLMFL Circulant Matrix Involving Perrin Sequence

Theorem 5. *Let* $\mathbf{C} = RFMLRcircfr(R_1, R_2, \ldots, R_n)$. *If* n *is odd, then*

$$\det \mathbf{C} = \frac{(-\mathscr{X}_1 + \mathscr{X}_2 + \mathscr{X}_3)}{\mathscr{Y}_1}, \quad (27)$$

and if n *is even, then*

$$\det \mathbf{C} = \frac{(\mathscr{X}_1 - \mathscr{X}_2 - \mathscr{X}_3)}{\mathscr{Y}_1}, \quad (28)$$

where

$$\begin{aligned}
\mathscr{X}_1 &= R_n^n \left(X_n^2 - X_{2n} - 2X_{n+1}\right), \\
\mathscr{X}_2 &= R_{n+1} R_n^{n-1} \left(X_{n-1}^2 - X_{2(n-1)} + 2X_{n-1}\right), \\
\mathscr{X}_3 &= 2R_n^{n-1} \left(R_{n+2} X_n + 3X_n - 5\right) + 2R_{n+1}^n, \\
\mathscr{Y}_1 &= Y_n^2 - Y_{n-1}^2 - Y_{2n} + Y_{2(n-1)} \\
&\quad -2Y_{n+1} - 4Y_n - 2Y_{n-1}, \\
X_n &= x_1^n + x_2^n + x_3^n, \\
Y_n &= y_1^n + y_2^n + y_3^n,
\end{aligned} \quad (29)$$

where x_1, x_2, x_3 *and* y_1, y_2, y_3 *are the roots of the equation* $R_n x^3 + (R_{n-1} + 3)x^2 - (R_{n-3} - 2)x - R_{n+1} = 0$, $x^3 + x^2 - 1 = 0$, *respectively.*

Proof. Obviously, $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{pmatrix} R_1 & R_2 & \ldots & R_n \\ R_n & R_1 - R_n & \ldots & R_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_3 & R_4 - R_3 & \ldots & R_2 \\ R_2 & R_3 - R_2 & \ldots & R_1 - R_n \end{pmatrix}. \quad (30)$$

In the light of Lemma 3 and the Binet form (11) and (10), we have

$$\begin{aligned}
\det \mathbf{C} &= \prod_{i=1}^{n} \left(R_1 + R_2 \varepsilon_i + \cdots + R_n \varepsilon_i^{n-1}\right) \\
&= \prod_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{3} r_j^k \varepsilon_i^{k-1} \\
&= \prod_{i=1}^{n} \sum_{j=1}^{3} \frac{r_j \left(1 - r_j^n \varepsilon_i^n\right)}{1 - r_j \varepsilon_i} \\
&= \prod_{i=1}^{n} \frac{R_n \varepsilon_i^3 + \left(R_{n-1} + 3\right) \varepsilon_i^2}{-\varepsilon_i^3 - \varepsilon_i^2 + 1} \\
&\quad - \frac{\left(R_{n-3} - 2\right) \varepsilon_i + R_{n+1}}{-\varepsilon_i^3 - \varepsilon_i^2 + 1}.
\end{aligned} \quad (31)$$

By Lemma 4 and recurrence (6), we obtain

$$\begin{aligned}
&\prod_{i=1}^{n} \left[R_n \varepsilon_i^3 + \left(R_{n-1} + 3\right) \varepsilon_i^2 - \left(R_{n-3} - 2\right) \varepsilon_i - R_{n+1}\right] \\
&\quad = \left(-R_n\right)^{n-1} \left[R_n X_n^2 - R_{n+1} X_{n-1}^2 - R_n X_{2n}\right. \\
&\qquad + R_{n+1} X_{2(n-1)} - 2R_n X_{n+1} \\
&\qquad \left. - 2\left(R_{n+2} + 3\right) X_n - 2R_{n+1} X_{n-1} + 10\right] \times 2^{-1} \\
&\quad + \left(-R_{n+1}\right)^n,
\end{aligned} \quad (32)$$

where $X_n = x_1^n + x_2^n + x_3^n$ and x_1, x_2, x_3 are the roots of the equation $R_n x^3 + (R_{n-1} + 3)x^2 - (R_{n-3} - 2)x - R_{n+1} = 0$. And

$$\begin{aligned}
\prod_{i=1}^{n} \left(-\varepsilon_i^3 - \varepsilon_i^2 + 1\right) &= \frac{1}{2} \left(-Y_n^2 + Y_{n-1}^2 + Y_{2n} - Y_{2(n-1)}\right) \\
&\quad + Y_{n+1} + 2Y_n + Y_{n-1},
\end{aligned} \quad (33)$$

where $Y_n = y_1^n + y_2^n + y_3^n$ and y_1, y_2, y_3 are the roots of the equation $x^3 + x^2 - 1 = 0$. Consequently, if n is odd, then

$$\det \mathbf{C} = \frac{(-\mathscr{X}_1 + \mathscr{X}_2 + \mathscr{X}_3)}{\mathscr{Y}_1}, \quad (34)$$

and if n is even, then

$$\det \mathbf{C} = \frac{(\mathscr{X}_1 - \mathscr{X}_2 - \mathscr{X}_3)}{\mathscr{Y}_1}, \quad (35)$$

where

$$\begin{aligned}
\mathscr{X}_1 &= R_n^n \left(X_n^2 - X_{2n} - 2X_{n+1}\right), \\
\mathscr{X}_2 &= R_{n+1} R_n^{n-1} \left(X_{n-1}^2 - X_{2(n-1)} + 2X_{n-1}\right), \\
\mathscr{X}_3 &= 2R_n^{n-1} \left(R_{n+2} X_n + 3X_n - 5\right) + 2R_{n+1}^n, \\
\mathscr{Y}_1 &= Y_n^2 - Y_{n-1}^2 - Y_{2n} + Y_{2(n-1)} \\
&\quad - 2Y_{n+1} - 4Y_n - 2Y_{n-1}.
\end{aligned} \quad (36)$$

The proof is completed. □

Theorem 6. *Let* $\mathbf{D} = RFMLRcircfr(R_n, \ldots, R_1)$. *Then*

$$\det \mathbf{D} = \frac{2\left[\alpha_4 + \left(R_n - 3\right)\left(\alpha_1^{n-1} + \alpha_2^{n-1}\right) - \left(\alpha_1^n + \alpha_2^n\right)\right]}{-R_{n+2} R_{n-5} + R_{2n-3} + 2R_{n+4} + 4}, \quad (37)$$

where

$$\alpha_1 = \frac{\left(-R_{n+2} - 1 + \sqrt{\alpha_3}\right)}{2},$$
$$\alpha_2 = \frac{\left(-R_{n+2} - 1 - \sqrt{\alpha_3}\right)}{2},$$
$$\alpha_3 = \left(R_{n+2} + 1\right)^2 - 4\left(R_{n+1} + 2\right)\left(R_n - 3\right),$$
$$\alpha_4 = \left(R_n - 3\right)^n + R_{n+5}\left(R_{n+1} + 2\right)^{n-1}. \quad (38)$$

Proof. The matrix **D** has the form

$$\mathbf{D} = \begin{pmatrix} R_n & R_{n-1} & \cdots & R_1 \\ R_1 & R_n - R_1 & \cdots & R_2 \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-2} & R_{n-3} - R_{n-2} & \cdots & R_{n-1} \\ R_{n-1} & R_{n-2} - R_{n-1} & \cdots & R_n - R_{n-1} \end{pmatrix}. \quad (39)$$

According to Lemma 3 and the Binet form (11) and (10), we have

$$\begin{aligned} \det \mathbf{D} &= \prod_{i=1}^{n} \left(R_n + R_{n-1}\varepsilon_i + \cdots + R_1 \varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n} \sum_{k=0}^{n-1} \sum_{j=1}^{3} r_j^{n-k} \varepsilon_i^k \\ &= \prod_{i=1}^{n} \sum_{j=1}^{3} \frac{r_j^{n+1} - r_j \varepsilon_i^n}{r_j - \varepsilon_i} \\ &= \prod_{i=1}^{n} \frac{\left(R_{n+1} + 2\right)\varepsilon_i^2 + \left(R_{n+2} + 1\right)\varepsilon_i + R_n - 3}{-\varepsilon_i^3 + \varepsilon_i + 1}. \end{aligned} \quad (40)$$

Using Lemma 4 and recurrence (6), we obtain

$$\begin{aligned} &\prod_{i=1}^{n} \left[\left(R_{n+1} + 2\right)\varepsilon_i^2 + \left(R_{n+2} + 1\right)\varepsilon_i + R_n - 3\right] \\ &\quad = \alpha_4 + \left(R_n - 3\right)\left(\alpha_1^{n-1} + \alpha_2^{n-1}\right) - \left(\alpha_1^n + \alpha_2^n\right), \end{aligned} \quad (41)$$

where

$$\alpha_1 = \frac{\left(-R_{n+2} - 1 + \sqrt{\alpha_3}\right)}{2},$$
$$\alpha_2 = \frac{\left(-R_{n+2} - 1 - \sqrt{\alpha_3}\right)}{2},$$
$$\alpha_3 = \left(R_{n+2} + 1\right)^2 - 4\left(R_{n+1} + 2\right)\left(R_n - 3\right),$$
$$\alpha_4 = \left(R_n - 3\right)^n + R_{n+5}\left(R_{n+1} + 2\right)^{n-1}, \quad (42)$$

$$\begin{aligned} &\prod_{i=1}^{n} \left(-\varepsilon_i^3 + \varepsilon_i + 1\right) \\ &\quad = -\frac{1}{2}\left(R_{n+2}R_{n-5} - R_{2n-3} - 2R_{n+4} - 4\right). \end{aligned} \quad (43)$$

Therefore,

$$\det \mathbf{D} = \frac{2\left[\alpha_4 + \left(R_n - 3\right)\left(\alpha_1^{n-1} + \alpha_2^{n-1}\right) - \left(\alpha_1^n + \alpha_2^n\right)\right]}{-R_{n+2}R_{n-5} + R_{2n-3} + 2R_{n+4} + 4}. \quad (44)$$

The proof is completed. □

Corollary 7. *Let* $\mathbf{E} = RLMFLcircfr(R_1, R_2, \ldots, R_n)$. *If* $n \equiv 0\,(mod\,4)$ *or* $n \equiv 1\,(mod\,4)$, *then*

$$\det \mathbf{E} = \frac{2\left[\alpha_4 + \left(R_n - 3\right)\left(\alpha_1^{n-1} + \alpha_2^{n-1}\right) - \left(\alpha_1^n + \alpha_2^n\right)\right]}{-R_{n+2}R_{n-5} + R_{2n-3} + 2R_{n+4} + 4}, \quad (45)$$

and if $n \equiv 2\,(mod\,4)$ *or* $n \equiv 3\,(mod\,4)$, *then*

$$\det \mathbf{E} = \frac{2\left[\alpha_4 + \left(R_n - 3\right)\left(\alpha_1^{n-1} + \alpha_2^{n-1}\right) - \left(\alpha_1^n + \alpha_2^n\right)\right]}{R_{n+2}R_{n-5} - R_{2n-3} - 2R_{n+4} - 4}, \quad (46)$$

where α_1, α_2, α_3, α_4 *are defined the same as Theorem 6.*

Proof. Since

$$\det \hat{\mathbf{I}} = (-1)^{n(n-1)/2}, \quad (47)$$

the result can be derived from Theorem 6 and relation (4). □

2.2. Determinants of the RFMLR and RLMFL Circulant Matrix Involving Padovan Sequence

Theorem 8. *Let* $\mathbf{F} = RFMLRcircfr(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$. *If n is odd, then*

$$\det \mathbf{F} = \frac{\left(-\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3\right)}{\mathcal{Y}_1}, \quad (48)$$

and if n is even, then

$$\det \mathbf{F} = \frac{\left(\mathcal{Q}_1 - \mathcal{Q}_2 - \mathcal{Q}_3\right)}{\mathcal{Y}_1}, \quad (49)$$

where

$$\mathcal{Q}_1 = \mathbb{P}_n^n \left(\mathrm{Q}_n^2 - \mathrm{Q}_{2n} - 2\mathrm{Q}_{n+1}\right),$$
$$\mathcal{Q}_2 = \left(\mathbb{P}_{n+1} - 1\right)\mathbb{P}_n^{n-1}\left(\mathrm{Q}_{n-1}^2 - \mathrm{Q}_{2(n-1)} + 2\mathrm{Q}_{n-1}\right),$$
$$\mathcal{Q}_3 = 2\mathbb{P}_n^{n-1}\left(\mathbb{P}_{n+2}\mathrm{Q}_n + \mathrm{Q}_n - 3\right) + 2\left(\mathbb{P}_{n+1} - 1\right)^n,$$
$$\mathrm{Q}_n = q_1^n + q_2^n + q_3^n, \quad (50)$$

where q_1, q_2, q_3 *are the roots of the equation* $\mathbb{P}_n x^3 + (1 + \mathbb{P}_{n-1})x^2 + (1 - \mathbb{P}_{n-3})x + 1 - \mathbb{P}_{n+1} = 0$ *and* $\mathcal{Y}_1$ *is defined as Theorem 5.*

Proof. The matrix **F** has the form

$$\mathbf{F} = \begin{pmatrix} \mathbb{P}_1 & \mathbb{P}_2 & \cdots & \mathbb{P}_n \\ \mathbb{P}_n & \mathbb{P}_1 - \mathbb{P}_n & \cdots & \mathbb{P}_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}_3 & \mathbb{P}_4 - \mathbb{P}_3 & \cdots & \mathbb{P}_2 \\ \mathbb{P}_2 & \mathbb{P}_3 - \mathbb{P}_2 & \cdots & \mathbb{P}_1 - \mathbb{P}_n \end{pmatrix}. \quad (51)$$

The determinant of **F** is

$$\det \mathbf{F} = \prod_{i=1}^{n}\left(\mathbb{P}_1 + \mathbb{P}_2\varepsilon_i + \cdots + \mathbb{P}_n\varepsilon_i^{n-1}\right)$$
$$= \prod_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{3} a_j r_j^k \varepsilon_i^{k-1}$$
$$= \prod_{i=1}^{n}\sum_{j=1}^{3}\frac{a_j r_j\left(1 - r_j^n\varepsilon_i^n\right)}{1 - r_j\varepsilon_i} \tag{52}$$
$$= \prod_{i=1}^{n}\left[\frac{\mathbb{P}_n\varepsilon_i^3 + \left(1 + \mathbb{P}_{n-1}\right)\varepsilon_i^2 + \left(1 - \mathbb{P}_{n-3}\right)\varepsilon_i}{-\varepsilon_i^3 - \varepsilon_i^2 + 1} + \frac{1 - \mathbb{P}_{n+1}}{-\varepsilon_i^3 - \varepsilon_i^2 + 1}\right]$$

from Lemma 3 and the Binet form (12) and (10).

Using Lemma 4 and recurrence (7), we obtain

$$\prod_{i=1}^{n}\left[\mathbb{P}_n\varepsilon_i^3 + \left(1 + \mathbb{P}_{n-1}\right)\varepsilon_i^2 + \left(1 - \mathbb{P}_{n-3}\right)\varepsilon_i + 1 - \mathbb{P}_{n+1}\right]$$
$$= \left(-\mathbb{P}_n\right)^{n-1}\left[\mathbb{P}_n Q_n^2 + \left(1 - \mathbb{P}_{n+1}\right)Q_{n-1}^2 - \mathbb{P}_n Q_{2n} - \left(1 - \mathbb{P}_{n+1}\right)Q_{2(n-1)} - 2\mathbb{P}_n Q_{n+1} - 2\left(1 + \mathbb{P}_{n+2}\right)Q_n + 2\left(1 - \mathbb{P}_{n+1}\right)Q_{n-1}\right] \times 2^{-1} + 3\left(-\mathbb{P}_n\right)^{n-1} + \left(1 - \mathbb{P}_{n+1}\right)^n, \tag{53}$$

where $Q_n = q_1^n + q_2^n + q_3^n$, q_1, q_2, q_3 are the roots of the equation $\mathbb{P}_n x^3 + (1 + \mathbb{P}_{n-1})x^2 + (1 - \mathbb{P}_{n-3})x + 1 - \mathbb{P}_{n+1} = 0$. According to (33), we have the following results: if n is odd, then

$$\det \mathbf{F} = \frac{\left(-\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3\right)}{\mathcal{Y}_1}, \tag{54}$$

and if n is even, then

$$\det \mathbf{F} = \frac{\left(\mathcal{Q}_1 - \mathcal{Q}_2 - \mathcal{Q}_3\right)}{\mathcal{Y}_1}, \tag{55}$$

where

$$\mathcal{Q}_1 = \mathbb{P}_n^n\left(Q_n^2 - Q_{2n} - 2Q_{n+1}\right),$$
$$\mathcal{Q}_2 = \left(\mathbb{P}_{n+1} - 1\right)\mathbb{P}_n^{n-1}\left(Q_{n-1}^2 - Q_{2(n-1)} + 2Q_{n-1}\right), \tag{56}$$
$$\mathcal{Q}_3 = 2\mathbb{P}_n^{n-1}\left(\mathbb{P}_{n+2}Q_n + Q_n - 3\right) + 2\left(\mathbb{P}_{n+1} - 1\right)^n,$$

and $\mathcal{Y}_1$ is defined as Theorem 5. □

Theorem 9. *Let* $\mathbf{G} = RFMLRcircfr(\mathbb{P}_n, \ldots, \mathbb{P}_1)$. *If n is odd, then*

$$\det \mathbf{G} = \frac{\mathcal{Z}_1 + \mathcal{Z}_2 - \mathcal{Z}_3 + 2\left(\mathbb{P}_n - 1\right)^n}{-R_{n+2}R_{n-5} + R_{2n-3} + 2R_{n+4} + 4}, \tag{57}$$

and if n is even, then

$$\det \mathbf{G} = \frac{\mathcal{Z}_1 + \mathcal{Z}_2 - \mathcal{Z}_3 - 2\left(\mathbb{P}_n - 1\right)^n}{R_{n+2}R_{n-5} - R_{2n-3} - 2R_{n+4} - 4}, \tag{58}$$

where

$$\mathcal{Z}_1 = \left(\mathbb{P}_n - 1\right)\left(Z_{n-1}^2 - Z_{2(n-1)} + 2Z_{n-1}\right),$$
$$\mathcal{Z}_2 = Z_n^2 - Z_{2n} - 2Z_{n+1}, \tag{59}$$
$$\mathcal{Z}_3 = 2\left[\left(1 + \mathbb{P}_{n+1}\right)Z_n - \mathbb{P}_{n+5}\right],$$
$$Z_n = z_1^n + z_2^n + z_3^n,$$

where z_1, z_2, z_3 are the roots of the equation $x^3 + \mathbb{P}_{n+1}x^2 + \mathbb{P}_{n+2}x + \mathbb{P}_n - 1 = 0$.

Proof. The matrix **G** has the form

$$\mathbf{G} = \begin{pmatrix} \mathbb{P}_n & \mathbb{P}_{n-1} & \cdots & \mathbb{P}_1 \\ \mathbb{P}_1 & \mathbb{P}_n - \mathbb{P}_1 & \cdots & \mathbb{P}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}_{n-2} & \mathbb{P}_{n-3} - \mathbb{P}_{n-2} & \cdots & \mathbb{P}_{n-1} \\ \mathbb{P}_{n-1} & \mathbb{P}_{n-2} - \mathbb{P}_{n-1} & \cdots & \mathbb{P}_n - \mathbb{P}_{n-1} \end{pmatrix}. \tag{60}$$

According to Lemma 3 and the Binet form (12) and (10), we have

$$\det \mathbf{G} = \prod_{i=1}^{n}\left(\mathbb{P}_n + \mathbb{P}_{n-1}\varepsilon_i + \cdots + \mathbb{P}_1\varepsilon_i^{n-1}\right)$$
$$= \prod_{i=1}^{n}\sum_{k=0}^{n-1}\sum_{j=1}^{3} a_j r_j^{n-k}\varepsilon_i^k$$
$$= \prod_{i=1}^{n}\sum_{j=1}^{3}\frac{a_j r_j^{n+1} - a_j r_j \varepsilon_i^n}{r_j - \varepsilon_i} \tag{61}$$
$$= \prod_{i=1}^{n}\frac{\varepsilon_i^3 + \mathbb{P}_{n+1}\varepsilon_i^2 + \mathbb{P}_{n+2}\varepsilon_i + \mathbb{P}_n - 1}{-\varepsilon_i^3 + \varepsilon_i + 1}.$$

Using Lemma 4 and (12), we obtain

$$\prod_{i=1}^{n}\left(\varepsilon_i^3 + \mathbb{P}_{n+1}\varepsilon_i^2 + \mathbb{P}_{n+2}\varepsilon_i + \mathbb{P}_n - 1\right)$$
$$= (-1)^{n-1}\left[\left(\mathbb{P}_n - 1\right)\left(Z_{n-1}^2 - Z_{2(n-1)} + 2Z_{n-1}\right) + Z_n^2 - Z_{2n} - 2Z_{n+1} - 2\left(1 + \mathbb{P}_{n+1}\right)Z_n + 2\mathbb{P}_{n+5}\right] \times 2^{-1} + \left(\mathbb{P}_n - 1\right)^n, \tag{62}$$

where $Z_n = z_1^n + z_2^n + z_3^n$, z_1, z_2, z_3 are the roots of the equation $x^3 + \mathbb{P}_{n+1}x^2 + \mathbb{P}_{n+2}x + \mathbb{P}_n - 1 = 0$. Employing (43), we have the following results: if n is odd, then

$$\det \mathbf{G} = \frac{\mathcal{Z}_1 + \mathcal{Z}_2 - \mathcal{Z}_3 + 2\left(\mathbb{P}_n - 1\right)^n}{-R_{n+2}R_{n-5} + R_{2n-3} + 2R_{n+4} + 4}, \tag{63}$$

and if n is even, then

$$\det \mathbf{G} = \frac{\mathcal{Z}_1 + \mathcal{Z}_2 - \mathcal{Z}_3 - 2(\mathbb{P}_n - 1)^n}{R_{n+2}R_{n-5} - R_{2n-3} - 2R_{n+4} - 4}, \quad (64)$$

where

$$\begin{aligned} \mathcal{Z}_1 &= (\mathbb{P}_n - 1)\left(Z_{n-1}^2 - Z_{2(n-1)} + 2Z_{n-1}\right), \\ \mathcal{Z}_2 &= Z_n^2 - Z_{2n} - 2Z_{n+1}, \\ \mathcal{Z}_3 &= 2\left[(1 + \mathbb{P}_{n+1}) Z_n - \mathbb{P}_{n+5}\right]. \end{aligned} \quad (65)$$

□

Corollary 10. *Let* $\mathbf{H} = RLMFLcircfr(\mathbb{P}_1, \ldots, \mathbb{P}_n)$*. If* $n \equiv 0\,(mod\,4)$*, then*

$$\det \mathbf{H} = \frac{\mathcal{Z}_1 + \mathcal{Z}_2 - \mathcal{Z}_3 - 2(\mathbb{P}_n - 1)^n}{R_{n+2}R_{n-5} - R_{2n-3} - 2R_{n+4} - 4}, \quad (66)$$

and if $n \equiv 1\,(mod\,4)$*, then*

$$\det \mathbf{H} = \frac{\mathcal{Z}_1 + \mathcal{Z}_2 - \mathcal{Z}_3 + 2(\mathbb{P}_n - 1)^n}{-R_{n+2}R_{n-5} + R_{2n-3} + 2R_{n+4} + 4}, \quad (67)$$

and if $n \equiv 2\,(mod\,4)$*, then*

$$\det \mathbf{H} = \frac{-\mathcal{Z}_1 - \mathcal{Z}_2 + \mathcal{Z}_3 + 2(\mathbb{P}_n - 1)^n}{R_{n+2}R_{n-5} - R_{2n-3} - 2R_{n+4} - 4}, \quad (68)$$

and if $n \equiv 3\,(mod\,4)$*, then*

$$\det \mathbf{H} = \frac{-\mathcal{Z}_1 - \mathcal{Z}_2 + \mathcal{Z}_3 - 2(\mathbb{P}_n - 1)^n}{-R_{n+2}R_{n-5} + R_{2n-3} + 2R_{n+4} + 4}, \quad (69)$$

where $\mathcal{Z}_1$*,* $\mathcal{Z}_2$*,* $\mathcal{Z}_3$ *are defined the same as Theorem 9.*

Proof. The theorem can be proved by using Theorem 9 and relation (4). □

2.3. Determinants of the RFMLR and RLMFL Circulant Matrix Involving Tribonacci Numbers

Theorem 11. *Let* $\mathbf{J} = RFMLRcircfr(T_1, T_2, \ldots, T_n)$*. If n is odd, then*

$$\det \mathbf{J} = \frac{\left(-\mathcal{U}_1 - \mathcal{U}_2 - \mathcal{U}_3 - 2(1 - T_{n+1})^n\right)}{\mathcal{V}_1}, \quad (70)$$

and if n is even, then

$$\det \mathbf{J} = \frac{\left(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3 - 2(1 - T_{n+1})^n\right)}{\mathcal{V}_1}, \quad (71)$$

where

$$\begin{aligned} \mathcal{U}_1 &= T_n^n\left(U_n^2 - U_{2n} - 2U_{n+1} - 2U_n\right), \\ \mathcal{U}_2 &= 2T_n^{n-1}\left(1 - T_{n-1}U_n\right), \\ \mathcal{U}_3 &= T_n^{n-1}\left(1 - T_{n+1}\right)\left(U_{n-1}^2 - U_{2(n-1)} + 2U_{n-1}\right), \\ \mathcal{V}_1 &= V_n^2 - V_{n-1}^2 - V_{2n} + V_{2(n-1)} - 2V_{n+1} \\ &\quad - 4V_n - 2V_{n-1} + 2, \\ U_n &= u_1^n + u_2^n + u_3^n, \\ V_n &= v_1^n + v_2^n + v_3^n, \end{aligned} \quad (72)$$

where u_1*,* u_2*,* u_3 *and* v_1*,* v_2*,* v_3 *are the roots of the equations* $T_nx^3 + T_{n-1}x^2 + T_{n-2}x + 1 - T_{n+1} = 0$*,* $x^3 + x^2 + x - 1 = 0$*, respectively.*

Proof. Obviously, $\mathbf{J}$ has the form

$$\mathbf{J} = \begin{pmatrix} T_1 & T_2 & \ldots & T_n \\ T_n & T_1 - T_n & \ldots & T_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ T_3 & T_4 - T_3 & \ldots & T_2 \\ T_2 & T_3 - T_2 & \ldots & T_1 - T_n \end{pmatrix}. \quad (73)$$

According to Lemma 3 and the Binet form (16) and (14), we have

$$\begin{aligned} \det \mathbf{J} &= \prod_{i=1}^{n}\left(T_1 + T_2\varepsilon_i + \cdots + T_n\varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{3} b_j t_j^k \varepsilon_i^{k-1} \\ &= \prod_{i=1}^{n}\sum_{i=1}^{3}\frac{b_j t_j\left(1 - t_j^n\varepsilon_i^n\right)}{1 - t_j\varepsilon_i} \\ &= \prod_{i=1}^{n}\frac{T_n\varepsilon_i^3 + T_{n-1}\varepsilon_i^2 + T_{n-2}\varepsilon_i + 1 - T_{n+1}}{-\varepsilon_i^3 - \varepsilon_i^2 - \varepsilon_i + 1}. \end{aligned} \quad (74)$$

From Lemma 4 it follows that

$$\begin{aligned} &\prod_{i=1}^{n}\left(T_n\varepsilon_i^3 + T_{n-1}\varepsilon_i^2 + T_{n-2}\varepsilon_i + 1 - T_{n+1}\right) \\ &\quad = \frac{1}{2}(-T_n)^n\left(-U_n^2 + U_{2n} + 2U_{n+1} + 2U_n\right) \\ &\quad + (-T_n)^{n-1}\left(1 - T_{n-1}U_n\right) + \frac{1}{2}(-T_n)^{n-1}\left(1 - T_{n+1}\right) \\ &\quad \times\left(U_{n-1}^2 - U_{2(n-1)} + 2U_{n-1}\right) + \left(1 - T_{n+1}\right)^n, \end{aligned} \quad (75)$$

where $U_n = u_1^n + u_2^n + u_3^n$, u_1, u_2, u_3 are the roots of the equation $T_n x^3 + T_{n-1}x^2 + T_{n-2}x + 1 - T_{n+1} = 0$. And

$$\prod_{i=1}^{n}\left(-\varepsilon_i^3 - \varepsilon_i^2 - \varepsilon_i + 1\right) = \frac{1}{2}\left(-V_n^2 + V_{n-1}^2 + V_{2n} - V_{2(n-1)} + 2V_{n+1}, +4V_n + 2V_{n-1} - 2\right), \tag{76}$$

where $V_n = v_1^n + v_2^n + v_3^n$, v_1, v_2, v_3 are the roots of the equation $x^3+x^2+x-1 = 0$. Consequently, we have the following results: if n is odd, then

$$\det \mathbf{J} = \frac{\left(-\mathscr{U}_1 - \mathscr{U}_2 - \mathscr{U}_3 - 2\left(1 - T_{n+1}\right)^n\right)}{\mathscr{V}_1}, \tag{77}$$

and if n is even, then

$$\det \mathbf{J} = \frac{\left(\mathscr{U}_1 + \mathscr{U}_2 + \mathscr{U}_3 - 2\left(1 - T_{n+1}\right)^n\right)}{\mathscr{V}_1}, \tag{78}$$

where

$$\begin{aligned}
\mathscr{U}_1 &= T_n^n\left(U_n^2 - U_{2n} - 2U_{n+1} - 2U_n\right),\\
\mathscr{U}_2 &= 2T_n^{n-1}\left(1 - T_{n-1}U_n\right),\\
\mathscr{U}_3 &= T_n^{n-1}\left(1 - T_{n+1}\right)\left(U_{n-1}^2 - U_{2(n-1)} + 2U_{n-1}\right),\\
\mathscr{V}_1 &= V_n^2 - V_{n-1}^2 - V_{2n} + V_{2(n-1)} - 2V_{n+1}\\
&\quad - 4V_n - 2V_{n-1} + 2.
\end{aligned} \tag{79}$$

The proof is completed. □

Theorem 12. *Let* $\mathbf{K} = RFMLRcircfr(T_n, \dots, T_1)$. *If n is odd, then*

$$\det \mathbf{K} = \frac{\left(-\mathscr{W}_1 - \mathscr{W}_2 + \mathscr{W}_3\right)}{\mathscr{L}_1}, \tag{80}$$

and if n is even, then

$$\det \mathbf{K} = \frac{\left(\mathscr{W}_1 + \mathscr{W}_2 - \mathscr{W}_3\right)}{\mathscr{L}_1}, \tag{81}$$

where

$$\begin{aligned}
\mathscr{W}_1 &= W_n^2 - W_{2n} - 2W_{n+1},\\
\mathscr{W}_2 &= T_n\left(W_{n-1}^2 - W_{2(n-1)} + 2W_{n-1}\right),\\
\mathscr{W}_3 &= 2\left(T_{n+1}W_n - T_{n+2} - T_n + T_n^n\right),\\
\mathscr{L}_1 &= \mathbb{L}_n^2 - \mathbb{L}_{n-1}^2 - \mathbb{L}_{2n} + \mathbb{L}_{2(n-1)} - 2\mathbb{L}_{n+1}\\
&\quad - 2\mathbb{L}_{n-1} - 6,\\
W_n &= w_1^n + w_2^n + w_3^n,
\end{aligned} \tag{82}$$

where w_1, w_2, w_3 are the roots of the equation $x^3 + (T_{n+1} - 1)x^2 + (T_{n+2} - T_{n+1})x + T_n = 0$.

Proof. The matrix $\mathbf{K}$ has the form

$$\mathbf{K} = \begin{pmatrix} T_n & T_{n-1} & \dots & T_1 \\ T_1 & T_n - T_1 & \dots & T_2 \\ \vdots & \vdots & \ddots & \vdots \\ T_{n-2} & T_{n-3} - T_{n-2} & \dots & T_{n-1} \\ T_{n-1} & T_{n-2} - T_{n-1} & \dots & T_n - T_1 \end{pmatrix}. \tag{83}$$

According to Lemma 3 and the Binet form (16) and (14), we have

$$\begin{aligned}
\det \mathbf{K} &= \prod_{i=1}^{n}\left(T_n + T_{n-1}\varepsilon_i + \dots + T_1\varepsilon_i^{n-1}\right)\\
&= \prod_{i=1}^{n}\sum_{k=0}^{n-1}\sum_{j=1}^{3} b_j t_j^{n-k}\varepsilon_i^k\\
&= \prod_{i=1}^{n}\sum_{j=1}^{3}\frac{b_j t_j^{n+1} - b_j t_j \varepsilon_i^n}{t_j - \varepsilon_i}\\
&= \prod_{i=1}^{n}\frac{\varepsilon_i^3 + \left(T_{n+1} - 1\right)\varepsilon_i^2 + \left(T_{n+2} - T_{n+1}\right)\varepsilon_i + T_n}{-\varepsilon_i^3 + \varepsilon_i^2 + \varepsilon_i + 1}.
\end{aligned} \tag{84}$$

Considering Lemma 4 and (17), we obtain

$$\begin{aligned}
&\prod_{i=1}^{n}\left[\varepsilon_i^3 + \left(T_{n+1} - 1\right)\varepsilon_i^2 + \left(T_{n+2} - T_{n+1}\right)\varepsilon_i + T_n\right]\\
&\quad = \frac{1}{2}(-1)^{n-1}\left[W_n^2 - W_{2n} - 2W_{n+1} - 2W_n + T_n\left(W_{n-1}^2 - W_{2(n-1)} + 2W_{n-1}\right)\right]\\
&\qquad + (-1)^{n-1}\left(-T_{n+1}W_n + W_n + T_{n+2} + T_n\right) + T_n^n,
\end{aligned} \tag{85}$$

where $W_n = w_1^n + w_2^n + w_3^n$, w_1, w_2, w_3 are the roots of the equation $x^3 + (T_{n+1} - 1)x^2 + (T_{n+2} - T_{n+1})x + T_n = 0$. And

$$\prod_{i=1}^{n}\left(-\varepsilon_i^3 + \varepsilon_i^2 + \varepsilon_i + 1\right) = \left(-\mathbb{L}_n^2 + \mathbb{L}_{n-1}^2 + \mathbb{L}_{2n} - \mathbb{L}_{2(n-1)} + 2\mathbb{L}_{n+1} + 2\mathbb{L}_{n-1} + 6\right) \times 2^{-1}. \tag{86}$$

Consequently, if n is odd, then

$$\det \mathbf{K} = \frac{\left(-\mathscr{W}_1 - \mathscr{W}_2 + \mathscr{W}_3\right)}{\mathscr{L}_1}, \tag{87}$$

and if n is even, then

$$\det \mathbf{K} = \frac{\left(\mathscr{W}_1 + \mathscr{W}_2 - \mathscr{W}_3\right)}{\mathscr{L}_1}, \tag{88}$$

where

$$\begin{aligned}\mathscr{W}_1 &= W_n^2 - W_{2n} - 2W_{n+1},\\ \mathscr{W}_2 &= T_n\left(W_{n-1}^2 - W_{2(n-1)} + 2W_{n-1}\right),\\ \mathscr{W}_3 &= 2\left(T_{n+1}W_n - T_{n+2} - T_n + T_n^n\right),\\ \mathscr{L}_1 &= \mathbb{L}_n^2 - \mathbb{L}_{n-1}^2 - \mathbb{L}_{2n} + \mathbb{L}_{2(n-1)}\\ &\quad - 2\mathbb{L}_{n+1} - 2\mathbb{L}_{n-1} - 6.\end{aligned} \tag{89}$$

□

Corollary 13. *Let* $\mathbf{L} = RLMFLcircfr(T_1, T_2, \ldots, T_n)$*. If* $n \equiv 0\,(mod 4)$*, then*

$$\det \mathbf{L} = \frac{(\mathscr{W}_1 + \mathscr{W}_2 - \mathscr{W}_3)}{\mathscr{L}_1}, \tag{90}$$

and if $n \equiv 1\,(mod\ 4)$*, then*

$$\det \mathbf{L} = \frac{(-\mathscr{W}_1 - \mathscr{W}_2 + \mathscr{W}_3)}{\mathscr{L}_1}, \tag{91}$$

and if $n \equiv 2\,(mod\ 4)$*, then*

$$\det \mathbf{L} = \frac{(-\mathscr{W}_1 - \mathscr{W}_2 + \mathscr{W}_3)}{\mathscr{L}_1}, \tag{92}$$

and if $n \equiv 3\,(mod\ 4)$*, then*

$$\det \mathbf{L} = \frac{(\mathscr{W}_1 + \mathscr{W}_2 - \mathscr{W}_3)}{\mathscr{L}_1}, \tag{93}$$

where $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3, \mathscr{L}_1$ *are defined as Theorem 12.*

Proof. The theorem can be proved by using Theorem 12 and relation (4). □

2.4. Determinants of the RFMLR and RLMFL Circulant Matrix Involving Generalized Lucas Numbers

Theorem 14. *Let* $\mathbf{M} = RFMLRcircfr(\mathbb{L}_1, \ldots, \mathbb{L}_n)$*. If n is odd, then*

$$\det \mathbf{M} = \frac{\left(-\mathscr{S}_1 + \mathscr{S}_2 - \mathscr{S}_3 - 2(1-\mathbb{L}_{n+1})^n\right)}{\mathscr{V}_1}, \tag{94}$$

and if n is even, then

$$\det \mathbf{M} = \frac{\left(\mathscr{S}_1 - \mathscr{S}_2 + \mathscr{S}_3 - 2(1-\mathbb{L}_{n+1})^n\right)}{\mathscr{V}_1}, \tag{95}$$

where

$$\begin{aligned}\mathscr{S}_1 &= \mathbb{L}_n^n\left(S_n^2 - S_{2n} - 2S_{n+1} - 2S_n\right),\\ \mathscr{S}_2 &= 2\mathbb{L}_n^{n-1}\left(\mathbb{L}_{n-1}S_n + 3S_n - 6\right),\\ \mathscr{S}_3 &= \left(1-\mathbb{L}_{n+1}\right)\left(S_{n-1}^2 - S_{2(n-1)} + 2S_{n-1}\right),\\ S_n &= s_1^n + s_2^n + s_3^n,\end{aligned} \tag{96}$$

where s_1, s_2, s_3 *are the roots of the equation* $\mathbb{L}_n x^3 + (\mathbb{L}_{n-1}+3)x^2 + (\mathbb{L}_{n-2}+2)x + 1 - \mathbb{L}_{n+1} = 0$*, and* $\mathscr{V}_1$ *is defined as Theorem 11.*

Proof. The matrix $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{pmatrix} \mathbb{L}_1 & \mathbb{L}_2 & \cdots & \mathbb{L}_n \\ \mathbb{L}_n & \mathbb{L}_1 - \mathbb{L}_n & \cdots & \mathbb{L}_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{L}_3 & \mathbb{L}_4 - \mathbb{L}_3 & \cdots & \mathbb{L}_2 \\ \mathbb{L}_2 & \mathbb{L}_3 - \mathbb{L}_2 & \cdots & \mathbb{L}_1 - \mathbb{L}_n \end{pmatrix}. \tag{97}$$

According to Lemma 3 and the Binet form (17) and (14), we have

$$\begin{aligned}\det \mathbf{M} &= \prod_{i=1}^{n}\left(\mathbb{L}_1 + \mathbb{L}_2\varepsilon_i + \cdots + \mathbb{L}_n\varepsilon_i^{n-1}\right)\\ &= \prod_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{3} t_j^k \varepsilon_i^{k-1}\\ &= \prod_{i=1}^{n}\sum_{j=1}^{3}\frac{t_j\left(1 - t_j^n\varepsilon_i^n\right)}{1 - t_j^n\varepsilon_i}\\ &= \prod_{i=1}^{n}\left[\frac{\mathbb{L}_n\varepsilon_i^3 + (\mathbb{L}_{n-1}+3)\varepsilon_i^2 + (\mathbb{L}_{n-2}+2)\varepsilon_i}{-\varepsilon_i^3 - \varepsilon_i^2 - \varepsilon_i + 1}\right.\\ &\qquad\left. + \frac{1 - \mathbb{L}_{n+1}}{-\varepsilon_i^3 - \varepsilon_i^2 - \varepsilon_i + 1}\right].\end{aligned} \tag{98}$$

From Lemma 4 and (17), we obtain

$$\begin{aligned}&\prod_{i=1}^{n}\left[\mathbb{L}_n\varepsilon_i^3 + (\mathbb{L}_{n-1}+3)\varepsilon_i^2 + (\mathbb{L}_{n-2}+2)\varepsilon_i + 1 - \mathbb{L}_{n+1}\right]\\ &= \frac{1}{2}(-\mathbb{L}_n)^n\left(-S_n^2 + S_{2n} + 2S_{n+1} + 2S_n\right) + (-\mathbb{L}_n)^{n-1}\\ &\quad\times\left(-\mathbb{L}_{n-1}S_n - 3S_n + 6\right) + \frac{1}{2}(-\mathbb{L}_n)^{n-1}\left(1 - \mathbb{L}_{n+1}\right)\\ &\quad\times\left(S_{n-1}^2 - S_{2(n-1)} + 2S_{n-1}\right) + \left(1 - \mathbb{L}_{n+1}\right)^n,\end{aligned} \tag{99}$$

where $S_n = s_1^n + s_2^n + s_3^n$, s_1, s_2, s_3 are the roots of the equation $\mathbb{L}_n x^3 + (\mathbb{L}_{n-1}+3)x^2 + (\mathbb{L}_{n-2}+2)x + 1 - \mathbb{L}_{n+1} = 0$. And

$$\begin{aligned}\prod_{i=1}^{n}\left(-\varepsilon_i^3 - \varepsilon_i^2 - \varepsilon_i + 1\right) &= \frac{1}{2}\left(-V_n^2 + V_{n-1}^2 + V_{2n}\right.\\ &\quad - V_{2(n-1)} + 2V_{n+1}\\ &\quad\left. + 4V_n + 2V_{n-1} - 2\right),\end{aligned} \tag{100}$$

where $V_n = v_1^n + v_2^n + v_3^n$, v_1, v_2, v_3 are the roots of the equation $x^3 + x^2 + x - 1 = 0$. Hence, if n is odd, then

$$\det \mathbf{M} = \frac{-\mathscr{S}_1 + \mathscr{S}_2 - \mathscr{S}_3 - 2(1-\mathbb{L}_{n+1})^n}{\mathscr{V}_1}, \tag{101}$$

and if n is even, then

$$\det \mathbf{M} = \frac{\mathscr{S}_1 - \mathscr{S}_2 + \mathscr{S}_3 - 2(1-\mathbb{L}_{n+1})^n}{\mathscr{V}_1}, \tag{102}$$

where

$$\mathcal{S}_1 = \mathbb{L}_n^n\left(S_n^2 - S_{2n} - 2S_{n+1} - 2S_n\right),$$
$$\mathcal{S}_2 = 2\mathbb{L}_n^{n-1}\left(\mathbb{L}_{n-1}S_n + 3S_n - 6\right), \quad (103)$$
$$\mathcal{S}_3 = \left(1 - \mathbb{L}_{n+1}\right)\left(S_{n-1}^2 - S_{2(n-1)} + 2S_{n-1}\right),$$

and $\mathcal{V}_1$ is defined as Theorem 11. □

Theorem 15. *Let* $\mathbf{N} = RFMLRcircfr(\mathbb{L}_n, \dots, \mathbb{L}_1)$*. If n is odd, then*

$$\det \mathbf{N} = \frac{\left(-\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 - 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (104)$$

and if n is even, then

$$\det \mathbf{N} = \frac{\left(\mathcal{T}_1 - \mathcal{T}_2 + \mathcal{T}_3 - 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (105)$$

where

$$\mathcal{T}_1 = \Gamma_n^2 - \Gamma_{2n} - 2\Gamma_{n+1},$$
$$\mathcal{T}_2 = 2\left(\mathbb{L}_{n+1} + 2\right)\Gamma_n - 2\left(\mathbb{L}_{n+2} + \mathbb{L}_n\right),$$
$$\mathcal{T}_3 = \left(\mathbb{L}_n - 3\right)\left(\Gamma_{n-1}^2 - \Gamma_{2(n-1)} + 2\Gamma_{n-1}\right), \quad (106)$$
$$\Gamma_n = \gamma_1^n + \gamma_2^n + \gamma_3^n,$$

where $\gamma_1, \gamma_2, \gamma_3$ *are the roots of the equation* $x^3 + (\mathbb{L}_{n+1}+1)x^2 + (\mathbb{L}_{n+2} - \mathbb{L}_{n+1} + 1)x + \mathbb{L}_n - 3 = 0$*, and* $\mathcal{L}_1$ *is defined as Theorem 12.*

Proof. The matrix $\mathbf{N}$ has the form

$$\mathbf{N} = \begin{pmatrix} \mathbb{L}_n & \mathbb{L}_{n-1} & \cdots & \mathbb{L}_1 \\ \mathbb{L}_1 & \mathbb{L}_n - \mathbb{L}_1 & \cdots & \mathbb{L}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{L}_{n-2} & \mathbb{L}_{n-3} - \mathbb{L}_{n-2} & \cdots & \mathbb{L}_{n-1} \\ \mathbb{L}_{n-1} & \mathbb{L}_{n-2} - \mathbb{L}_{n-1} & \cdots & \mathbb{L}_n - \mathbb{L}_1 \end{pmatrix}. \quad (107)$$

According to Lemma 3, (17), and (14), we have

$$\begin{aligned} \det \mathbf{N} &= \prod_{i=1}^{n}\left(\mathbb{L}_n + \mathbb{L}_{n-1}\varepsilon_i + \cdots + \mathbb{L}_1\varepsilon_i^{n-1}\right) \\ &= \prod_{i=1}^{n}\sum_{k=0}^{n-1}\sum_{j=1}^{3} t_j^{n-k}\varepsilon_i^k \\ &= \prod_{i=1}^{n}\sum_{j=1}^{3}\frac{t_j^{n+1} - t_j\varepsilon_i^n}{t_j - \varepsilon_i} \\ &= \prod_{i=1}^{n}\left[\frac{\varepsilon_i^3 + \left(\mathbb{L}_{n+1}+1\right)\varepsilon_i^2 + \mathbb{L}_n - 3}{-\varepsilon_i^3 + \varepsilon_i^2 + \varepsilon_i + 1} + \frac{\left(\mathbb{L}_{n+2} - \mathbb{L}_{n+1} + 1\right)\varepsilon_i}{-\varepsilon_i^3 + \varepsilon_i^2 + \varepsilon_i + 1}\right]. \end{aligned} \quad (108)$$

By Lemma 4 and the Binet form (17), we obtain

$$\begin{aligned} &\prod_{i=1}^{n}\left[\varepsilon_i^3 + \left(\mathbb{L}_{n+1}+1\right)\varepsilon_i^2 + \left(\mathbb{L}_{n+2} - \mathbb{L}_{n+1} + 1\right)\varepsilon_i + \mathbb{L}_n - 3\right] \\ &= \frac{1}{2}(-1)^n\left(-\Gamma_n^2 + \Gamma_{2n} + 2\Gamma_{n+1} + 2\Gamma_n\right) + (-1)^{n-1} \\ &\quad \times\left(-\mathbb{L}_{n+1}\Gamma_n - \Gamma_n + \mathbb{L}_{n+2} + \mathbb{L}_n\right) + \frac{1}{2}(-1)^{n-1} \\ &\quad \times\left(\mathbb{L}_n - 3\right)\left(\Gamma_{n-1}^2 - \Gamma_{2(n-1)} + 2\Gamma_{n-1}\right) + \left(\mathbb{L}_n - 3\right)^n, \end{aligned} \quad (109)$$

where $\Gamma_n = \gamma_1^n + \gamma_2^n + \gamma_3^n$, $\gamma_1, \gamma_2, \gamma_3$ are the roots of the equation $x^3 + (\mathbb{L}_{n+1}+1)x^2 + (\mathbb{L}_{n+2} - \mathbb{L}_{n+1} + 1)x + \mathbb{L}_n - 3 = 0$. And

$$\prod_{i=1}^{n}\left(-\varepsilon_i^3 + \varepsilon_i^2 + \varepsilon_i + 1\right) = \left(-\mathbb{L}_n^2 + \mathbb{L}_{n-1}^2 + \mathbb{L}_{2n} - \mathbb{L}_{2(n-1)} + 2\mathbb{L}_{n+1} + 2\mathbb{L}_{n-1} + 6\right) \times 2^{-1}. \quad (110)$$

Thus, if n is odd, then

$$\det \mathbf{N} = \frac{\left(-\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 - 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (111)$$

and if n is even, then

$$\det \mathbf{N} = \frac{\left(\mathcal{T}_1 - \mathcal{T}_2 + \mathcal{T}_3 - 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (112)$$

where

$$\mathcal{T}_1 = \Gamma_n^2 - \Gamma_{2n} - 2\Gamma_{n+1},$$
$$\mathcal{T}_2 = 2\left(\mathbb{L}_{n+1} + 2\right)\Gamma_n - 2\left(\mathbb{L}_{n+2} + \mathbb{L}_n\right), \quad (113)$$
$$\mathcal{T}_3 = \left(\mathbb{L}_n - 3\right)\left(\Gamma_{n-1}^2 - \Gamma_{2(n-1)} + 2\Gamma_{n-1}\right),$$

and $\mathcal{L}_1$ is defined as Theorem 12. □

Corollary 16. *Let* $\mathbf{P} = RLMFLcircfr(\mathbb{L}_1, \dots, \mathbb{L}_n)$*. If* $n \equiv 0$ *(mod 4), then*

$$\det \mathbf{P} = \frac{\left(\mathcal{T}_1 - \mathcal{T}_2 + \mathcal{T}_3 - 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (114)$$

and if $n \equiv 1$ *(mod 4), then*

$$\det \mathbf{P} = \frac{\left(-\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 - 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (115)$$

and if $n \equiv 2$ *(mod 4), then*

$$\det \mathbf{P} = \frac{\left(-\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 + 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (116)$$

and if $n \equiv 3 \pmod 4$*, then*

$$\det \mathbf{P} = \frac{\left(\mathcal{T}_1 - \mathcal{T}_2 + \mathcal{T}_3 + 2(\mathbb{L}_n - 3)^n\right)}{\mathcal{L}_1}, \quad (117)$$

where $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{L}_1$ *are defined as Theorem 15.*

Proof. The theorem can be proved by using Theorem 15 and relation (4). □

3. Conclusions

The row first-minus-last right (RFMLR) circulant matrices and row last-minus-first left (RLMFL) circulant matrices are two kinds of matrices with specific structure. We explored the determinant problem of these two kinds of matrices when their entries are Perrin, Padovan, Tribonacci, and the generalized Lucas sequences, respectively. On the basis of the inverse factorization of polynomial and the third-order recurrence, Binet form, and other properties of these sequences, we present the exact formulae of determinants by some terms of these famous sequences. Chillag has studied some properties and applications of $f(x)$-circulant matrices in [7]. The RFMLR circulant and RLMFL circulant matrices have more explicit structures and better properties than the general $f(x)$-circulant matrices, so they will play more important roles than the general $f(x)$-circulant matrices in some fields of signal encoding, image processing, and so on. That is the reason why we focus our attentions on RFMLR circulant and RLMFL circulant matrices.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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