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Research Article

Derivation of Conservation Laws for the Magma Equation Using the Multiplier Method: Power Law and Exponential Law for Permeability and Viscosity

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The derivation of conservation laws for the magma equation using the multiplier method for both the power law and exponential law relating the permeability and matrix viscosity to the voidage is considered. It is found that all known conserved vectors for the magma equation and the new conserved vectors for the exponential laws can be derived using multipliers which depend on the voidage and spatial derivatives of the voidage. It is also found that the conserved vectors are associated with the Lie point symmetry of the magma equation which generates travelling wave solutions which may explain by the double reduction theorem for associated Lie point symmetries why many of the known analytical solutions are travelling waves.

1. Introduction

The one-dimensional migration of melt upwards through the mantle of the Earth is governed by the third order nonlinear partial differential equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[K(\phi)\left(1 - \frac{\partial}{\partial z}\left(G(\phi)\frac{\partial \phi}{\partial t}\right)\right)\right] = 0, \quad (1)$$

where $\phi(t,z)$ is the voidage or volume fraction of melt, t is time, z is the vertical spatial coordinate, K is the permeability of the medium, and G is the viscosity of the matrix phase. The variables ϕ, t, and z and the physical quantities $K(\phi)$ and $G(\phi)$ in (1) are dimensionless. The voidage $\phi(t,z)$ is scaled by the background voidage ϕ_0. The background state is therefore defined by $\phi = 1$. The characteristic length in the z-direction, which is vertically upwards, is the compaction length δ_c defined by

$$\delta_c = \left[\frac{K(\phi_0)\,G(\phi_0)}{\mu}\right]^{1/2}, \quad (2)$$

where μ is the coefficient of shear viscosity of the melt. The characteristic time is t_0 defined by

$$t_0 = \frac{\phi_0}{g\Delta\rho}\left[\frac{\mu G(\phi_0)}{K(\phi_0)}\right]^{1/2}, \quad (3)$$

where g is the acceleration due to gravity and $\Delta\rho$ is the difference between the density of the solid matrix and the density of the melt. The permeability is scaled by $K(\phi_0)$ and therefore

$$K(1) = 1. \quad (4)$$

When the voidage is zero the permeability must also be zero and therefore

$$K(0) = 0. \quad (5)$$

The viscosity $G(\phi)$ is scaled by $G(\phi_0)$ so that

$$G(1) = 1 \quad (6)$$

and $G(0)$ will be infinite because the matrix viscosity is infinite when the voidage vanishes. In the derivation of (1) it is assumed that the background voidage satisfies $\phi_0 \ll 1$.

The partially molten medium consists of a solid matrix and a fluid melt which are modelled as two immiscible fully connected fluids of constant but different densities. The density of the melt is less than the density of the solid matrix and the melt migrates through the compacting medium by the buoyancy force due to the difference in density between the melt and the solid matrix. Changes of phase are not included in the model. It is assumed that the melting has occurred and only migration of the melt under gravity is described by (1) [1].

In the model proposed by Scott and Stevenson [2], consider

$$K(\phi) = \phi^{n}, \qquad G(\phi) = \phi^{-m}, \tag{7}$$

where $n \geq 0$ and $m \geq 0$. Harris and Clarkson [3] have investigated this model using Painleve analysis. Mindu and Mason [4] showed that the magma equation also admits Lie point symmetries other than translations in time and space if the permeability is in the form of an exponential law:

$$K(\phi) = \exp[n(\phi - 1)]. \tag{8}$$

Conservation laws for (1) when the permeability and matrix viscosity satisfy the power laws (7) have been obtained using the direct method by Barcilon and Richter [5] and Harris [6] and using Lie point symmetry generators by Maluleke and Mason [7].

In this paper we will derive the conservation laws for the partial differential equation (1) using the multiplier method. We will consider power laws given by (7) and also the exponential laws

$$K(\phi) = \exp[n(\phi - 1)], \qquad G(\phi) = \exp[-m(\phi - 1)], \tag{9}$$

where $n \geq 0$ and $m \geq 0$, relating the permeability and matrix viscosity to the voidage. The permeability increases as the voidage increases while the viscosity of the matrix decreases as the voidage increases. The exponential laws are not suitable models when the voidage ϕ is small because $K(0) = \exp(-n) \neq 0$ and $G(0) = \exp(m) \neq \infty$. They are suitable for describing rarefaction for which $\phi > 1$.

An outline of the paper is as follows. In Section 2 we present the formulae and theory that we will use in the paper. In Section 3 conservation laws for the magma equation, with power laws relating the permeability and viscosity to the voidage, are derived using the multiplier method. Further in Section 4 conservation laws for the magma equation, with exponential laws relating the permeability and viscosity to the voidage, are derived using the multiplier method. Finally the conclusions are summarized in Section 5.

2. Formulae and Theory

Consider an sth order partial differential equation

$$F(x, \phi, \phi_{(1)}, \ldots, \phi_{(s)}) = 0, \tag{10}$$

in the variables $x = (x^1, x^2, \ldots, x^p)$, where $\phi_{(p)}$ denotes the collection of pth-order partial derivatives of ϕ. The equation

$$D_i T^i = 0, \tag{11}$$

evaluated on the surface given by (10), where i runs from 1 to r and D_i is the total derivative defined by

$$D_i = \frac{\partial}{\partial x_i} + \phi_i \frac{\partial}{\partial \phi} + \phi_{ki} \frac{\partial}{\partial \phi_k} + \cdots, \tag{12}$$

is called a conservation law for the differential equation (10). The vector $T = (T^1, \ldots, T^r)$ is a conserved vector for the partial differential equation and $T^1, \ldots, T^r$ are its components. Thus, a conserved vector gives rise to a conservation law. A Lie point symmetry generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, \phi) \frac{\partial}{\partial \phi}, \tag{13}$$

where i runs from 1 to r, is said to be associated with the conserved vector $T = (T^1, \ldots, T^r)$ for the partial differential equation (10) if [8, 9]

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2, \ldots r. \tag{14}$$

The association of a Lie point symmetry with a conserved vector can be used to integrate the partial differential equation twice by the double reduction theorem of Sjöberg [10].

Conserved vectors for a partial differential equation can be generated from known conserved vectors and Lie point symmetries of the partial differential equation. For

$$T_*^i = X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i), \quad i = 1, 2, \ldots r, \tag{15}$$

where k runs from 1 to r, is a conserved vector for the partial differential equation although it may be a linear combination of known conserved vectors [8, 9].

We now present the multiplier method for the derivation of conservation laws for partial differential equations. We will outline its application to the partial differential equation (1) in two independent variables.

(1) Multiply the partial differential equation (1) by the multiplier, Λ, to obtain the conservation law

$$\Lambda F = D_1 T^1 + D_2 T^2, \tag{16}$$

where $F = 0$ is the partial differential equation (1), $x^1 = t$ and $x^2 = z$, and

$$\begin{aligned} D_1 = D_t &= \frac{\partial}{\partial t} + \phi_t \frac{\partial}{\partial \phi} + \phi_{tt} \frac{\partial}{\partial_t} + \phi_{zt} \frac{\partial}{\partial_z} + \cdots, \\ D_2 = D_z &= \frac{\partial}{\partial z} + \phi_z \frac{\partial}{\partial \phi} + \phi_{tz} \frac{\partial}{\partial_t} + \phi_{zz} \frac{\partial}{\partial_z} + \cdots. \end{aligned} \tag{17}$$

The multiplier depends on t, z, ϕ, and the partial derivatives of ϕ. The more derivatives included in the multiplier the wider the range of conserved vectors that can be derived.

(2) The determining equation for the multiplier is obtained by operating on (16) by the Euler operator E_ϕ defined by [11]

$$\begin{aligned} E_\phi = \frac{\partial}{\partial \phi} - D_t \frac{\partial}{\partial \phi_t} - D_z \frac{\partial}{\partial \phi_z} + D_t^2 \frac{\partial}{\partial \phi_{tt}} \\ + D_t D_z \frac{\partial}{\partial \phi_{tz}} + D_z^2 \frac{\partial}{\partial \phi_{zz}} - \cdots. \end{aligned} \tag{18}$$

Since the Euler operator annihilates divergence expressions this gives [11]

$$E_\phi[\Lambda F] = 0. \tag{19}$$

(3) The determining equation (19) is separated by equating the coefficients of like powers and products of the derivatives of ϕ because ϕ is an arbitrary function.

(4) When ϕ is a solution of the partial differential equation, $F = 0$, (16) becomes a conservation law. The condition $F = 0$ is imposed on (16). The product of the multiplier and the partial differential equation is then written in conserved form by elementary manipulations. This yields the conserved vectors by setting all the constants equal to zero except one in turn.

3. Conservation Laws for the Magma Equation with Power Law Permeability and Viscosity by the Multiplier Method

When the permeability and viscosity are related to the voidage by the power laws (7) the magma equation becomes

$$\frac{\partial\phi}{\partial t} + \frac{\partial}{\partial z}\left[\phi^n\left(1 - \frac{\partial}{\partial z}\left(\phi^{-m}\frac{\partial\phi}{\partial t}\right)\right)\right] = 0. \tag{20}$$

3.1. Lower Order Conservation Laws. In order to derive conservation laws for (20) consider first a multiplier of the form

$$\Lambda = \Lambda(\phi). \tag{21}$$

A multiplier for the partial differential equation has the property

$$\Lambda(\phi) F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz}) = D_1 T^1 + D_2 T^2, \tag{22}$$

where

$$\begin{aligned} F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz}) &= \phi_t + n\phi^{n-1}\phi_z + m(n-m-1)\phi^{n-m-2}\phi_z^2\phi_t \\ &\quad + m\phi^{n-m-1}\phi_{zz}\phi_t + (2m-n)\phi^{n-m-1}\phi_z\phi_{tz} \\ &\quad - \phi^{n-m}\phi_{tzz}. \end{aligned} \tag{23}$$

The determining equation for the multiplier is

$$E_\phi[\Lambda(\phi) F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz})] = 0, \tag{24}$$

where E_ϕ is defined by (18). Separating (24) with respect to products and powers of the partial derivatives of ϕ we obtain the following system of equations:

$$\phi_z\phi_{tz} : \phi\frac{d^2\Lambda}{d\phi^2} + (m+n)\frac{d\Lambda}{d\phi} = 0, \tag{25}$$

$$\phi_t\phi_{zz} : \phi\frac{d^2\Lambda}{d\phi^2} + (m+n)\frac{d\Lambda}{d\phi} = 0, \tag{26}$$

$$\phi_t\phi_z^2 : \phi^2\frac{d^3\Lambda}{d\phi^3} + 2n\phi\frac{d^2\Lambda}{d\phi^2} - (m+n)(m-n+1)\frac{d\Lambda}{d\phi} = 0. \tag{27}$$

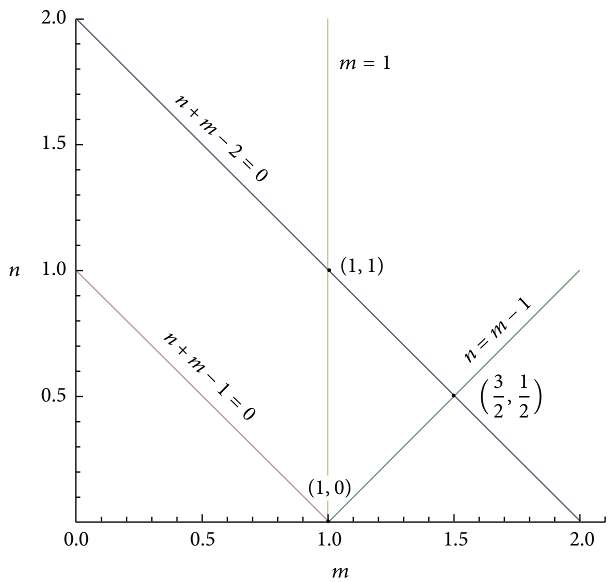

Figure 1: The (m, n)-plane. The conservation laws are constrained to the region $m \geq 0$, $n \geq 0$. The special cases are the straight lines $n + m - 1 = 0$, $n + m - 2 = 0$, $m = 1$, and $n = m - 1$.

Equation (26) is the same as (25). It is readily verified that every solution of (25) is a solution of (27). We therefore need to consider only (25). The general solution of (25) is

$$\Lambda(\phi) = c_2\phi^{1-m-n} + c_1, \quad \text{if } n + m - 1 \neq 0, \tag{28}$$

$$\Lambda(\phi) = c_2\ln\phi + c_1, \quad \text{if } n + m - 1 = 0. \tag{29}$$

There are several cases to consider depending on the values of m and n. The special cases are illustrated as lines and points in the (m, n) plane in Figure 1.

(i) $n + m - 1 \neq 0$, $n + m - 2 \neq 0$, $m \neq 1$. From (22) and (28),

$$\begin{aligned} &\left(c_1 + c_2\phi^{1-m-n}\right)\left(\phi_t + n\phi^{n-1}\phi_z + m(n-m-1)\right. \\ &\qquad \times \phi^{n-m-2}\phi_z^2\phi_t + m\phi^{n-m-1}\phi_{zz}\phi_t \\ &\qquad \left. + (2m-n)\phi^{n-m-1}\phi_z\phi_{tz} - \phi^{n-m}\phi_{tzz}\right) \\ &= D_1\left[c_1\phi + c_2\left(\frac{1}{2-m-n}\left(\phi^{2-m-n} - 1\right)\right.\right. \\ &\qquad \left.\left. + \frac{(1-m-n)}{2}\phi^{-2m}\phi_z^2\right)\right] \\ &\quad + D_2\left[c_1\left(\phi^n\left(1 + m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz}\right)\right)\right. \\ &\qquad \left. + c_2\left(\frac{n}{m-1}\phi^{1-m} - \phi^{1-2m}\phi_{tz} + m\phi^{-2m}\phi_t\phi_z\right)\right]. \end{aligned} \tag{30}$$

Equation (30) is satisfied for arbitrary functions $\phi(t,z)$. When $\phi(t,z)$ is a solution of the partial differential equation (20), then

$$D_1\left[c_1\phi+c_2\left(\frac{1}{2-m-n}\left(\phi^{2-m-n}-1\right)+\frac{(1-m-n)}{2}\phi^{-2m}\phi_z^2\right)\right] + D_2\left[c_1\left(\phi^n\left(1+m\phi^{-m-1}\phi_t\phi_z-\phi^{-m}\phi_{tz}\right)\right) + c_2\left(\frac{n}{m-1}\phi^{1-m}-\phi^{1-2m}\phi_{tz}+m\phi^{-2m}\phi_t\phi_z\right)\right]=0. \quad (31)$$

Hence, any conserved vector of the partial differential equation (20) with m and n satisfying the conditions of this case and with multiplier of the form $\Lambda = \Lambda(\phi)$ is a linear combination of the two conserved vectors

$$T^1=\phi, \qquad T^2=\phi^n\left(1+m\phi^{-m-1}\phi_t\phi_z-\phi^{-m}\phi_{tz}\right), \quad (32)$$

$$T^1=\frac{1}{2-m-n}\left(\phi^{2-m-n}-1\right)+\frac{(1-m-n)}{2}\phi^{-2m}\phi_z^2, \quad T^2=\frac{n}{1-m}\phi^{1-m}-\phi^{1-2m}\phi_{tz}+m\phi^{-2m}\phi_t\phi_z. \quad (33)$$

The conserved vector (32) is the elementary conserved vector.

(ii) $n+m=1$, $m\neq 1$. Proceeding as before we obtain

$$T^1=\phi, \qquad T^2=\phi^{1-m}\left(1+m\phi^{-m-1}\phi_t\phi_z-\phi^{-m}\phi_{tz}\right), \quad (34)$$

$$T^1=-\frac{1}{2}\phi^{-2m}\phi_z^2+\ln\phi, \quad T_2=\phi^{1-2m}\ln\phi-\frac{1}{1-m}\phi^{1-m} - \left(\phi^{1-2m}\ln\phi\right)\phi_{tz}+\left(m\phi^{-2m}\ln\phi\right)\phi_t\phi_z. \quad (35)$$

The conserved vector (34) is the elementary conserved vector with $n=1-m$. The multiplier for (35) is, from (29),

$$\Lambda(\phi)=\ln\phi. \quad (36)$$

(iii) $n+m=2$, $m\neq 1$. We find that

$$T^1=\phi, \qquad T^2=\phi^{2-m}\left(1+m\phi^{-m-1}\phi_t\phi_z-\phi^{-m}\phi_{tz}\right), \quad (37)$$

$$T^1=-\frac{1}{2}\phi^{-2m}\phi_z^2+\ln\phi, \quad T^2=\frac{2-m}{1-m}\phi^{1-m}-\phi^{1-2m}\phi_{tz}+m\phi^{-2m}\phi_t\phi_z. \quad (38)$$

The conserved vector (37) is the elementary conserved vector with $n=2-m$. The multiplier for (38) is, from (28),

$$\Lambda(\phi)=\frac{1}{\phi}. \quad (39)$$

(iv) $m=1$, $n=0$. We obtain

$$T^1=\phi, \qquad T^2=1+\phi^{-2}\phi_t\phi_z-\phi^{-1}\phi_{tz}, \quad (40)$$

$$T^1=-\frac{1}{2}\phi^{-2}\phi_z^2+\phi\ln\phi-\phi, \quad T^2=-\left(\phi^{-1}\ln\phi\right)\phi_{tz}+\left(\phi^{-2}\ln\phi\right)\phi_t\phi_z. \quad (41)$$

The conserved vector (40) is the elementary conserved vector with $m=1$, $n=0$. The multiplier for (41) is given by (29).

(v) $m=n=1$. We obtain

$$T^1=\phi, \qquad T^2=\phi\left(1+\phi^{-2}\phi_t\phi_z-\phi^{-1}\phi_{tz}\right), \quad (42)$$

$$T^1=-\frac{1}{2}\phi^{-2}\phi_z^2+\ln\phi, \quad T^2=\ln\phi-\phi^{-1}\phi_{tz}+\phi^{-2}\phi_t\phi_z. \quad (43)$$

The conserved vector (42) is the elementary conserved vector with $m=n=1$. The multiplier for (43) is (39).

(vi) $m=1$, $n\neq 0$, $n\neq 1$. We obtain

$$T^1=\phi, \qquad T^2=\phi^n\left(1+\phi^{-2}\phi_t\phi_z-\phi^{-1}\phi_{tz}\right), \quad (44)$$

$$T^1=\frac{1}{1-n}\phi^{1-n}-\frac{n}{2}\phi^{-2}\phi_z^2, \quad T^2=\ln\phi-\phi^{-1}\phi_{tz}+\phi^{-2}\phi_t\phi_z. \quad (45)$$

The conserved vector (44) is the elementary conserved vector with $m=1$. The multiplier for (45) is

$$\Lambda(\phi)=\phi^{-n}. \quad (46)$$

3.2. The Search for Higher Order Conservation Laws. We now consider a multiplier of the form

$$\Lambda=\Lambda(\phi,\phi_z). \quad (47)$$

As before the determining equation for the multiplier is

$$E_\phi\left[\Lambda(\phi,\phi_z)F(\phi,\phi_t,\phi_z,\phi_{tz},\phi_{zz},\phi_{tzz})\right]=0, \quad (48)$$

where F is given by (23). By equating the coefficient of the highest order derivative term, ϕ_{tzzz}, to zero in (48) we have

$$\frac{\partial\Lambda}{\partial\phi_z}=0, \quad (49)$$

and therefore

$$\Lambda(\phi,\phi_z)=\Lambda(\phi). \quad (50)$$

Hence, (47) does not give a new multiplier or a new conserved vector.

Consider next the multiplier

$$\Lambda=\Lambda(\phi,\phi_z,\phi_{zz}). \quad (51)$$

The determining equation for the multiplier is

$$E_\phi\left[\Lambda\left(\phi,\phi_z,\phi_{zz}\right)F\left(\phi,\phi_t,\phi_z,\phi_{tz},\phi_{zz},\phi_{tzz}\right)\right]=0, \quad (52)$$

where F is given by (23). By Equating the coefficients of $\phi_{tz}\phi_{zzzz}$, $\phi_t\phi_{zzzz}$, and ϕ_{zzzz}^2 to zero in (52), we obtain the following system of equations:

$$\phi_{tz}\phi_{zzzz}:(2m-n)\,\phi_z\frac{\partial^2\Lambda}{\partial\phi_{zz}^2}+\phi\frac{\partial^2\Lambda}{\partial\phi_z\phi_{zz}}=0, \quad (53)$$

$$\begin{aligned}\phi_t\phi_{zzzz}:&\left(\phi^{m+2}+m\left(n-m-1\right)\phi^n+m\phi^{n+1}\right)\frac{\partial^2\Lambda}{\partial\phi_{zz}^2}\\&+\left(m+n\right)\phi^{n+1}\frac{\partial\Lambda}{\partial\phi_{zz}}+\phi^{n+2}\frac{\partial^2\Lambda}{\partial\phi\partial\phi_{zz}}=0,\end{aligned} \quad (54)$$

$$\phi_{zzzz}^2:\frac{\partial^2\Lambda}{\partial\phi_{zz}^2}=0. \quad (55)$$

From (55) it follows that

$$\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=A\left(\phi,\phi_z\right)\phi_{zz}+B\left(\phi,\phi_z\right). \quad (56)$$

Substituting (56) into (53) we find that

$$\frac{\partial A}{\partial\phi_z}=0, \quad (57)$$

and therefore

$$A\left(\phi,\phi_z\right)=A\left(\phi\right). \quad (58)$$

Thus, (51) becomes

$$\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=A\left(\phi\right)\phi_{zz}+B\left(\phi,\phi_z\right). \quad (59)$$

Now substitute (59) into (54) which gives

$$\phi\frac{dA}{d\phi}+\left(m+n\right)A=0. \quad (60)$$

The solution to (60) is

$$A=c_1\phi^{-(m+n)}, \quad (61)$$

where c_1 is a constant. Equation (59) becomes

$$\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=c_1\phi^{-(m+n)}\phi_{zz}+B\left(\phi,\phi_z\right). \quad (62)$$

Now substitute (62) into (48) and then equate the coefficient of ϕ_{tzzz} in (48) to zero. This gives

$$\frac{\partial B}{\partial\phi_z}=c_1\left(n-2m\right)\phi^{-(m+n+1)}\phi_z \quad (63)$$

and integrating (63) we have

$$B\left(\phi,\phi_z\right)=\frac{1}{2}c_1\left(n-2m\right)\phi^{-(m+n+1)}\phi_z^2+P\left(\phi\right). \quad (64)$$

Thus, (62) becomes

$$\begin{aligned}\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=&\,c_1\phi^{-(m+n)}\phi_{zz}\\&+\frac{1}{2}c_1\left(n-2m\right)\phi^{-(m+n+1)}\phi_z^2+P\left(\phi\right).\end{aligned} \quad (65)$$

Lastly, substitute (65) into (48) and equate the coefficient of $\phi_t\phi_z^2\phi_{zz}$ to zero, which gives

$$\left(m-n-1\right)c_1=0. \quad (66)$$

It follows from (66) that there are two cases.

Case 1 ($c_1=0$). If $m-n-1\neq0$, then from (66) we have $c_1=0$. Therefore,

$$\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=P\left(\phi\right). \quad (67)$$

Thus, $\Lambda(\phi,\phi_z,\phi_{zz})$ does not give a new multiplier and therefore new conservation laws will not be derived.

Case 2 ($n=m-1$). If $c_1\neq0$, then from (66), we have $n=m-1$. Thus, (65) becomes

$$\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=c_1\phi^{1-2m}\phi_{zz}-\frac{1}{2}\left(m+1\right)c_1\phi^{-2m}\phi_z^2+P\left(\phi\right). \quad (68)$$

Now substitute (68) into (48) and equate the coefficient of $\phi_z\phi_{tz}$ in (48) to zero, which gives

$$\frac{d^2P}{d\phi^2}+\frac{\left(2m-1\right)}{\phi}\frac{dP}{d\phi}=\left(m-2\right)c_1\phi^{1-2m}. \quad (69)$$

Solving (69) we have

$$P\left(\phi\right)=\frac{m-2}{3-2m}\phi^{3-2m}c_1+\frac{\phi^{2(1-m)}}{2\left(1-m\right)}c_2+c_3 \quad (70)$$

provided that $m\neq3/2$ and $m\neq1$. Since $n=m-1$, these two special cases correspond to the points $(3/2,1/2)$ and $(1,0)$ on the (m,n) plane in Figure 1.

Consider first the general case $n=m-1$ excluding the points $(3/2,1/2)$ and $(1,0)$. Equation (68) becomes

$$\begin{aligned}\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=&\,c_1\phi^{1-2m}\phi_{zz}-\frac{1}{2}\left(m+1\right)c_1\phi^{-2m}\phi_z^2\\&+\frac{m-2}{3-2m}\phi^{3-2m}c_1+\frac{\phi^{2(1-m)}}{2\left(1-m\right)}c_2+c_3.\end{aligned} \quad (71)$$

Substituting (71) into (48) we find that $c_2=0$. Hence,

$$\begin{aligned}\Lambda\left(\phi,\phi_z,\phi_{zz}\right)=&\,c_1\phi^{1-2m}\phi_{zz}-\frac{1}{2}\left(m+1\right)c_1\phi^{-2m}\phi_z^2\\&+\frac{m-2}{3-2m}\phi^{3-2m}c_1+c_3.\end{aligned} \quad (72)$$

Since the multiplier (72) contains two constants, c_1 and c_3, it leads to two conserved vectors. The conserved vector corresponding to $c_3 = 1, c_1 = 0$ is the elementary conserved vector (32). The constants $c_1 = 1, c_3 = 0$ lead to the conserved vector

$$T_1 = \frac{1}{2}\phi^{-2m}\phi_{zz}^2 - \frac{1}{6}m(m+2)\phi^{-2m-2}\phi_z^4 + \frac{1}{2}(3-m)\phi^{1-2m}\phi_z^2 + \frac{1}{2(3-2m)}\phi^{4-2m}, \quad (73)$$

$$\begin{aligned} T_2 = &\left[\frac{1}{6}m(m+5)\phi^{-2m-2}\phi_z^3 - \frac{(m-3)(m-1)}{(3-2m)}\phi^{1-2m}\phi_z\right]\phi_t \\ &- \left[\frac{1}{2}(m+1)\phi^{-2m-1}\phi_z^2 + \frac{(2-m)}{(3-2m)}\phi^{2-2m}\right]\phi_{tz} \\ &- \frac{1}{2}(m-1)\phi^{-m-1}\phi_z^2 + \frac{(m-1)}{(3-2m)}\phi^{2-m}. \end{aligned} \quad (74)$$

The case $n = m-1$ with $m = 1$ and $n = 0$ has already been considered. The multiplier is given by (29) and the conserved vectors by (40) and (41).

Consider $n = m - 1$ with $m = 3/2$ and $n = 1/2$. The differential equation (69) becomes

$$\frac{d^2P}{d\phi^2} + \frac{2}{\phi}\frac{dP}{d\phi} = -\frac{1}{2}\frac{c_1}{\phi^2}. \quad (75)$$

The general solution of (75) is

$$P(\phi) = -\frac{1}{2}c_1 \ln\phi + \frac{c_3}{\phi} + c_4. \quad (76)$$

When $m = 3/2$ the multiplier (68) becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1\phi^{-2}\phi_{zz} - \frac{5}{4}c_1\phi^{-3}\phi_z^2 - \frac{1}{2}c_1\ln\phi + \phi^{-1}c_3 + c_4. \quad (77)$$

On substituting (77) into the determining equation (48) we find that $c_3 = 0$ and the multiplier reduces to

$$\Lambda = c_1\phi^{-2}\phi_{zz}^2 - \frac{5}{4}c_1\phi^{-3}\phi_z^2 - \frac{1}{2}c_1\ln\phi + c_4. \quad (78)$$

The multiplier (78) again contains two arbitrary constants, c_1 and c_4. Setting the constants $c_4 = 1, c_1 = 0$ gives the elementary conserved vector (32). Setting $c_1 = 1, c_4 = 0$ leads to the conserved vector

$$T_1 = \frac{1}{2}\phi^{-3}\phi_{zz}^2 - \frac{7}{8}\phi^{-5}\phi_z^4 + \frac{3}{4}\phi^{-2}\phi_z^2\frac{1}{2}\phi\ln\phi - \frac{1}{2}\phi, \quad (79)$$

$$\begin{aligned} T^2 = &\left[\frac{13}{8}\phi^{-5}\phi_z^3 - \phi^{-2}\phi_z + \frac{3}{4}\left(\phi^{-2}\ln\phi\right)\phi_z\right]\phi_t \\ &- \left[\frac{5}{4}\phi^{-4}\phi_z^2 + \frac{1}{2}\phi^{-1}\ln\phi\right]\phi_{tz} - \frac{1}{4}\phi^{-5/2}\phi_z^2 + \frac{1}{2}\phi^{1/2}\ln\phi. \end{aligned} \quad (80)$$

Consider next multipliers of the form

$$\Lambda(\phi, \phi_z, \phi_{zz}, \phi_{zzz}). \quad (81)$$

The determining equation for the multiplier is

$$E_\phi\left(\Lambda(\phi, \phi_z, \phi_{zz}, \phi_{zzz}) F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz})\right] = 0, \quad (82)$$

where F is given by (23). Equating the coefficient of ϕ_{tzzzzz} in (82) to zero, we find that

$$\frac{\partial\Lambda}{\partial\phi_{zzz}} = 0 \quad (83)$$

and therefore

$$\Lambda = \Lambda(\phi, \phi_z, \phi_{zz}), \quad (84)$$

which has already been considered.

Harris [6] proved that, except possibly for the two special cases $n = m-1$ with $m \neq 1$ and $m = 1$ with $n \neq 0$, there are no more independent conserved vectors. She proved this result using the direct method for conservation laws.

The multipliers and the corresponding conserved vectors for the partial differential equation (20) are listed in Table 1. This table was presented by Maluleke and Mason [7] without the multipliers. These conserved vectors agree with the results obtained by Barcilon and Richter [5] and Harris [6].

3.3. Association of Lie Point Symmetries with Conserved Vectors. The Lie point symmetries for the partial differential equation (20) are listed in Table 2. These Lie point symmetries were derived by Maluleke and Mason [7, 12]. Using (14) we will investigate which of the Lie point symmetries are associated with the conserved vectors for the Magma equation (20).

(i) $0 \leq n < \infty$, $0 \leq m < \infty$. Consider first the Lie symmetry generator

$$X = (c_1 + (2-m-n)c_3 t)\frac{\partial}{\partial t} + (c_2 + (n-m)c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial\phi} \quad (85)$$

and the elementary conserved vector (32). Applying (14) we find that (85) is associated with the conserved vector (32) provided that $c_3 = 0$, that is, provided that

$$X = c_1\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial z}. \quad (86)$$

(ii) $n + m \neq 2$, $m \neq 1$, $n + m \neq 1$. Consider next the Lie point symmetry generator (85), with the conserved vector (33). Applying (14) we find that (85) is associated with (33) provided that $c_3 = 0$, that is, provided X is given by (86).

(iii) $m = 1$, $n \neq 0$, $n \neq 1$. Now consider

$$X = (-c_3 t + c_1)\frac{\partial}{\partial t} + (c_3 z + c_2)\frac{\partial}{\partial z} + \frac{2c_3\phi}{n-1}\frac{\partial}{\partial\phi} \quad (87)$$

Table 1: Multipliers and conserved vectors for the partial differential equation (20).

Case A. $0 \le n < \infty, 0 \le m < \infty$ Multiplier: $\Lambda = 1$ $T^1 = \phi$ $T^2 = \phi^n(1 + m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz})$	Case B.5. $n + m = 1, m \neq 1$ Multiplier: $\Lambda = \phi^{-2(n-2)}$ $T^1 = -\frac{1}{2}\phi^{-2m}\phi_z^2 + \ln\phi$ $T^2 = \phi^{1-m}\ln\phi - \frac{1}{1-m}\phi^{1-m} - (\phi^{1-2m}\ln\phi)\phi_{tz} + (m\phi^{-2m}\ln\phi)\phi_t\phi_z$
Case B.1. $m \neq 1, m + n \neq 1, m + n \neq 2$ Multiplier: $\Lambda = \phi^{1-m-n}$ $T^1 = \frac{1}{2-m-n}(\phi^{2-m-n} - 1) + \frac{1}{2}(1-m-n)\phi^{-2m}\phi_z^2$ $T^2 = \frac{n}{1-m}\phi^{1-m} - \phi^{1-2m}\phi_{tz} + m\phi^{-2m}\phi_t\phi_z$	Case B.6. $n + m = 2, m \neq 1$ Multiplier: $\Lambda = \frac{1}{\phi}$ $T^1 = -\frac{1}{2}\phi^{-2m}\phi_z^2 + \ln\phi$ $T^2 = \frac{2-m}{1-m}\phi^{1-m} - \phi^{1-2m}\phi_{tz} + m\phi^{-2m}\phi_t\phi_z$
Case B.2. $m = 1, n \neq 0, n \neq 1$ Multiplier: $\Lambda = \phi^{-n}$ $T^1 = \frac{1}{1-n} - \frac{n}{2}\phi^{-2}\phi_z^2$ $T^2 = n\ln\phi - \phi^{-1}\phi_{tz} + \phi^{-2}\phi_t\phi_z$	Case C.1. $n = m - 1, m \neq \frac{3}{2}, m \neq 1$ Multiplier: $\Lambda = \phi^{1-2m}\phi_{zz} - \frac{1}{2}(m+1)\phi^{-2m}\phi_z^2 + \frac{m-2}{3-2m}\phi^{3-2m}$ $T^1 = \frac{1}{1-n}\phi^{1-n} - \frac{n}{2}\phi^{-2}\phi_z^2 + \frac{1}{2}(3-m)\phi^{1-2m}\phi_z^2 + \frac{1}{2(3-2m)}\phi^{4-2m}$ $T^2 = \left[\frac{1}{6}m(m+5)\phi^{-2m-2}\phi_z^3 - \frac{(m-3)(m-1)}{(3-2m)}\phi^{1-2m}\phi_z\right]\phi_t - \left[\frac{1}{2}(m+1)\phi^{-2m-1}\phi_z^2 + \frac{(2-m)}{(3-2m)}\phi^{2-2m}\right]\phi_{tz} - \frac{1}{2}(m-1)\phi^{-m-1}\phi_z^2 + \frac{(m-1)}{(3-2m)}\phi^{2-m}$
Case B.3. $m = n = 1$ Multiplier: $\Lambda = \frac{1}{\phi}$ $T^1 = -\frac{1}{2}\phi^{-2}\phi_z^2 + \ln\phi$ $T^2 = \ln\phi - \phi^{-1}\phi_{tz} + \phi^{-2}\phi_t\phi_z$	Case C.2. $n = m - 1, m = \frac{3}{2}, n = \frac{1}{2}$ Multiplier: $\Lambda = \phi^{-2}\phi_{zz}^2 - \frac{5}{4}\phi^{-3}\phi_z^2 - \frac{1}{2}\ln\phi$ $T^1 = \frac{1}{2}\phi^{-3}\phi_{zz}^2 - \frac{7}{8}\phi^{-5}\phi_z^4 + \frac{3}{4}\phi^{-2}\phi_z^2 + \frac{1}{2}\phi\ln\phi - \frac{1}{2}\phi$ $T^2 = \left[\frac{13}{8}\phi^{-5}\phi_z^3 - \phi^{-2}\phi_z + \frac{3}{4}(\phi^{-2}\ln\phi)\ \phi_z\right]\phi_t - \left[\frac{5}{4}\phi^{-4}\phi_z^2 + \frac{1}{2}\phi^{-1}\ln\phi\right]\phi_{tz} - \frac{1}{4}\phi^{-(5/2)}\phi_z^2 + \frac{1}{2}\phi^{1/2}\ln\phi$
Case B.4. $m = 1, n = 0$ Multiplier: $\Lambda = \phi^2$ $T^1 = \frac{1}{2}\phi^{-2}\phi_z^2 + \phi\ln\phi - \phi$ $T^2 = -(\phi^{-1}\ln\phi)\phi_{tz} + (\phi^{-2}\ln\phi)\phi_t\phi_z$	

and the conserved vector (45). Applying (14) we find that (87) is associated with (45) provided that $c_3 = 0$, that is, provided that X is given by (86).

(iv) $m = n = 1$. Consider next

$$X = c_1\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial\phi} \tag{88}$$

and the conserved vector (42). Applying (14) we find that (88) is associated with (42) provided that $c_3 = 0$, that is, provided that X is (86).

(v) $m = 1$, $n = 0$. Consider

$$X = \xi^1(t,z)\frac{\partial}{\partial t} + (c_2 + c_3 z)\frac{\partial}{\partial z} - 2c_3\phi\frac{\partial}{\partial\phi} \tag{89}$$

and the conserved vector (41). Applying (14) we find that (89) is associated with (41) provided that $c_3 = 0$, that is, provided that

$$X = \xi^1(t,z)\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial z}. \tag{90}$$

(vi) $n + m = 1$, $m \neq 1$. Consider next

$$X = (c_1 + c_3 t)\frac{\partial}{\partial t} + (c_2 + (1-2m)c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial\phi} \tag{91}$$

and the conserved vector (35). Applying (14) we find that (91) is associated with (35) provided that $c_3 = 0$, that is, provided that X is given by (86).

(vii) $n + m = 2, m \neq 1$. Consider

$$X = c_1 \frac{\partial}{\partial t} + (c_2 + 2(1-m) c_3 z) \frac{\partial}{\partial z} + 2c_3 \phi \frac{\partial}{\partial \phi} \tag{92}$$

with the conserved vector (38). Applying (14) we find that (92) is associated with (38) provided that $c_3 = 0$, that is, provided that X is given by (86).

(viii) $n = m - 1, m \neq 3/2, m \neq 1$. Consider

$$X = (c_1 + (3-2m) c_3) \frac{\partial}{\partial t} + (c_2 - c_3 z) \frac{\partial}{\partial z} + 2c_3 \phi \frac{\partial}{\partial \phi} \tag{93}$$

with the conserved vector given by (73) and (74). Applying (14) we find that (93) is associated with the conserved vector with components (73) and (74) provided that $c_3 = 0$, that is, provided that X is given by (86).

(ix) $n = m-1, m = 3/2, n = 1/2$. Finally consider the Lie point symmetry (93) and the conserved vector with components (79) and (80). Applying (14) we find that (93) is associated with this conserved vector provided that $c_3 = 0$, that is, provided that X is given by (86).

Except for the conserved vector (41) ($n = 0, m = 1$) the conserved vectors are all associated with the Lie point symmetry which generates travelling wave solutions. The Lie point symmetry with which the conserved vector (41) is associated contains (86) as a special case. In all cases new conserved vectors are not generated by (15).

Next we derive the conservation laws for the magma equation with an exponential law for the permeability and viscosity using the multiplier method.

4. Conservation Laws for the Magma Equation with an Exponential Law for the Permeability and Viscosity by the Multiplier Method

When the permeability and viscosity are related to the voidage by exponential laws the magma equation becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[\exp[n(\phi-1)]\left(1 - \frac{\partial}{\partial z}\left(\exp[-m(\phi-1)] \frac{\partial \phi}{\partial t}\right)\right)\right] = 0. \tag{94}$$

4.1. Lower Order Conservation Laws. In order to derive conservation laws for (94) consider a multiplier of the form (21). A multiplier for the partial differential equation has the property (22), where now

$$\begin{aligned} F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz}) &= \phi_t + n\phi_z \exp[n(\phi-1)] \\ &\quad + mn\phi_t \phi_z^2 \exp[(n-m)(\phi-1)] \\ &\quad - n\phi_z \phi_{tz} \exp[(n-m)(\phi-1)] \\ &\quad - m^2 \phi_t \phi_z^2 \exp[(n-m)(\phi-1)] \\ &\quad + m\phi_t \phi_{zz} \exp[(n-m)(\phi-1)] \\ &\quad + 2m\phi_z \phi_{tz} \exp[(n-m)(\phi-1)] \\ &\quad - \exp[(n-m)(\phi-1)] \phi_{tzz}. \end{aligned} \tag{95}$$

The determining equation for the multiplier is given by (24), where E_ϕ is given by (18). Separating (24) with respect to products and powers of the partial derivatives of ϕ we obtain the following system of equations:

$$\phi_z \phi_{tz} : \frac{d^2\Lambda}{d\phi^2} + (m+n) \frac{d\Lambda}{d\phi} = 0, \tag{96}$$

$$\phi_t \phi_{zz} : \frac{d^2\Lambda}{d\phi^2} + (m+n) \frac{d\Lambda}{d\phi} = 0, \tag{97}$$

$$\phi_t \phi_z^2 : \frac{d^3\Lambda}{d\phi^3} + 2n \frac{d^2\Lambda}{d\phi^2} + (n^2 - m^2) \frac{d\Lambda}{d\phi} = 0. \tag{98}$$

Equation (96) is the same as (97). It is readily verified that every solution of (96) is a solution of (98). We therefore need to consider only (96). The general solution of (96) is

$$\Lambda(\phi) = c_2 \exp[-(m+n)\phi] + c_1, \quad \text{if } n + m \neq 0, \tag{99}$$

$$\Lambda(\phi) = c_2 \phi + c_1, \quad \text{if } n + m = 0. \tag{100}$$

We are considering $n \geq 0$ and $m \geq 0$ and therefore $n+m = 0$ only if $n = 0$ and $m = 0$. Proceeding as before we have for various combinations of m and n different conserved vectors.

(i) $n + m \neq 0, m \neq 0$. This gives the conserved vectors

$$\begin{aligned} T^1 &= \phi, \\ T^2 &= \exp[n(\phi-1)] + m \exp[(n-m)(\phi-1)] \phi_t \phi_z \\ &\quad - \exp[(n-m)(\phi-1)] \phi_{tz}, \end{aligned} \tag{101}$$

$$\begin{aligned} T^1 &= -\frac{1}{m+n} \exp[-(m+n)\phi] \\ &\quad - \frac{(n+m)}{2} \phi_z^2 \exp[-2m\phi + m - n], \\ T^2 &= -\frac{n}{m} \exp[-(m\phi + n)] \\ &\quad + \exp[-2m\phi + m - n](m\phi_t \phi_z - \phi_{tz}). \end{aligned} \tag{102}$$

TABLE 2: Lie point symmetries of the partial differential equation (20).

Case 1. $m \neq 1, m \neq n, n \neq 0$	Case 4. $n = 0, m \neq 0, m \neq \frac{4}{3}$
$X_1 = (2 - n - m)t\frac{\partial}{\partial t} + (n - m)z\frac{\partial}{\partial z} + 2\phi\frac{\partial}{\partial \phi}$	$X_1 = \xi(t)\frac{\partial}{\partial t}$
$X_2 = \frac{\partial}{\partial t}$	$X_2 = -\frac{m}{2}z\frac{\partial}{\partial z} + \phi\frac{\partial}{\partial \phi}$
$X_3 = \frac{\partial}{\partial z}$	$X_3 = \frac{\partial}{\partial z}$
Case 2. $n \neq 0, m = 1, n \neq 1$	Case 5. $m = 0, n = 0$
$X_1 = -t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} + \frac{2\phi}{n-1}\frac{\partial}{\partial \phi}$	$X_1 = \xi(t)\frac{\partial}{\partial t}$
$X_2 = \frac{\partial}{\partial t}$	$X_2 = \frac{\partial}{\partial z}$
$X_3 = \frac{\partial}{\partial z}$	$X_3 = \sinh(2z)\frac{\partial}{\partial z} + \phi\cosh(2z)\frac{\partial}{\partial \phi}$
Case 3. $m = \frac{4}{3}, n = 0$	$X_4 = \cosh(2z)\frac{\partial}{\partial z} + \phi\sinh(2z)\frac{\partial}{\partial \phi}$
$X_1 = -\frac{1}{3}z^2\frac{\partial}{\partial z} + z\phi\frac{\partial}{\partial \phi}$	$X_5 = \phi\frac{\partial}{\partial \phi}$
$X_2 = \xi(t)\frac{\partial}{\partial t}$	$X_A = A(t,z)\frac{\partial}{\partial \phi}$
$X_3 = z\frac{\partial}{\partial z} - \frac{3}{2}\phi\frac{\partial}{\partial \phi}$	$A(t,z)$ satisfies (20) with $n = m = 0$
$X_4 = \frac{\partial}{\partial z}$	

The conserved vector (101) is the elementary conserved vector. The multiplier for (102) is, from (99),

$$\Lambda(\phi) = \exp[-(m+n)\phi]. \tag{103}$$

(ii) $n = m = 0$. We obtain two conserved vectors

$$T^1 = \phi, \qquad T^2 = 1 - \phi_{tz}, \tag{104}$$

$$T^1 = \frac{\phi^2}{2}, \qquad T^2 = \phi_t - \phi_{tz}. \tag{105}$$

The conserved vector (104) is the elementary conserved vector with multiplier c_1 and $m = n = 0$. The multiplier for (105) is, from (100),

$$\Lambda(\phi) = \phi. \tag{106}$$

(iii) $m = 0$ and $n > 0$. We again obtain two conserved vectors

$$T^1 = \phi, \qquad T^2 = \exp[n(\phi - 1)](1 - \phi_{tz}), \tag{107}$$

$$\begin{aligned} T^1 &= -\frac{1}{n}\exp[-n\phi] - \frac{1}{2}n\exp[-n\phi]\phi_z^2, \\ T^2 &= n\exp[-n\phi] - \phi_{tz}. \end{aligned} \tag{108}$$

The conserved vector (107) is the elementary conserved vector with multiplier c_1 and $m = 0$. The multiplier of the conserved vector (108) is by (99)

$$\Lambda(\phi) = \exp[-n\phi]. \tag{109}$$

(iv) $n = 0$, $m > 0$. Finally we obtain the conserved vectors

$$\begin{aligned} T^1 &= \phi, \\ T^2 &= 1 + m\exp[-m(\phi - 1)]\phi_t\phi_z - \exp[-m(\phi - 1)]\phi_{tz}, \end{aligned} \tag{110}$$

$$\begin{aligned} T_1 &= -\frac{1}{m}\exp(-m\phi) - \frac{1}{2}m\exp[m(1 - 2\phi)]\phi_z^2, \\ T_2 &= \exp[m(1 - 2\phi)](m\phi_t\phi_z - \phi_{tz}). \end{aligned} \tag{111}$$

The conserved vector (110) is the elementary conserved vector with multiplier c_1 and $n = 0$, while the multiplier for (111) is by (99):

$$\Lambda = \exp[-m\phi]. \tag{112}$$

4.2. The Search for Higher Order Conservation Laws. We now consider a multiplier of the form

$$\Lambda = \Lambda(\phi, \phi_z). \tag{113}$$

The determining equation for the multiplier is (48), where F is given by (95). By equating the coefficient of the highest order derivative term, ϕ_{tzzz}, to zero in (48) we obtain again (49), so that $\Lambda(\phi, \phi_z) = \Lambda(\phi)$. The multiplier therefore reduces to that of the previous case and new conserved vectors are not derived.

Consider next the multiplier

$$\Lambda = \Lambda(\phi, \phi_z, \phi_{zz}). \tag{114}$$

As before, the determining equation for the multiplier is (48), where F is given by (95). By equating the coefficients of $\phi_{tz}\phi_{zzzz}$, $\phi_t\phi_{zzzz}$, and ϕ_{zzzz}^2 to zero in (48), the following system of equations is obtained:

$$\phi_{tz}\phi_{zzzz} : (2m-n)\,\phi_z \frac{\partial^2\Lambda}{\partial\phi_{zz}^2} + \frac{\partial^2\Lambda}{\partial\phi_z\phi_{zz}} = 0, \tag{115}$$

$$\begin{aligned}\phi_t\phi_{zzzz} : m\left((n-m)\,\phi_z^2+\phi_{zz}\right)\frac{\partial^2\Lambda}{\partial\phi_{zz}^2}\\ +(m+n)\frac{\partial\Lambda}{\partial\phi_{zz}}+\frac{\partial^2\Lambda}{\partial\phi\partial\phi_{zz}} = 0,\end{aligned} \tag{116}$$

$$\phi_{zzzz}^2 : \frac{\partial^2\Lambda}{\partial\phi_{zz}^2} = 0. \tag{117}$$

By using (115) and (117), it is readily shown that (56) again holds. Substituting (56) into (116) gives

$$\frac{dA}{d\phi} + (m+n)\,A = 0, \tag{118}$$

and therefore

$$A(\phi) = c_1 \exp\left[-(m+n)\,\phi\right]. \tag{119}$$

Equation (56) now becomes

$$\Lambda(\phi,\phi_z,\phi_{zz}) = c_1 \exp\left[-(m+n)\,\phi\right]\phi_{zz} + B(\phi,\phi_z). \tag{120}$$

Substituting (120) into the determining equation (48) and then equating the coefficients of ϕ_{tzzz} in (48) to zero gives

$$\frac{\partial B}{\partial\phi_z} = \frac{1}{2}(n-2m)\,c_1\exp\left[-(m+n)\,\phi\right]\phi_z \tag{121}$$

and hence

$$B(\phi,\phi_z) = \frac{1}{4}(n-2m)\,c_1\exp\left[-(m+n)\,\phi\right]\phi_z^2 + P(\phi). \tag{122}$$

The multiplier becomes

$$\begin{aligned}\Lambda(\phi,\phi_z,\phi_{zz}) = c_1\phi^{-(m+n)}\phi_{zz} + \frac{1}{4}(n-2m)\\ \times c_1\exp\left[-(m+n)\,\phi\right]\phi_z^2 + P(\phi).\end{aligned} \tag{123}$$

Finally we substitute (123) back into (48) and equate the coefficient of $\phi_t\phi_z^2\phi_{zz}$ to zero. This yields

$$m(m-n)\,c_1 = 0. \tag{124}$$

There are three cases to consider, $m=0$, $m=n$, and $c_1=0$. The conserved vectors for $m=0$, $n>0$ are given by (107) and (108). We now consider the two remaining cases.

Case 1 ($m\neq n$, $m>0$). Then, $c_1=0$ and (123) reduces to

$$\Lambda(\phi,\phi_z,\phi_{zz}) = P(\phi). \tag{125}$$

The multiplier is therefore a function of ϕ only which does not yield new conserved vectors.

Case 2 ($c_1\neq 0$, $m>0$). Then, $m=n$ and the multiplier (123) becomes

$$\begin{aligned}\Lambda(\phi,\phi_z,\phi_{zz}) = c_1\exp\left[-2n\phi\right]\phi_{zz}\\ -\frac{1}{4}nc_1\exp\left[-2n\phi\right]\phi_z^2 + P(\phi).\end{aligned} \tag{126}$$

Equation (126) is substituted back into the determining equation (48) and by equating the coefficient of $\phi_z\phi_{tz}$ to zero we obtain

$$\frac{d^2P}{d\phi^2} + 2n\frac{dP}{d\phi} = n\exp\left[-n(2\phi+1)+1\right]c_1. \tag{127}$$

The general solution to (127) is

$$\begin{aligned}P(\phi) = -\frac{c_1}{4n}(1+2n\phi)\exp\left[1-n-2n\phi\right]\\ + c_2\exp\left[-2n\phi\right] + c_3,\end{aligned} \tag{128}$$

where $n\neq 0$ since $n=m$ and $m\neq 0$. Thus, (123) becomes

$$\begin{aligned}\Lambda(\phi,\phi_z,\phi_{zz}) = c_1\exp\left[-2n\phi\right]\phi_{zz} - \frac{1}{4}nc_1\exp\left[-2n\phi\right]\phi_z^2\\ -\frac{c_1}{4n}(1+2n\phi)\exp\left[1-n-2n\phi\right]\\ + c_2\exp\left[-2n\phi\right] + c_3.\end{aligned} \tag{129}$$

Finally substituting (129) into the determining equation (48) gives $c_2=0$ and therefore

$$\begin{aligned}\Lambda(\phi,\phi_z,\phi_{zz}) = c_1\exp\left[-2n\phi\right]\phi_{zz} - \frac{1}{4}nc_1\exp\left[-2n\phi\right]\phi_z^2\\ -\frac{c_1}{4n}(1+2n\phi)\exp\left[1-n-2n\phi\right] + c_3.\end{aligned} \tag{130}$$

Two conserved vectors are obtained since the multiplier (130) contains two arbitrary constants. The constant c_3 gives the elementary conserved vector (100) while the constant c_1 gives the new conserved vector

$$\begin{aligned}T^1 = \frac{1}{2}\exp\left[-2n\phi\right]\phi_{zz}^2 - \frac{1}{2}n^2\exp\left[-2n\phi\right]\phi_z^4\\ + \frac{1}{4}n(1+2n\phi)\,\phi_z^2 + \frac{1}{8n^2}(1+2n\phi)\exp\left[1-n-2n\phi\right],\end{aligned} \tag{131}$$

$$\begin{aligned}T_2 = \left[n^2\exp\left[-2n\phi\right]\phi_z^3 - \frac{n}{4}(2n\phi+1)\exp\left[1-n-2n\phi\right]\phi_z\right]\\ \times\phi_t - n\exp\left[-n(\phi+1)\right]\phi_{tz}\\ -\frac{1}{4}(2n\phi+1)\exp\left[1-n-2n\phi\right],\end{aligned} \tag{132}$$

which exists if $m=n$ and $m>0$, $n>0$.

Table 3: Multipliers and conserved vectors for the partial differential equation (94).

Case A. $0 \le n < \infty, 0 \le m < \infty$ Multiplier: $\Lambda(\phi) = 1$ $T^1 = \phi$ $T^2 = \exp[n(\phi - 1)] + m\exp[(n-m)(\phi-1)]\phi_t\phi_z - \exp[(n-m)(\phi-1)]\phi_{tz}$
Case B.1. $m + n \neq 0, m > 0, n > 0$ Multiplier: $\Lambda(\phi) = \exp[-(m+n)\phi]$ $T^1 = -\frac{1}{m+n}\exp[-(m+n)\phi] - \frac{1}{2}(n+m)\exp[m-n-2m\phi]\phi_z^2$ $T^2 = -\frac{n}{m}\exp[-(m\phi+n)] + \exp[m-n-2m\phi](m\phi_t\phi_z - \phi_{tz})$
Case B.2. $m = 0, n = 0$ Multiplier: $\Lambda(\phi) = \phi$ $T^1 = \frac{1}{2}\phi^2$ $T^2 = \phi_t - \phi_{tz}$
Case B.3. $m = 0, n > 0$ Multiplier: $\Lambda(\phi) = \exp(-n\phi)$ $T^1 = -\frac{1}{n}\exp(-n\phi) - \frac{1}{2}n\exp[-n\phi]\phi_z^2$ $T^2 = n\exp[-n\phi] - \phi_{tz}$
Case B.4. $n = 0, m > 0$ Multiplier: $\Lambda(\phi) = \exp[-m\phi]$ $T^1 = -\frac{1}{m}\exp[-m\phi] - \frac{1}{2}m\exp[m(1-2\phi)]\phi_z^2$ $T^2 = \exp[m(1-2\phi)](m\phi_t\phi_z - \phi_{tz})$
Case C. $m = n, n \neq 0$ Multiplier: $\Lambda(\phi, \phi_z, \phi_{zz}) = \exp[-2n\phi]\phi_{zz} - \frac{1}{2}n\exp[-2n\phi]\phi_z^2 - \frac{1}{4n}(2n\phi+1)\exp[1-n-2n\phi]$ $T_1 = \frac{1}{2}\exp[-2n\phi]\phi_{zz}^2 - \frac{1}{2}n^2\exp[-2n\phi]\phi_z^4 + \frac{1}{4}n(2n\phi+1)\phi_z^2 + \frac{1}{8n^2}(2n\phi+1)\exp[1-n-2n\phi]$ $T_2 = \left[n^2\exp(-2n\phi)\phi_z^3 - \frac{1}{4}n(2n\phi+1)\exp[1-n-2n\phi)]\phi_z\right]\phi_t - n\exp(-n(\phi+1))\phi_{tz}$ $-\frac{1}{4}(2n\phi+1)\exp[1-n-2n\phi]$

Consider next multipliers of the form (81). The determining equation for the multiplier is (82), where F is given by (95). By equating to zero the coefficient of ϕ_{tzzzzz} in (82) we again derive (83) and the multiplier therefore reduces to the form (84) which has already been considered.

The multipliers and the corresponding conserved vectors for the partial differential equation (94) are listed in Table 3. The (m, n) plane is illustrated in Figure 2.

4.3. Association of Lie Point Symmetries with Conserved Vectors. The Lie point symmetries of the partial differential equation (94) are given in Table 4. We use (14) to investigate which Lie point symmetries of (94) are associated with the conserved vectors for (94).

(i) $m \neq 0, n > 0, m > 0$. Consider first the Lie point symmetry generator

$$X = \left(c_3\frac{m+n}{m-n}t + c_1\right)\frac{\partial}{\partial t} + (c_3 z + c_2)\frac{\partial}{\partial z} - \frac{2c_3}{m-n}\frac{\partial}{\partial \phi} \tag{133}$$

and the elementary conserved vector (101). We find that (133) is associated with the elementary conserved vector provided that $c_3 = 0$, that is, provided that X is given by (86).

(ii) $m + n \neq 0, m > 0, n > 0$. Consider next the Lie point symmetry generator (133) and the conserved vector (102). It can be verified that (102) is associated with (133) provided that $c_3 = 0$, that is, provided that X is given by (86).

(iii) $m = 0, n > 0$. Now consider the Lie point symmetry

$$X = (c_1 - c_3 t)\frac{\partial}{\partial t} + (c_3 z + c_2)\frac{\partial}{\partial z} + \frac{2}{n}c_3\frac{\partial}{\partial \phi} \tag{134}$$

and the conserved vector (108). Using again (14) we find that (134) is associated with (108) provided that $c_3 = 0$, that is, provided that X is given by (86).

(iv) $n = 0, m > 0$. Consider the Lie point symmetry

$$X = \xi^1(t, z)\frac{\partial}{\partial t} + (c_2 + c_3 z)\frac{\partial}{\partial z} - \frac{2c_3}{m}\frac{\partial}{\partial \phi} \tag{135}$$

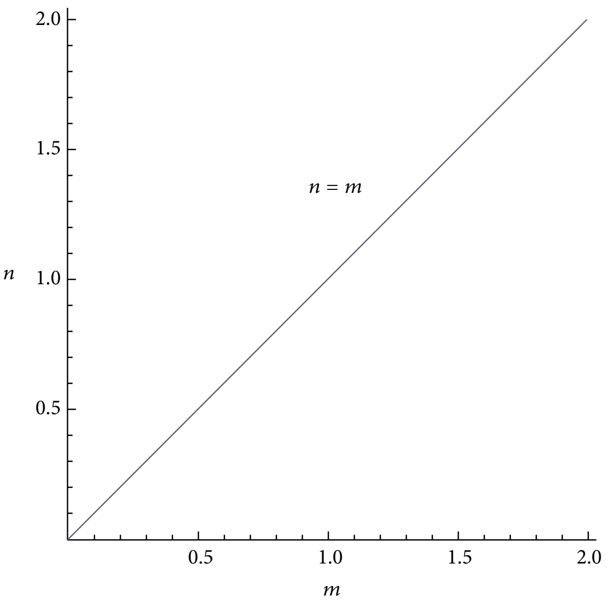

Figure 2: The (m, n)-plane. The special cases lie on the line $n = m$.

and the conserved vector (111). We find that (135) is associated with (111) provided that $c_3 = 0$, that is, provided that

$$X = \xi^1 (t, z) + c_2 \frac{\partial}{\partial z}. \tag{136}$$

We see that, except for (135) ($n = 0, m > 0$), the conserved vector is associated with the Lie point symmetry (86) which generates a travelling wave solution. The conserved vector (111) is associated with (136) which includes (86) as a special case. In all cases, (15) does not yield a new conserved vector.

5. Conclusion

In this paper the multiplier method was used to derive the conservation laws for the magma equation for the case in which the permeability and viscosity satisfy a power law. The results agree with those of Harris [6], who derived the conserved vectors using the direct method. Unlike the direct method the functional form of the conserved vector does not need to be assumed with the multiplier method. Instead the variables on which the multiplier depend have to be chosen but this can be done by starting with a simple form and including higher order partial derivatives later to derive higher order conservation laws. The determining equation for the multiplier is readily obtained with the aid of the Euler operator.

Conserved vectors for the magma equation when the permeability and matrix viscosity depend on the voidage by exponential laws were derived using the multiplier method. Their properties are similar to the properties of the conserved vectors for the power law relations.

We investigated the association of Lie point symmetries of the magma equation with the conserved vectors. For all

Table 4: Lie point symmetries of the partial differential equation (94).

Case 1. $m, n \neq 0, m \neq n$	Case 2. $n = 0, m \neq 0$
$X_1 = \frac{m+n}{m-n} t \frac{\partial}{\partial t} + z \frac{\partial}{\partial z} - \frac{2}{m-n} \frac{\partial}{\partial \phi}$	$X_1 = \xi(t) \frac{\partial}{\partial t}$
$X_2 = \frac{\partial}{\partial t}$	$X_2 = \frac{\partial}{\partial z}$
$X_3 = \frac{\partial}{\partial z}$	$X_3 = z \frac{\partial}{\partial z} - \frac{2}{m} \frac{\partial}{\partial \phi}$
Case 3. $n \neq 0, m \neq 0, m = n$	Case 4. $m = 0, n \neq 0$
$X_1 = t \frac{\partial}{\partial t} - \frac{1}{n} \frac{\partial}{\partial \phi}$	$X_1 = -t \frac{\partial}{\partial t} + z \frac{\partial}{\partial z} + \frac{2}{n} \frac{\partial}{\partial \phi}$
$X_2 = \frac{\partial}{\partial t}$	$X_2 = \frac{\partial}{\partial z}$
$X_3 = \frac{\partial}{\partial z}$	$X_3 = \frac{\partial}{\partial t}$

conserved vectors considered except two the associated Lie point symmetry was the Lie point symmetry which generates travelling wave solutions [4, 5].

We were not able to derive new conservation laws for the partial differential equation (20) or determine if the number of conservation laws for (20) is finite or infinite. Harris [6] has proved that except possibly for the two special cases, $n = m - 1$ with $m \neq 1$ and $m = 1$ with $n \neq 0$, there are no more independent conserved vectors. Our results derived using multipliers are consistent with the results of Harris. All known conserved vectors of (20) and also the new conserved vectors for (94) can be derived from multipliers which depend only on ϕ and the partial derivatives of ϕ with respect to z. We find that the multipliers $\Lambda(\phi, \phi_z)$ and $\Lambda(\phi, \phi_z, \phi_{zz}, \phi_{zzz})$ whose variables ended in odd order partial derivatives of ϕ with respect to z did not generate new conserved vectors but instead reduced to the multipliers $\Lambda(\phi)$ and $\Lambda(\phi, \phi_z, \phi_{zz})$, respectively. This also applies to the multipliers for the conserved vectors for (94).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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