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Research Article

On the Existence and Uniqueness of R_ν-Generalized Solution for Dirichlet Problem with Singularity on All Boundary

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The existence and uniqueness of the R_ν-generalized solution for the first boundary value problem and a second order elliptic equation with coordinated and uncoordinated degeneracy of input data and with strong singularity solution on all boundary of a two-dimensional domain are established.

1. Introduction

The singularity of solution for boundary value problems to two-dimensional closed domain can be due to the degeneration of the input data (coefficients and right-hand sides of equations and boundary conditions), availability of the reentrant corners, and change of the kind of the boundary conditions or by the internal properties of the solution. A boundary value problem is said to possess strong singularity if its solution $u(x)$ does not belong to Sobolev space W_2^1 (H^1) or, in other words, the Dirichlet integral of the solution $u(x)$ diverges. In the case if the solution belongs to the space W_2^1 (H^1) but does not belong to the space W_2^2 (H^2), a boundary value problem is called the problem with a weak singularity.

Boundary value problems with strong singularity are found in the physics of plasma and gas discharge, electrodynamics, nuclear physics, nonlinear optics, and other branches of physics. In particular cases, numerical methods for problems of electrodynamics and quantum mechanics with strong singularity were constructed, based on separation of singular and regular components, mesh refinement near singular points, multiplicative extraction of singularities, and so forth, (see, e.g., [1–6]).

In [7], it was suggested to define the solution of boundary value problem for second-order elliptic equation with singularity on a finite set of points belonging to boundary of a two-dimensional domain as an R_ν-generalized solution in the weighted Sobolev space. Such a new concept of solution led to the distinction of two classes of boundary value problems: problems with coordinated and uncoordinated degeneracy of input data; it also made it possible to study the existence and uniqueness of solutions as well as its coercivity and differential properties in the weighted Sobolev spaces (see [8, 9]).

For boundary value problems for elliptic equations, Maxwell equations and Lamé system, we constructed the numerical methods with rate of convergence independent of the singularity based on the concept of an R_ν-generalized solution (see, e.g., [10–12]).

In this paper, we consider the first boundary value problem for a second-order elliptic equation with strong singularity solution on all boundary of a two-dimensional domain. We distinguish two classes of the boundary value problems: problems with coordinated and uncoordinated degeneracy of input data. For this problem we define the solution as an R_ν-generalized one in a weighted Sobolev space $H^1_{2,\nu+\beta/2}(\Omega)$ and in a weighted set $W^1_{2,\nu+\beta/2}(\Omega,\delta)$, respectively. We prove its existence and uniqueness in the corresponding weighted space and weighted set. It was established that, for all values of parameter ν for which the R_ν-generalized solution exists, it is unique for all of these parameters.

2. Notation and Auxiliary Statements

We denote the two-dimensional Euclidean space by $\mathbb{R}^2$ with $x = (x_1, x_2)$ and $dx = dx_1 dx_2$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and let $\overline{\Omega}$ be the closure of Ω; that is, $\overline{\Omega} = \Omega \cup \partial\Omega$. We denote by Ω' the adjoining streak of the boundary $\partial\Omega$ of width $\delta > 0$ and $\Omega' \subset \Omega$.

We introduce a weight function $\rho(x)$ that coincides in Ω' with the distance from point x to the boundary $\partial\Omega$ and is equal to δ for $x \in \overline{\Omega} \setminus \Omega'$.

Let $H^k_{2,\alpha}(\Omega)$ and $W^k_{2,\alpha}(\Omega)$ be the weighted spaces with norms:

$$\|u\|_{H^k_{2,\alpha}(\Omega)} = \left(\sum_{|\lambda| \le k} \int_\Omega \rho^{2(\alpha+|\lambda|-k)} \left|D^\lambda u\right|^2 dx \right)^{1/2}, \tag{1}$$

$$\|u\|_{W^k_{2,\alpha}(\Omega)} = \left(\sum_{|\lambda| \le k} \int_\Omega \rho^{2\alpha} \left|D^\lambda u\right|^2 dx \right)^{1/2}, \tag{2}$$

where $D^\lambda = \partial^{|\lambda|}/\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}$, $\lambda = (\lambda_1, \lambda_2)$, and $|\lambda| = \lambda_1 + \lambda_2$; λ_1, λ_2 are integer nonnegative numbers, α is some real nonnegative number, and k is an integer nonnegative number. For $k = 0$ we use the notation $H^0_{2,\alpha}(\Omega) = W^0_{2,\alpha}(\Omega) = L_{2,\alpha}(\Omega)$.

By $W^1_{2,\alpha}(\Omega, \delta)$ for $\alpha > 0$, we denote a set of functions satisfying the following conditions:

(a) $|D^k u(x)| \le c_1 (\delta/\rho(x))^{\alpha+k}$ for $x \in \Omega'$, where $k = 0, 1$, c_1 is positive constant independent of k,

(b) $\|u\|_{L_{2,\alpha}(\Omega \setminus \Omega')} \ge c_2 > 0$,

and with the norm (2).

The spaces $\overset{\circ}{H}{}^k_{2,\alpha}(\Omega) \subset H^k_{2,\alpha}(\Omega)$ and $\overset{\circ}{W}{}^k_{2,\alpha}(\Omega) \subset W^k_{2,\alpha}(\Omega)$ and the set $\overset{\circ}{W}{}^1_{2,\alpha}(\Omega, \delta) \subset W^1_{2,\alpha}(\Omega, \delta)$ are defined as the closures of the set of infinitely differentiable and finite in Ω functions in norms (1) and (2), respectively.

Let $H^k_{\infty,-\alpha}(\Omega, c_3)$ $(k \ge 0, \alpha \in R)$ be the set of functions with the norm satisfying the inequality

$$\|u\|_{H^k_{\infty,-\alpha}(\Omega, c_3)} = \max_{|\lambda| \le k} \operatorname*{ess\,sup}_{x \in \Omega} \left|\rho^{-\alpha+|\lambda|} D^\lambda u\right| \le c_3, \tag{3}$$

with a positive constant c_3 independent of u. For $k = 0$, we have $H^0_{\infty,-\alpha}(\Omega, c_4) = L_{\infty,-\alpha}(\Omega, c_4)$.

Lemma 1. *For each function u in the set $W^1_{2,\alpha}(\Omega, \delta)$ and for any $\alpha^* > \alpha$, the estimate*

$$\|u\|_{L_{2,\alpha^*-1}(\Omega', \delta)} \le c_5 \|u\|_{L_{2,\alpha^*}(\Omega, \delta)} \tag{4}$$

holds, where $c_5 = c_6 (\delta^\alpha / \sqrt{\alpha^* - \alpha})$, $c_6 = const > 0$.

Proof. Taking into account condition (a), one can show that, for $\alpha^* > \alpha$, we have

$$\begin{aligned} \|u\|^2_{L_{2,\alpha^*-1}(\Omega', \delta)} &= \int_{\Omega'} \rho^{2(\alpha^*-1)} u^2 dx \\ &\le c_1^2 \delta^{2\alpha} \int_{\Omega'} \rho^{2(\alpha^*-1)} \rho^{-2\alpha} dx \\ &\le \frac{c_1^2 \delta^{2\alpha} c_7 \delta^{2(\alpha^*-\alpha)}}{2(\alpha^* - \alpha)}, \end{aligned} \tag{5}$$

where c_7 is a constant dependent of $\operatorname{mes} \Omega'$. Considering condition (b), we write the inequality for the function u as follows:

$$\begin{aligned} \|u\|^2_{L_{2,\alpha^*}(\Omega)} &\ge \|u\|^2_{L_{2,\alpha^*}(\Omega \setminus \Omega')} \\ &= \delta^{2(\alpha^*-\alpha)} \|u\|^2_{L_{2,\alpha}(\Omega \setminus \Omega')} \ge c_2^2 \delta^{2(\alpha^*-\alpha)}. \end{aligned} \tag{6}$$

From inequalities (5) and (6) we get the estimate (4) with $c_6 = (c_1/c_2)\sqrt{c_7/2}$. □

3. The Boundary Value Problem with Coordinated Degeneration of the Input Data on All Boundary of the Domain

In the domain Ω, we consider the differential equation

$$\begin{gathered} -\sum_{k,l=1}^{2} a_{kl}(x) \frac{\partial^2 u}{\partial x_k \partial x_l} + \sum_{k=1}^{2} a_k(x) \frac{\partial u}{\partial x_k} \\ + a(x) u = f(x), \quad x \in \Omega, \end{gathered} \tag{7}$$

with the boundary condition

$$u = 0, \qquad x \in \partial\Omega. \tag{8}$$

Definition 2. The boundary value problem (7) and (8) is called the Dirichlet problem with coordinated degeneration of the input data on all boundary of the domain or Problem A, if $a_{kl}(x) = a_{lk}(x)$ $(k, l = 1, 2)$ and, for some real number β,

$$\begin{gathered} a_{kl} \in H^1_{\infty,-\beta}(\Omega, c_8), \quad a_k \in L_{\infty,-(\beta-1)}(\Omega, c_9), \quad (k, l = 1, 2), \\ a \in L_{\infty,-(\beta-2)}(\Omega, c_{10}), \end{gathered} \tag{9}$$

$$\sum_{k,l=1}^{2} a_{kl}(x) \xi_k \xi_l \ge c_{11} \rho^\beta(x) \sum_{k=1}^{2} \xi_k^2, \tag{10}$$

$$a(x) > c_{12} \rho^{\beta-2}(x) \ \text{ almost everywhere on } \ \Omega \tag{11}$$

and right-hand side of (7) satisfies

$$f \in L_{2,\mu}(\Omega), \tag{12}$$

where c_i $(i = 8, \ldots, 12)$ are positive constants independent of x; ξ_1 and ξ_2 are any real parameters; μ is some nonnegative real number.

Denote by

$$a(u,v)=\int_{\Omega}\left[\sum_{k,l=1}^{2}a_{kl}\rho^{2\nu}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_l}+a_{kl}\frac{\partial\rho^{2\nu}}{\partial x_k}\frac{\partial u}{\partial x_l}v+\frac{\partial a_{kl}}{\partial x_k}\rho^{2\nu}\frac{\partial u}{\partial x_l}v+a_k\rho^{2\nu}\frac{\partial u}{\partial x_k}v+a\rho^{2\nu}uv\right]dx,$$
$$f(v)=\int_{\Omega}\rho^{2\nu}fv\,dx \tag{13}$$

the bilinear and linear forms, respectively.

Definition 3. A function u_ν from the space $\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega)$ is called an R_ν-generalized solution of the Dirichlet problem with coordinated degeneration of the input data on all boundary of the domain or Problem A, if, for any v in $\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega)$, the identity

$$a(u_\nu,v)=f(v) \tag{14}$$

holds, where ν is arbitrary but fixed and satisfies the inequality

$$\nu\geq\mu+\frac{\beta}{2}-1. \tag{15}$$

For Problem A, we prove the main result.

Theorem 4. *Let conditions* (9)–(12) *and* (15) *hold and let*

$$2\left(c_8(2|\nu|+1)+\frac{1}{2}c_9\right)^2<c_{11}c_{12} \tag{16}$$

be satisfied.

Then, the R_ν-generalized solution u_ν of the Dirichlet problem with coordinated degeneration of the input data on all boundary of the domain exists and is unique in the space $\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega)$ and the following estimate is valid:

$$\|u_\nu\|_{H^1_{2,\nu+\beta/2}(\Omega)}\leq c_{13}\|f\|_{L_{2,\mu}(\Omega)}, \tag{17}$$

where c_{13} is a positive constant not depending on u_ν and f.

Proof. First, we show that the forms $a(u,v)$ and $f(v)$ are continuous on $\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega)$. In fact, by virtue of conditions (9), (12), and (15) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|a(u,v)|\\
&\leq\int_{\Omega}\left|\sum_{k,l=1}^{2}a_{kl}\rho^{2\nu}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_l}+\frac{\partial a_{kl}}{\partial x_k}\rho^{2\nu}\frac{\partial u}{\partial x_l}v+a_k\rho^{2\nu}\frac{\partial u}{\partial x_k}v+a\rho^{2\nu}uv\right|dx\\
&\quad+\int_{\Omega'}\left|a_{kl}\frac{\partial\rho^{2\nu}}{\partial x_k}\frac{\partial u}{\partial x_l}v\right|dx\\
&\leq 2c_8\left(\int_{\Omega}\rho^{2\nu+\beta}\sum_{k=1}^{2}\left(\frac{\partial u}{\partial x_k}\right)^2dx\right)^{1/2}\times\left(\int_{\Omega}\rho^{2\nu+\beta}\sum_{k=1}^{2}\left(\frac{\partial v}{\partial x_k}\right)^2dx\right)^{1/2}\\
&\quad+2\sqrt{2}c_8\left(\int_{\Omega}\rho^{2\nu+\beta}\sum_{l=1}^{2}\left(\frac{\partial u}{\partial x_l}\right)^2dx\right)^{1/2}\times\left(\int_{\Omega}\rho^{2\nu+\beta-2}v^2dx\right)^{1/2}\\
&\quad+\sqrt{2}c_9\left(\int_{\Omega}\rho^{2\nu+\beta}\sum_{k=1}^{2}\left(\frac{\partial u}{\partial x_k}\right)^2dx\right)^{1/2}\times\left(\int_{\Omega}\rho^{2\nu+\beta-2}v^2dx\right)^{1/2}+c_{10}\left(\int_{\Omega}\rho^{2\nu+\beta-2}u^2dx\right)^{1/2}\times\left(\int_{\Omega}\rho^{2\nu+\beta-2}v^2dx\right)^{1/2}\\
&\quad+4\sqrt{2}|\nu|c_8\left(\int_{\Omega'}\rho^{2\nu+\beta}\sum_{l=1}^{2}\left(\frac{\partial u}{\partial x_l}\right)^2dx\right)^{1/2}\times\left(\int_{\Omega'}\rho^{2\nu+\beta-2}v^2dx\right)^{1/2}\\
&\leq 2c_8|u|_{H^1_{2,\nu+\beta/2}(\Omega)}|v|_{H^1_{2,\nu+\beta/2}(\Omega)}\\
&\quad+\left(2\sqrt{2}c_8+\sqrt{2}c_9\right)|u|_{H^1_{2,\nu+\beta/2}(\Omega)}\|v\|_{L_{2,\nu+\beta/2-1}(\Omega)}\\
&\quad+4\sqrt{2}|\nu|c_8|u|_{H^1_{2,\nu+\beta/2}(\Omega')}\|v\|_{L_{2,\nu+\beta/2-1}(\Omega')}\\
&\quad+c_{10}\|u\|_{L_{2,\nu+\beta/2-1}(\Omega)}\|v\|_{L_{2,\nu+\beta/2-1}(\Omega)},
\end{aligned} \tag{18}$$

or

$$|a(u,v)|\leq c_{14}\|u\|_{H^1_{2,\nu+\beta/2}(\Omega)}\|v\|_{H^1_{2,\nu+\beta/2}(\Omega)},\quad\forall u,v\in\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega) \tag{19}$$

$$\begin{aligned}
&|f(v)|\\
&\leq\left(\max_{x\in\Omega}\rho^{2\nu-2\mu-\beta+2}\right)^{1/2}\left(\int_{\Omega}\rho^{2\mu}f^2dx\right)^{1/2}\times\left(\int_{\Omega}\rho^{2\nu+\beta-2}v^2dx\right)^{1/2}\leq c_{15}\|f\|_{L_{2,\mu}(\Omega)}\|v\|_{H^1_{2,\nu+\beta/2}(\Omega)},\\
&\forall v\in\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega),\ \forall f\in L_{2,\mu}(\Omega).
\end{aligned} \tag{20}$$

Let us now prove the $\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega)$ ellipticity of the bilinear form $a(u,v)$; that is,

$$\exists c_{16} > 0, \quad \forall u \in \overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega), \quad a(u,u) \geq c_{16} \|u\|^2_{H^1_{2,\nu+\beta/2}(\Omega)}. \tag{21}$$

We substitute v by u in (13), and by means of the Cauchy-Schwarz inequality, ε-inequality, and conditions (9), we estimate the absolute values of the second, third, and fourth terms of the form $a(u,u)$:

$$\begin{aligned} &\left| \sum_{k,l=1}^{2} \int_{\Omega} a_{kl} \frac{\partial \rho^{2\nu}}{\partial x_k} \frac{\partial u}{\partial x_l} u dx \right| \\ &\quad \leq \left| \sum_{k,l=1}^{2} \int_{\Omega} a_{kl} 2\nu \rho^{2\nu-1} \frac{\partial u}{\partial x_l} u\, dx \right| \\ &\quad \leq 2\sqrt{2} c_8 |\nu| \left(\varepsilon_1 \sum_{l=1}^{2} \int_{\Omega'} \rho^{2\nu+\beta} \left(\frac{\partial u}{\partial x_l} \right)^2 dx \right. \\ &\qquad \left. + \frac{1}{\varepsilon_1} \int_{\Omega'} \rho^{2\nu+\beta-2} u^2 dx \right), \end{aligned} \tag{22}$$

$$\begin{aligned} &\left| \sum_{k,l=1}^{2} \int_{\Omega} \frac{\partial a_{kl}}{\partial x_k} \rho^{2\nu} \frac{\partial u}{\partial x_l} u dx \right| \\ &\quad \leq \sqrt{2} c_8 \left(\varepsilon_2 \sum_{l=1}^{2} \int_{\Omega} \rho^{2\nu+\beta} \left(\frac{\partial u}{\partial x_l} \right)^2 dx \right. \\ &\qquad \left. + \frac{1}{\varepsilon_2} \int_{\Omega} \rho^{2\nu+\beta-2} u^2 dx \right), \end{aligned} \tag{23}$$

$$\begin{aligned} &\left| \sum_{k=1}^{2} \int_{\Omega} a_k \rho^{2\nu} \frac{\partial u}{\partial x_k} u dx \right| \\ &\quad \leq \sqrt{2} c_9 \left(\frac{\varepsilon_3}{2} \sum_{k=1}^{2} \int_{\Omega} \rho^{2\nu+\beta} \left(\frac{\partial u}{\partial x_k} \right)^2 dx \right. \\ &\qquad \left. + \frac{1}{2\varepsilon_3} \int_{\Omega} \rho^{2\nu+\beta-2} u^2 dx \right). \end{aligned} \tag{24}$$

Here, ε_1, ε_2, and ε_3 are any positive numbers.

Using (10) and (11), we have

$$\sum_{k,l=1}^{2} \int_{\Omega} a_{kl} \rho^{2\nu} \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_l} dx \geq c_{11} \int_{\Omega} \rho^{2\nu+\beta} \sum_{k=1}^{2} \left(\frac{\partial u}{\partial x_k} \right)^2 dx, \tag{25}$$

$$\int_{\Omega} a \rho^{2\nu} u^2 dx \geq c_{12} \int_{\Omega} \rho^{2\nu+\beta-2} u^2 dx. \tag{26}$$

Then, from (22)–(26), we obtain

$$\begin{aligned} &a(u,u) \\ &\quad \geq \left(c_{11} - \sqrt{2} \left(c_8 2 |\nu| \varepsilon_1 + c_8 \varepsilon_2 + \frac{1}{2} c_9 \varepsilon_3 \right) \right) \\ &\quad\quad \times \int_{\Omega} \rho^{2\nu+\beta} \sum_{k=1}^{2} \left(\frac{\partial u}{\partial x_k} \right)^2 dx \\ &\quad\quad + \left(c_{12} - \sqrt{2} \left(c_8 2 |\nu| \varepsilon_1^{-1} + c_8 \varepsilon_2^{-1} + \frac{1}{2} c_9 \varepsilon_3^{-1} \right) \right) \\ &\quad\quad \times \int_{\Omega} \rho^{2\nu+\beta-2} u^2 dx. \end{aligned} \tag{27}$$

Note that if condition (16) is satisfied, then there exists a positive constant ε such that

$$\begin{aligned} &c_{11} - \sqrt{2} \left(c_8 (2|\nu| + 1) + \frac{1}{2} c_9 \right) \varepsilon > 0, \\ &c_{12} - \sqrt{2} \left(c_8 (2|\nu| + 1) + \frac{1}{2} c_9 \right) \varepsilon^{-1} > 0. \end{aligned} \tag{28}$$

Supposing that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$ in (27), we get

$$\begin{aligned} &a(u,u) \\ &\quad \geq c_{16} \left(\int_{\Omega} \rho^{2\nu+\beta} \sum_{k=1}^{2} \left(\frac{\partial u}{\partial x_k} \right)^2 dx \right. \\ &\qquad \left. + \int_{\Omega} \rho^{2\nu+\beta-2} u^2 dx \right) = c_{16} \|u\|_{H^1_{2,\nu+\beta/2}(\Omega)} \end{aligned} \tag{29}$$

with constant

$$\begin{aligned} c_{16} = \min &\left(c_{11} - \sqrt{2} \left(c_8 (2|\nu| + 1) + \frac{1}{2} c_9 \right) \varepsilon, \right. \\ &\left. c_{12} - \sqrt{2} \left(c_8 (2|\nu| + 1) + \frac{1}{2} c_9 \right) \varepsilon^{-1} \right). \end{aligned} \tag{30}$$

According to (19), (21), and (20), the bilinear form $a(u,v)$ is continuous and $\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega)$-elliptical, and the linear form $f(v)$ is continuous on $\overset{\circ}{H}{}^{1}_{2,\nu+\beta/2}(\Omega)$; then, the existence and uniqueness of an R_ν-generalized solution of Problem A follow from the Lax-Milgram theorem (see [13]).

Taking into account that

$$\begin{aligned} c_{16} \|u_\nu\|^2_{H^1_{2,\nu+\beta/2}(\Omega)} &\leq a(u_\nu, u_\nu) \\ &= f(v) \leq c_{15} \|f\|_{L_{2,\mu}(\Omega)} \|u\|_{H^1_{2,\nu+\beta/2}(\Omega)}, \end{aligned} \tag{31}$$

we get estimate (17). □

Corollary 5. *If there exists at least one ν for which there exists a unique R_ν-generalized solution of the Problem A, then one can* *always define a half-open interval* $[\nu_1, \nu_2)$ *such that, for each* $\nu \in [\nu_1, \nu_2)$*, there exists a unique* R_ν*-generalized solution. Here,*

$$\nu_1 = \max\left\{\mu + \frac{\beta}{2} - 1; \left(1 - \frac{(c_{11}c_{12})^{1/2} - c_9/2}{c_8}\right) + \varepsilon\right\},$$
$$\nu_2 = \frac{(c_{11}c_{12})^{1/2} - c_9/2}{c_8} - 1, \tag{32}$$

where ε *is a given sufficiently small positive number.*

Corollary follows from the proof of Theorem 4.

Theorem 6. *If the assumptions of Theorem 4 are valid, then, for all* ν *in the interval* $[\nu_1, \nu_2)$*, the* R_ν*-generalized solution of the Problem A is unique.*

The proof of this statement is similar to that of Theorem 2 in [14].

4. The Boundary Value Problem with Uncoordinated Degeneration of the Input Data on All Boundary of the Domain

We consider the boundary value problem

$$-\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(a_{kk}(x)\frac{\partial u}{\partial x_k}\right) + a(x)u = f(x), \quad x \in \Omega, \tag{33}$$

$$u = 0, \quad x \in \partial\Omega. \tag{34}$$

Definition 7. The boundary value problem (33) and (34) is called the Dirichlet problem with uncoordinated degeneration of the input data on all boundary of the domain or Problem B, if, for some real number β,

$$a_{kk} \in H^1_{\infty,-\beta}(\Omega, c_{16}), \qquad a \in L_{\infty,-\beta}(\Omega, c_{17}), \tag{35}$$

$$\sum_{k=1}^{2} a_{kk}(x)\xi_k^2 \geq c_{18}\rho^\beta(x)\sum_{k=1}^{2}\xi_k^2, \tag{36}$$

$$a(x) \geq c_{19}\rho^\beta(x) \ \text{ almost everywhere on } \ \Omega, \tag{37}$$

and the right-hand side of the equation satisfies the condition

$$f \in L_{2,\mu}(\Omega, \delta) \tag{38}$$

for some nonnegative real number μ. Here, c_i, $i = 16, \ldots, 19$, are positive constants not depending on x; ξ_1 and ξ_2 are arbitrary real parameters.

Set

$$b(u, v) = \int_\Omega \left[\sum_{k=1}^{2} a_{kk}\rho^{2\nu}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k} + a_{kk}\frac{\partial \rho^{2\nu}}{\partial x_k}\frac{\partial u}{\partial x_k}v + a\rho^{2\nu}uv\right]dx, \tag{39}$$

$$l(v) = \int_\Omega \rho^{2\nu} f v\,dx.$$

Definition 8. A function u_ν from the set $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega, \delta)$ is called an R_ν-generalized solution of the Problem B if the identity $b(u_\nu, v) = l(v)$ holds for all $v \in \overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega, \delta)$ and for any given value of ν satisfying the inequality

$$\nu \geq \mu + \frac{\beta}{2}. \tag{40}$$

Theorem 9. *Let conditions* (35)–(40) *hold and*

$$\nu + \frac{\beta}{2} > 0. \tag{41}$$

Then, for any ν *satisfying conditions* (40) *and* (41)*, there always exists parameter* δ *such that* R_ν*-generalized solution* u_ν *of the Dirichlet problem with uncoordinated degeneration of the input data on all boundary of the domain exists and is unique in the set* $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega, \delta)$*. In this case, the following estimate is valid:*

$$\|u_\nu\|_{W^1_{2,\nu+\beta/2}(\Omega,\delta)} \leq c_{20}\|f\|_{L_{2,\mu}(\Omega,\delta)}, \tag{42}$$

where c_{20} *is a positive constant independent of* f *and* u_ν*.*

Proof. First of all, we note that the bilinear and linear forms are continuous on the set $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega, \delta)$ and the inequalities

$$b(u, v) \leq c_{21}\|u\|_{W^1_{2,\nu+\beta/2}(\Omega,\delta)}\|v\|_{W^1_{2,\nu+\beta/2}(\Omega,\delta)}, \tag{43}$$

$$l(v) \leq c_{22}\|f\|_{L_{2,\mu}(\Omega,\delta)}\|v\|_{W^1_{2,\nu+\beta/2}(\Omega,\delta)} \tag{44}$$

hold. The proofs of estimates (43) and (44) are established by analogy with (19) and (20), which we obtain by using conditions (35), (38), and (40) and Lemma 1.

Let us show that the bilinear form is $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}$-elliptical in Ω. We have

$$b(u, u) = \sum_{k=1}^{2}\int_\Omega \left[a_{kk}\rho^{2\nu}\left(\frac{\partial u}{\partial x_k}\right)^2 + a_{kk}\frac{\partial \rho^{2\nu}}{\partial x_k}\frac{\partial u}{\partial x_k}u\right]dx + \int_\Omega a\rho^{2\nu}u^2dx \tag{45}$$

for any u from $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega, \delta)$. By means of condition (35), we estimate the absolute value of the second term on the right-hand side in (45):

$$\left|\sum_{k=1}^{2}\int_\Omega a_{kk}\frac{\partial \rho^{2\nu}}{\partial x_k}\frac{\partial u}{\partial x_k}u dx\right| \leq \left|\sum_{k=1}^{2}\int_{\Omega'} a_{kk}2\nu\rho^{2\nu-1}\frac{\partial u}{\partial x_k}u dx\right| \leq \varepsilon\sum_{k=1}^{2}\int_{\Omega'}\rho^{2\nu+\beta}\left(\frac{\partial u}{\partial x_k}\right)^2dx + \frac{2c_{16}^2\nu^2}{\varepsilon}\int_{\Omega'}\rho^{2\nu+\beta-1}u^2dx. \tag{46}$$

From (45) and the last inequality we get

$$b(u,u) \geq \sum_{k=1}^{2} \int_{\Omega} a_{kk}\rho^{2\nu}\left(\frac{\partial u}{\partial x_k}\right)^2 dx - \varepsilon \sum_{k=1}^{2} \int_{\Omega'} \rho^{2\nu+\beta}\left(\frac{\partial u}{\partial x_k}\right)^2 dx + \int_{\Omega} a\rho^{2\nu}u^2 dx - \frac{2c_{16}^2\nu^2}{\varepsilon}\int_{\Omega'} \rho^{2\nu+\beta-1}u^2 dx. \quad (47)$$

Supposing that α^* and α equal ν and $\nu/2$ in Lemma 1, respectively, we have

$$\|u\|^2_{L_{2,\nu+\beta/2-1}(\Omega',\delta)} \leq \frac{2c_6^2\delta^\nu}{\nu}\|u\|^2_{L_{2,\nu+\beta/2}(\Omega,\delta)}, \quad c_6 = \text{const} > 0. \quad (48)$$

Taking into account (36), (37), and (48), from estimate (47) we get

$$b(u,u) \geq (c_{18} - \varepsilon)\,|u|^2_{W^1_{2,\nu+\beta/2}(\Omega,\delta)} + \left(c_{19} - \frac{4c_{16}^2c_6^2\nu\delta^\nu}{\varepsilon}\right) \times \|u\|^2_{L_{2,\nu+\beta/2}(\Omega,\delta)}. \quad (49)$$

Obviously, we can always choose ε and δ such that the constants $c_{18} > \varepsilon$, $c_{19} > 4c_{16}^2c_6^2\nu\delta^\nu/\varepsilon$ and the inequality

$$b(u,u) \geq c_{23}\|u\|^2_{W^1_{2,\nu+\beta/2}(\Omega,\delta)} \quad (50)$$

are valid with constant $c_{23} = \min(c_{18} - \varepsilon, c_{19} - 4c_{16}^2c_6^2\nu\delta^\nu/\varepsilon)$. Therefore, bilinear form $b(u,u)$ is $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}$-elliptical.

According to (43), (44), and (50), the bilinear form $b(u,u)$ is continuous and $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}$-elliptical, and the linear form $l(v)$ is continuous on $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega,\delta)$; then, the existence and uniqueness of an R_ν-generalized solution of Problem B follow from the Lax-Milgram theorem (see [13]).

Taking into account that

$$c_{23}\|u_\nu\|^2_{W^1_{2,\nu+\beta/2}(\Omega,\delta)} \leq b(u_\nu,u_\nu) = l(u_\nu) \leq c_{22}\|u_\nu\|_{W^1_{2,\nu+\beta/2}(\Omega,\delta)}\|f\|_{L_{2,\mu}(\Omega,\delta)}, \quad (51)$$

we get estimate (42). □

Theorem 10. *If for some δ there is a set of values ν such that an R_ν-generalized solution of the Problem B exists in the set $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega,\delta)$, then this solution is unique for all such ν.*

The proof of this statement is similar to that of Theorem 2 in [14].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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