Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 563613, 8 pages
http://dx.doi.org/10.1155/2014/563613

Research Article

Shape-Preserving and Convergence Properties for the q-Szász-Mirakjan Operators for Fixed $q \in (0, 1)$

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Received 22 February 2014; Accepted 14 April 2014; Published 6 May 2014

Academic Editor: Sofiya Ostrovska

We introduce a q-generalization of Szász-Mirakjan operators $S_{n,q}$ and discuss their properties for fixed $q \in (0, 1)$. We show that the q-Szász-Mirakjan operators $S_{n,q}$ have good shape-preserving properties. For example, $S_{n,q}$ are variation-diminishing, and preserve monotonicity, convexity, and concave modulus of continuity. For fixed $q \in (0, 1)$, we prove that the sequence $\{S_{n,q}(f)\}$ converges to $B_{\infty,q}(f)$ uniformly on $[0, 1]$ for each $f \in C[0, 1/(1-q)]$, where $B_{\infty,q}$ is the limit q-Bernstein operator. We obtain the estimates for the rate of convergence for $\{S_{n,q}(f)\}$ by the modulus of continuity of f, and the estimates are sharp in the sense of order for Lipschitz continuous functions.

1. Introduction

Let $q > 0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] := [k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1, \end{cases}$$
$$[k]! := \begin{cases} [k]\,[k-1]\cdots[1], & k \geq 1, \\ 1, & k = 0. \end{cases} \tag{1}$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!}. \tag{2}$$

We give the following two q-analogues of exponential function e^x:

$$e_q(x) := \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \frac{1}{((1-q)x; q)_\infty}, \quad |x| < \frac{1}{1-q} \quad \text{for } q < 1;$$
$$E_q(x) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{[k]!} = (-(1-q)x; q)_\infty, \quad x \in \mathbb{R} \text{ for } q < 1, \tag{3}$$

where $(x; q)_\infty := \prod_{k=1}^{\infty}(1 - xq^{k-1})$. Clearly, we have

$$e_q(x)\,E_q(-x) = 1, \qquad \lim_{q \to 1-} e_q(x) = \lim_{q \to 1-} E_q(x) = e^x. \tag{4}$$

In [1], Phillips proposed the q-Bernstein polynomials: for each positive integer n and $f \in C[0,1]$, the q-Bernstein polynomial of f is

$$B_{n,q}(f)(x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x). \tag{5}$$

Note that for $q = 1$, $B_{n,q}(f)$ is the classical Bernstein polynomial. In [2], Il'inskiia and Ostrovska proved that, for each $f \in C[0,1]$ and $q \in (0,1)$, the sequence $\{B_{n,q}(f)(x)\}$

converges to $B_{\infty,q}(f)(x)$ as $n \to \infty$ uniformly on $x \in [0,1]$, where

$$B_{\infty,q}(f)(x) = \begin{cases} \sum_{k=0}^{\infty} f\left(1-q^k\right) p_{\infty k}(q;x), & 0 \le x < 1, \\ f(1), & x = 1. \end{cases} \tag{6}$$

The operators $B_{\infty,q}$ are called the limit q-Bernstein operators. They also arise as the limit for a sequence of q-Meyer-König Zeller operators (see [3]). For results about properties of $B_{\infty,q}(f,x)$ we refer to [2, 4, 5].

In [6], Aral introduced the following q-Szász-Mirakjan operator: for each positive integer n and $f \in C[0,\infty)$,

$$S^b_{n,q}(f)(x) := E_q\left(-[n]\frac{x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]\,b_n}{[n]}\right)\frac{([n]\,x)^k}{[k]!(b_n)^k}, \tag{7}$$

where $0 \le x < \alpha_q(n)$, $\alpha_q(n) := b_n/(1-q^n)$, and $b = \{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$. In this paper, we introduce the following q-Szász-Mirakjan operator: for each positive integer n and $f \in C[0, 1/(1-q^n)]$,

$$S_{n,q}(f)(x) := \begin{cases} E_q(-[n]\,x)\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right)\frac{([n]\,x)^k}{[k]!}, & x \in \left[0, \frac{1}{1-q^n}\right), \\ f(1), & x = \frac{1}{1-q^n}. \end{cases} \tag{8}$$

Obviously, the operators $S_{n,q}$ are equal to the operators $S^b_{n,q}$ with $b = \{b_n\}$, $b_n = 1$. When $q = 1$, the q-Szász-Mirakjan operators $S_{n,q}$ reduce to the classical Szász-Mirakjan operators.

In recent years, generalizations of linear operators connected with q-Calculus have been investigated intensively. The pioneer work has been made by Lupas [7] and Phillips [1] who proposed generalizations of Bernstein polynomials based on the q-integers. There are also other important q-operators, for example, the two-parametric generalization of q-Bernstein polynomials [8], the q-Bernstein-Durrmeyer operator [9], q-Meyer-König Zeller operators [10], q-Bleimann, Butzer and Hahn operators [11], and q-Szász-Mirakjan operators [6, 12–15]. Among these generalizations, q-Bernstein polynomials proposed by Phillips attracted the most attention and were studied widely by a number of authors (see [1, 2, 5, 16–24]).

In this paper, we will discuss convergence and shape-preserving properties of the q-Szász-Mirakjan operators $S_{n,q}$ for fixed $q \in (0,1)$. We will show that the operators $S_{n,q}$ share good shape-preserving properties such as the variation-diminishing properties, and for each $f \in C[0, 1/(1-q)]$ the sequence $\{S_{n,q}(f)(x)\}$ converges to the function $B_{\infty,q}(f)(x)$ uniformly on $[0,1]$, where $B_{\infty,q}$ are the limit q-Bernstein operators defined by (6). We also investigate the rate of convergence of the q-Szász-Mirakjan operators $S_{n,q}$ for fixed $q \in (0,1)$. Our results demonstrate that in general convergence properties of the q-Szász-Mirakjan operators $S_{n,q}$ are essentially different from those for the classical Szász-Mirakjan operators; however, they are very similar to those for the q-Bernstein polynomials. Notice that different q-generalizations of Szász-Mirakjan operators were introduced and studied by Aral and Gupta [6, 12], by Radu [13], and by Mahmudov [14, 15]. However, our q-Szász-Mirakjan operators have better convergence properties than the other q-generalizations of Szász-Mirakjan operators for fixed $q \in (0,1)$.

The paper is organized as follows. In Section 2, we recall some properties of the q-Szász-Mirakjan operators $S_{n,q}$ and discuss their shape-preserving properties. In Section 3 we investigate the convergence of $S_{n,q}(f)$ for fixed $q \in (0,1)$ and obtain the rate of convergence of $S_{n,q}(f)$ by the modulus of continuity of f, and the estimates are sharp in the sense of order for Lipschitz continuous functions.

2. Shape-Preserving Properties of $S_{n,q}$ for $0 < q < 1$

In the sequel we always assume that $q \in (0,1)$. First we show that the q-Szász-Mirakjan operators $S_{n,q}$ are the positive linear operators on $C[0, 1/(1-q^n)]$. Clearly, it suffices to prove that, for $f \in C[0, 1/(1-q^n)]$,

$$\lim_{x\to(1/(1-q^n))-} S_{n,q}(f)(x) = f\left(\frac{1}{1-q^n}\right). \tag{9}$$

Indeed, for arbitrary $\varepsilon > 0$, there exist a constant $M > 0$ and a $\delta > 0$ such that $|f(x)| \le M$ for all $x \in [0, 1/(1-q^n)]$, and $|f(x) - f(1/(1-q^n))| \le \varepsilon$ for $x \in (1/(1-q^n)-\delta, 1/(1-q^n))$. We choose A to be the minimum positive integer greater than $\log_q((1-q^n)\delta)$. Then, for any $k > A$,

$$\begin{gathered} \left|\frac{[k]}{[n]} - \frac{1}{1-q^n}\right| = \frac{q^k}{1-q^n} < \delta, \\ \left|f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right| \le \varepsilon. \end{gathered} \tag{10}$$

It follows from the Euler identity that

$$\begin{gathered} E_q(-[n]\,x)\sum_{k=0}^{\infty}\frac{([n]\,x)^k}{[k]!} = 1 \quad \text{for } x \in \left[0, \frac{1}{1-q^n}\right), \\ E_q(-[n]\,x) = \left((1-q^n)\,x; q\right)_\infty = \prod_{s=0}^{\infty}\left(1 - q^s(1-q^n)\,x\right) \longrightarrow 0+, \\ \text{as } x \longrightarrow \frac{1}{1-q^n}-. \end{gathered} \tag{11}$$

This implies that, for $x \in [0, 1/(1-q^n))$,

$$\begin{aligned} &\left| S_{n,q}(f)(x) - f\left(\frac{1}{1-q^n}\right)\right| \\ &= \left| E_q(-[n]x) \sum_{k=0}^{\infty} \left(f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right) \frac{([n]x)^k}{[k]!} \right| \\ &\le E_q(-[n]x) \left(\sum_{k=0}^{A} \left| f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right| \frac{([n]x)^k}{[k]!} \right. \\ &\qquad \left. + \sum_{k=A+1}^{\infty} \left| f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right| \frac{([n]x)^k}{[k]!} \right) \\ &\le 2ME_q(-[n]x) \sum_{k=0}^{A} \frac{1}{(1-q)^k [k]!} \\ &\quad + \varepsilon E_q(-[n]x) \sum_{k=A+1}^{\infty} \frac{([n]x)^k}{[k]!} \\ &\le BE_q(-[n]x) + \varepsilon, \end{aligned} \tag{12}$$

where $B := 2M \sum_{k=0}^{A}(1/(1-q)^k[k]!)$ is a constant independent of x and $E_q(-[n]x) \to 0$ as $x \to (1/(1-q^n))-$. This proves (9).

The q-Szász-Mirakjan operators $S_{n,q}$ possess the end-point interpolation property:

$$S_{n,q}(f)(0) = f(0), \qquad S_{n,q}(f)\left(\frac{1}{1-q^n}\right) = f\left(\frac{1}{1-q^n}\right), \quad n \in \mathbb{N}. \tag{13}$$

They leave invariant linear functions:

$$S_{n,q}(at+b)(x) = ax + b \tag{14}$$

and are degree-preserving on polynomials; that is, if T is a polynomial of degree m, then $S_{n,q}(T)$ is a polynomial of degree m (see [6, Lemma 1] or [25, Theorem 1]).

The following representation of the q-Szász-Mirakjan operators $S_{n,q}$, called the q-difference form, was obtained in [6, Corollary 4]:

$$S_{n,q}(f)(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} f\left(\left[\frac{[0]}{[n]}; \frac{[1]}{[n]}; \ldots; \frac{[k]}{[n]}\right]\right) x^k, \tag{15}$$

where $f([x_0; x_1; \ldots; x_k])$ denotes the usual divided difference; that is,

$$\begin{aligned} &f([x_0]) = f(x_0); \qquad f([x_0; x_1]) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \ldots, \\ &f([x_0; x_1; \ldots; x_k]) = \frac{f([x_1; \ldots; x_k]) - f([x_0; \ldots; x_{k-1}])}{x_k - x_0}. \end{aligned} \tag{16}$$

Aral and Gupta discussed the shape-preserving properties of the q-Szász-Mirakjan operators in [12, Corollary 3.2]. We say a function f on an interval I is *i-convex*, $i \ge 1$, if $f \in C(I)$ and all ith forward differences

$$\begin{aligned} &\Delta_h^i f(t) := \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} f(t+kh), \\ &0 \le h \le \frac{1}{i}, \quad t, t+kh \in I \end{aligned} \tag{17}$$

are nonnegative. Obviously, a 1-*convex* function is nondecreasing and a 2-*convex* function is convex. Aral and Gupta obtained that, for an i-convex function on $[0, \infty)$, there exists $\hat{q} \in (0,1)$ such that $S_{n,q}(f)$ is also i-convex on $[0, 1/(1-q^n))$ for $q \in (\hat{q}, 1)$.

In this section we also study the shape-preserving properties of the operators $S_{n,q}$. We use a completely different method from the one in [12], and our results hold for all $q \in (0,1)$. In order to state the results, we introduce some notations.

For any real sequence a, finite or infinite, we denote by $S^-(a)$ the number of strict sign changes in a. For $f \in C(I)$, where I is an interval, we define $S^-(f)$ to be the number of sign changes of f; that is,

$$S^-(f) = \sup S^-(f(x_0), \ldots, f(x_m)), \tag{18}$$

where the supremum is taken over all increasing sequences $x_0 < \cdots < x_m$ and $x_0, x_m \in I$ for all positive integers m.

Let L be a positive linear operator on $C(I)$. We say that L is variation-diminishing if, for all functions $f \in C(I)$, we have

$$S^-(L_n f) \le S^-(f). \tag{19}$$

A function $\omega(t)$ on $[0, A]$, $A > 0$ is called a modulus of continuity if $\omega(t)$ is continuous, nondecreasing, and semiadditive and $\omega(0) = 0$. We denote by H^ω the class of continuous functions f on $[0, A]$ satisfying the inequality $\omega(f, t) \le \omega(t)$, where $\omega(f,t) = \max_{|x_1 - x_2| \le t} |f(x_2) - f(x_1)|$ is the modulus of continuity of $f(x)$. Note that if $f(x)$ is a concave modulus of continuity, then $x^{-1} f(x)$ is nonincreasing on $(0, A]$. Also, if $f(x)$ is a nondecreasing function such that $f(0) = 0$ and $x^{-1} f(x)$ is nonincreasing on $(0, A]$, then $f(x)$ is a modulus of continuity.

Our main results of this section can be formulated as follows.

Theorem 1. *(i) The operators $S_{n,q}$ are variation-diminishing on $[0, 1/(1-q^n)]$.*

(ii) If a function f is i-convex on $[0, 1/(1-q^n)]$, then the functions $S_{n,q}(f)$ are also i-convex on $[0, 1/(1-q^n)]$. Specially, if a function f is nondecreasing (nonincreasing) on $[0, 1/(1-q^n)]$, then $S_{n,q}(f)$ are also nondecreasing (nonincreasing) on $[0, 1/(1-q^n)]$ and if f is convex (concave) on $[0, 1/(1-q^n)]$, then so are $S_{n,q}(f)$.

(iii) If a function f is convex on $[0, 1/(1-q^n)]$, then $S_{n,q}(f)(x) \ge f(x)$, $x \in [0, 1/(1-q^n)]$.

(iv) If $\omega(t)$ is a modulus of continuity, then $f \in H^{\omega}$ implies that, for each $n \geq 1$, $S_{n,q}(f) \in H^{2\omega}$; if $\omega(t)$ is concave, then, for each $n \geq 1$, $S_{n,q}(f) \in H^{\omega}$.

(v) If $\omega(t)$ is a concave modulus of continuity, then, for each $n \geq 1$, $S_{n,q}(\omega)$ is also a concave modulus of continuity and $S_{n,q}(\omega)(t) \leq \omega(t)$.

(vi) If $f(x)$ is a nonnegative function such that $x^{-1}f(x)$ is nonincreasing on $(0, 1/(1-q^n)]$, then, for each $n \geq 1$, $x^{-1}S_{n,q}(f)(x)$ is nonincreasing also.

Proof. (i) Let I be an interval, $I \subset [0, \infty)$. We assume that, for a real sequence $a = \{a_k\}_{k=0}^{\infty}$, the power series $\sum_{k=0}^{\infty} a_k x^k$ converges to the function g on I. By means of the well-known Descartes' rule of sign it is easy to prove that

$$S^{-}(g) = S^{-}\left(\sum_{k=0}^{\infty} a_k x^k\right) \leq S^{-}(a). \tag{20}$$

Obviously, if $h(x) > 0$ for any $x \in I$ and $b_k > 0$ for $k \geq 0$, then

$$S^{-}(f) = S^{-}(f \cdot h), \qquad S^{-}\left(\{a_k b_k\}_{k=0}^{\infty}\right) = S^{-}\left(\{a_k\}_{k=0}^{\infty}\right). \tag{21}$$

It follows that

$$\begin{aligned} S^{-}\left(S_{n,q}(f)\right) &= S^{-}\left(\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right)\frac{[n]^k}{[k]!}x^k\right) \\ &\leq S^{-}\left(\left\{f\left(\frac{[k]}{[n]}\right)\right\}_{k=0}^{\infty}\right) \leq S^{-}(f), \end{aligned} \tag{22}$$

which implies that $S_{n,q}$ are variation-diminishing.

(ii) The operators $S_{n,q}$ possess the end-point interpolation property and are degree-preserving on polynomials and variation-diminishing. Then, (ii) follows from [26, Lemma 15].

(iii) It follows from [27, p. 281] that if a positive operator L on $C[0, A]$ reproduces linear functions, then $L(f, x) \geq f(x)$ for any convex function f and for any $x \in [0, A]$. Since $S_{n,q}$ are the positive linear operators and reproduce linear functions, we obtain (iii).

(iv) From [26, Corollary 8], we know that if a positive linear operator L on $C[0, A]$ $(A > 0)$ is variation-diminishing and reproduces linear functions, then, for all $f \in C[0, A]$ and $t \in (0, A]$,

$$\omega(Lf, t) \leq \tilde{\omega}(f, t). \tag{23}$$

Thus, if $f \in H^{\omega}$, then

$$\omega\left(S_{n,q}(f), t\right) \leq \tilde{\omega}(f, t) \leq \tilde{\omega}(t), \tag{24}$$

where $\tilde{\omega}(t)$ and $\tilde{\omega}(f, t)$ denote the least concave majorant of $\omega(t)$ and $\omega(f, t)$, respectively. It is well known that for each modulus of continuity ω there exists a concave modulus of continuity $\tilde{\omega}$ such that $\omega(t) \leq \tilde{\omega}(t) \leq 2\omega(t)$ for $t \in [0, A]$. Thence, $S_{n,q}(f) \in H^{2\omega}$ and furthermore $S_{n,q}(f) \in H^{\omega}$ if ω is concave, which means (iv) holds.

(v) From (i) we know that, for a concave modulus of continuity ω and each $n \geq 1$, the function $S_{n,q}(\omega)$ is nondecreasing and concave on $(0, A]$, where $A = 1/(1-q^n)$. We also have $S_{n,q}(\omega)(0) = 0$. This means that $S_{n,q}(\omega)$ is a concave modulus of continuity. The inequality $S_{n,q}(\omega)(t) \leq \omega(t)$ follows directly from (iii).

(vi) Since, for any constant c,

$$\begin{aligned} &S^{-}\left(\frac{S_{n,q}(f)(x)}{x} - c\right) \\ &= S^{-}\left(S_{n,q}(f)(x) - cx\right) = S^{-}\left(S_{n,q}(f(t) - ct)(x)\right) \\ &\leq S^{-}(f(x) - cx) = S^{-}\left(\frac{f(x)}{x} - c\right) \leq 1 \end{aligned} \tag{25}$$

we get that $S_{n,q}(f)(x)/x$ is nondecreasing or nonincreasing on $(0, A]$, where $A = 1/(1-q^n)$. For any $t \in (0, A)$, $f(t)/t \geq f(A)/A$, we have $f(t) \geq f(A)t/A$, and thus $S_{n,q}(f)(x) \geq S_{n,q}(f(A)t/A)(x) = f(A)x/A$. Hence,

$$\frac{S_{n,q}(f)(x)}{x} \geq \frac{f(A)}{A} = \frac{S_{n,q}(f)(A)}{A}, \tag{26}$$

which implies that $S_{n,q}(f)(x)/x$ is nonincreasing on $[0, A]$.

Theorem 1 is proved. □

3. The Rate of Convergence for the q-Szász-Mirakjan Operators $S_{n,q}$ for Fixed $q \in (0, 1)$

The approximation properties of the sequence $\{S_{n,q_n}^{b}(f)\}$ in weighted spaces as $\lim_{n\to\infty} q_n = 1-$ were investigated in [6, Theorem 2] and [25, Theorem 6]. The obtained results are similar to the ones of the classical Szász-Mirakjan operators. However, there are few results about convergence properties of $S_{n,q}$ for fixed $q \in (0, 1)$. This section is devoted to discussing the convergence properties of the q-Szász-Mirakjan operators $S_{n,q}$ for fixed $q \in (0, 1)$.

We set

$$s_{n,k}(q; x) = E_q(-[n]x)\frac{([n]x)^k}{[k]!} = \frac{([n]x)^k}{[k]!}((1-q^n)x; q)_{\infty}, \tag{27}$$

$$p_{\infty,k}(q; x) = \frac{x^k}{(1-q)^k [k]!}(x; q)_{\infty}. \tag{28}$$

Formerly, for $f \in C[0, 1/(1-q)]$ and each $k \geq 0$, $\{f([k]/[n])\}$ converges to $f(1-q^k)$, $\{s_{n,k}(q; x)\}$ converges to $p_{\infty,k}(q; x)$, and

$$\begin{aligned} S_{n,q}(f)(x) &= \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right) s_{n,k}(q; x) \\ &\longrightarrow \sum_{k=0}^{\infty} f\left(1-q^k\right) p_{\infty,k}(q; x) \\ &= B_{\infty,q}(f)(x), \end{aligned} \tag{29}$$

as $n \to \infty$. Indeed, the above conclusion holds. We have the following stronger results.

Theorem 2. *Let $f \in C[0, 1/(1-q)]$. Then, we have*

$$\sup_{x\in[0,1]} \left| S_{n,q}(f)(x) - B_{\infty,q}(f)(x) \right| \le C_q \omega(f, q^n), \tag{30}$$

where $C_q = 4 + q/(1-q) + (1/(1-q))e^{q^2/(1-q)^2}$. This estimate is sharp in the following sense of order: for each α, $0 < \alpha \le 1$, there exists a function $f_\alpha(x)$ which belongs to the Lipschitz class $\operatorname{Lip}\alpha := \{f \in C[0,1] \mid \omega(f;t) \le t^\alpha\}$ such that

$$\sup_{x\in[0,1]} \left| S_{n,q}(f_\alpha)(x) - B_{\infty,q}(f_\alpha)(x) \right| \ge C q^{n\alpha}, \tag{31}$$

where C is a positive constant independent of n.

Remark 3. It follows from (30) that, for $f \in C[0, 1/(1-q)]$, $\lim_{n\to\infty} S_{n,q}(f)(x) = B_{\infty,q}(f)(x)$ uniformly on $x \in [0,1]$ as $n \to \infty$. Since $B_{\infty,q}(f)(x) = f(x)$, $x \in [0,1]$, if and only if f is linear on $[0,1]$ (see [2, Theorem 6]), we get that the sequence $S_{n,q}(f)(x)$ converges to f uniformly on $[0,1]$ if and only if f is linear on $[0,1]$.

Remark 4. It should be emphasized that the proof of Theorem 2 requires estimation techniques involving the infinite product. Also, it is a little more difficult than the one used for q-Bernstein polynomials (see [23]), since $S_{n,q}(f)(1) \ne f(1) = B_{\infty,q}(f)(1)$.

Proof. Since the operators $S_{n,q}$ and $B_{\infty,q}$ reproduce linear functions, we get that, for $x \in [0,1)$,

$$\sum_{k=0}^{\infty} s_{n,k}(q;x) = 1, \qquad \sum_{k=0}^{\infty} \frac{[k]}{[n]} s_{n,k}(q;x) = x, \qquad x \in \left[0, \frac{1}{1-q^n}\right), \tag{32}$$

$$\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1, \qquad \sum_{k=0}^{\infty} \left(1-q^k\right) p_{\infty,k}(q;x) = x, \qquad x \in [0,1), \tag{33}$$

where $s_{n,k}(q;x)$ and $p_{\infty,k}(q;x)$ are defined by (27) and (28), respectively. By means of (32) and (33), direct calculations give that

$$\sum_{k=0}^{n} q^k s_{n,k}(q;x) = 1 - x + q^n x, \qquad \sum_{k=0}^{\infty} q^k p_{\infty,k}(q;x) = 1 - x. \tag{34}$$

For $x = 0$, we have

$$\left| S_{n,q}(f)(0) - B_{\infty,q}(f)(0) \right| = |f(0) - f(0)| = 0. \tag{35}$$

For $x = 1$, it follows that

$$\begin{aligned}
&\left| S_{n,q}(f)(1) - B_{\infty,q}(f)(1) \right| \\
&\quad = \left| \sum_{k=0}^{\infty} \left(f\left(\frac{[k]}{[n]}\right) - f(1) \right) s_{n,k}(q;1) \right| \\
&\quad \le \sum_{k=0}^{\infty} \left(\left| f\left(\frac{[k]}{[n]}\right) - f\left(1-q^k\right) \right| \right. \\
&\qquad\qquad \left. + \left| f(1) - f\left(1-q^k\right) \right| \right) s_{n,k}(q;1) \\
&\quad \le \sum_{k=0}^{\infty} \left(\omega\left(f, \frac{[k]q^n}{[n]}\right) + \omega\left(f, q^k\right) \right) s_{n,k}(q;1) \\
&\quad \le \omega(f, q^n) \sum_{k=0}^{\infty} \left(2 + \frac{[k]}{[n]} + \frac{q^k}{q^n} \right) s_{n,k}(q;1) \\
&\quad = 4\omega(f, q^n),
\end{aligned} \tag{36}$$

where in the first equality we used (32); in the last inequality we used the inequality $\omega(f, \lambda t) \le (1+\lambda)\omega(f,t)$ for any $\lambda, t > 0$; in the last equality we used (32) and (34).

Now for $x \in (0,1)$, by (32) and (33), we have

$$\begin{aligned}
&\left| S_{n,q}(f,x) - B_{\infty,q}(f,x) \right| \\
&\quad = \left| \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right) s_{n,k}(q;x) - \sum_{k=0}^{\infty} f\left(1-q^k\right) p_{\infty,k}(q;x) \right| \\
&\quad = \left| \sum_{k=0}^{\infty} \left(f\left(\frac{[k]}{[n]}\right) - f\left(1-q^k\right) \right) s_{n,k}(q;x) \right. \\
&\qquad\qquad \left. + \sum_{k=0}^{\infty} \left(f\left(1-q^k\right) - f(1) \right) \left(s_{n,k}(q;x) - p_{\infty,k}(q;x) \right) \right| \\
&\quad \le \sum_{k=0}^{\infty} \left| f\left(\frac{[k]}{[n]}\right) - f\left(1-q^k\right) \right| s_{n,k}(q;x) \\
&\qquad + \sum_{k=0}^{\infty} \left| f\left(1-q^k\right) - f(1) \right| \left| s_{n,k}(q;x) - p_{\infty,k}(q;x) \right| \\
&\quad =: J_1 + J_2.
\end{aligned} \tag{37}$$

Since

$$\left| \frac{[k]}{[n]} - \left(1-q^k\right) \right| = \frac{[k]q^n}{[n]}, \qquad \omega(f;\lambda t) \le (1+\lambda)\,\omega(f;t), \quad \lambda, t > 0, \tag{38}$$

we get by (32)

$$\begin{aligned} J_1 &\le \sum_{k=0}^{\infty}\omega\left(f,\frac{[k]\,q^n}{[n]}\right)s_{n,k}(q;x) \\ &\le \omega(f,q^n)\sum_{k=0}^{\infty}\left(1+\frac{[k]}{[n]}\right)s_{n,k}(q;x) \\ &= (1+x)\,\omega(f,q^n)\le 2\omega(f,q^n). \end{aligned} \tag{39}$$

In order to estimate J_2, we need to estimate $|s_{n,k}(q;x)-p_{\infty,k}(q;x)|$. We have

$$\begin{aligned} &|s_{n,k}(q;x)-p_{\infty,k}(q;x)| \\ &\quad=\left|\frac{([n]\,x)^k}{[k]!}\prod_{s=0}^{\infty}(1-(1-q^n)\,q^s x) -\frac{x^k}{(1-q)^k\,[k]!}\prod_{s=0}^{\infty}(1-q^s x)\right| \\ &\quad\le\frac{x^k}{(1-q)^k\,[k]!}\left|\prod_{s=0}^{\infty}(1-q^s(1-q^n)\,x)(1-q^n)^k -\prod_{s=0}^{\infty}(1-q^s x)(1-q^n)^k\right| \\ &\qquad+\frac{x^k}{(1-q)^k\,[k]!}\prod_{s=0}^{\infty}(1-q^s x)\left|1-(1-q^n)^k\right| \\ &\quad\le p_{\infty,k}(q;x)\left(\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right|+\left|1-(1-q^n)^k\right|\right). \end{aligned} \tag{40}$$

We note that

$$\begin{aligned} q^k\left(1-(1-q^n)^k\right) &= q^{k+n}\left(1+(1-q^n)+\cdots+(1-q^n)^{k-1}\right) \\ &\le kq^{k+n}\le\frac{q^{n+1}}{1-q}. \end{aligned} \tag{41}$$

It follows that

$$\begin{aligned} J_2 &\le \sum_{k=0}^{\infty}\omega(f,q^k)\,|s_{n,k}(q;x)-p_{\infty,k}(q;x)| \\ &\le \omega(f,q^n)\sum_{k=0}^{\infty}\left(1+\frac{q^k}{q^n}\right)|s_{n,k}(q;x)-p_{\infty,k}(q;x)| \\ &\le \omega(f,q^n)\left(\sum_{k=0}^{\infty}(s_{n,k}(q;x)+p_{\infty,k}(q;x)) +\sum_{k=0}^{\infty}\frac{q^k}{q^n}|s_{n,k}(q;x)-p_{\infty,k}(q;x)|\right) \\ &\le \omega(f,q^n)\left(2+\sum_{k=0}^{\infty}\frac{q^k}{q^n}p_{\infty,k}(q;x) \times\left(\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right| +\left|1-(1-q^n)^k\right|\right)\right) \\ &\le \omega(f,q^n)\left(2+\sum_{k=0}^{\infty}p_{\infty,k}(q;x) \times\left(q^{k-n}\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right|+\frac{q}{1-q}\right)\right) \\ &\le \omega(f,q^n)\left(2+q^{-n}(1-x) \times\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right|+\frac{q}{1-q}\right) \\ &=: \omega(f,q^n)\left(2+H+\frac{q}{1-q}\right), \end{aligned} \tag{42}$$

where in the fourth inequality we used (32) and (33); in the last inequality we used (34) and (33). We estimate H. We have

$$\begin{aligned} H &= q^{-n}\left|(1-x+q^n x)\prod_{s=1}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-(1-x)\right| \\ &= x\prod_{s=1}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)+q^{-n}(1-x) \times\left|\prod_{s=1}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right| \\ &=: xe^K+q^{-n}(1-x)\left(e^K-1\right), \end{aligned} \tag{43}$$

where $K:=\sum_{s=1}^{\infty}\ln(1+q^{s+n}x/(1-q^s x))$. Using the inequality $\ln(1+t)\le t$, $t\ge 0$, we get that

$$\begin{aligned} K &\le \sum_{s=1}^{\infty}\frac{q^{s+n}x}{1-q^s x}\le\sum_{s=1}^{\infty}\frac{q^{s+n}}{1-qx}\le\frac{q^{n+1}}{(1-q)(1-qx)} \\ &\le\frac{q^2}{(1-q)^2}. \end{aligned} \tag{44}$$

It follows that

$$\begin{gathered} e^K\le e^{q^2/(1-q)^2}, \\ e^K-1=Ke^{\xi}\le Ke^K\le\frac{q^{n+1}}{(1-q)(1-qx)}e^{q^2/(1-q)^2}, \\ \xi\in[0,K]. \end{gathered} \tag{45}$$

This deduces that, for $x \in (0,1)$,

$$H \le e^{q^2/(1-q)^2} + (1-x)\frac{q}{(1-q)(1-qx)}e^{q^2/(1-q)^2} \le \frac{1}{1-q}e^{q^2/(1-q)^2}, \tag{46}$$

and thence

$$J_2 \le \omega(f, q^n)\left(2 + \frac{q}{1-q} + \frac{1}{1-q}e^{q^2/(1-q)^2}\right). \tag{47}$$

We conclude from (39) and (47) that, for $x \in (0,1)$,

$$\left|S_{n,q}(f,x) - B_{\infty,q}(f,x)\right| \le J_1 + J_2 \le \omega(f, q^n)\left(4 + \frac{q}{1-q} + \frac{1}{1-q}e^{q^2/(1-q)^2}\right). \tag{48}$$

Hence, (30) follows from (35), (36), and (48).

At last we show that the estimate (30) is sharp. For each α, $0 < \alpha \le 1$, suppose that $f_\alpha^*(x)$ is a continuous function, which is equal to zero in $[0, 1-q]$ and $[1-q^2, 1]$, equal to $(x-(1-q))^\alpha$ in $[1-q, 1-q+q(1-q)/2]$, and linear in the rest of $[0,1]$. It is easy to see that $\omega(f_\alpha^*, t) \le At^\alpha$. We set $f_\alpha(t) = (1/A)f_\alpha^*(t)$. Then, $f_\alpha \in \operatorname{Lip}\alpha$, and for sufficiently large n, we have

$$\begin{aligned}\sup_{x\in[0,1]}&\left|S_{n,q}(f_\alpha)(x) - B_{\infty,q}(f_\alpha)(x)\right| \\ &= \frac{1}{A}\frac{(1-q)^\alpha q^{n\alpha}}{(1-q^n)^\alpha}\sup_{x\in[0,1]}\left|s_{n,1}(q;x)\right| \\ &\ge \frac{(1-q)^\alpha q^{n\alpha}}{A}\left|s_{n,1}\left(q;\frac{1}{2}\right)\right| \\ &\ge \frac{(1-q)^\alpha}{2A(1-q)}\prod_{s=0}^{\infty}\left(1-\frac{q^s}{2}\right)q^{n\alpha} =: Cq^{n\alpha}.\end{aligned} \tag{49}$$

The proof of Theorem 2 is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors were supported by the National Natural Science Foundation of China (Project no. 11271263), the Beijing Natural Science Foundation (1132001), and BCMIIS.

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