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Research Article

Fractional Integral Inequalities via Hadamard's Fractional Integral

Weerawat Sudsutad,[1] Sotiris K. Ntouyas,[2] and Jessada Tariboon[1]

[1] *Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand*
[2] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Correspondence should be addressed to Jessada Tariboon; jessadat@kmutnb.ac.th

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We establish new fractional integral inequalities, via Hadamard's fractional integral. Several new integral inequalities are obtained, including a Grüss type Hadamard fractional integral inequality, by using Young and weighted AM-GM inequalities. Many special cases are also discussed.

1. Introduction

Inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of mathematics. The study of mathematical inequalities plays very important role in classical differential and integral equations which has applications in many fields. Fractional inequalities are important in studying the existence, uniqueness, and other properties of fractional differential equations. Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville and Caputo derivative; see [1–7] and the references therein.

Another kind of fractional derivative that appears in the literature is the fractional derivative due to Hadamard introduced in 1892 [8], which differs from the Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [9–14]. Recently in the literature, there appeared some results on fractional integral inequalities using Hadamard fractional integral; see [15–17].

In this paper we present some new fractional integral inequalities using the Hadamard fractional integral. Several new integral inequalities are obtained by using Young and weighted AM-GM inequalities. Many special cases are also discussed. Moreover, a Grüss type Hadamard fractional integral inequality is obtained.

2. Preliminaries

In this section we give some preliminaries and basic proposition used in our subsequent discussion. The necessary background details are given in the book by Kilbas et al. [9].

Definition 1. The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 1$, is defined as

$$ {}_{H}J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \tag{1}$$

where Γ is the standard gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-s} s^{\alpha-1} ds$, provided the integral exists, where $\log(\cdot) = \log_e(\cdot)$.

Definition 2. The Hadamard fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$, of a function $f(t)$ is given by

$$ {}_{H}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^{n} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}. \tag{2}$$

Next, we recall a proposition concerning a Hadamard integral and derivative.

Proposition 3 (see [9]). *If $\alpha > 0$, the following relations hold:*

$$ {}_HJ^{\alpha}(\log t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\log t)^{\beta+\alpha-1}, \quad (3)$$

$$ {}_HD^{\alpha}(\log t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\log t)^{\beta-\alpha-1}, \quad (4)$$

respectively.

For the convenience of establishing our results, we give the semigroup property

$$ {}_HJ^{\alpha}{}_HJ^{\beta}f(t) = {}_HJ^{\alpha+\beta}f(t), \quad \alpha \geq 0,\ \beta \geq 0, \quad (5)$$

which implies the commutative property

$$ {}_HJ^{\alpha}{}_HJ^{\beta}f(t) = {}_HJ^{\beta}{}_HJ^{\alpha}f(t). \quad (6)$$

3. Main Results

Now, we are in a position to give our main results.

Theorem 4. *Let f be an integrable function on $[1,\infty)$. Assume the following.*

(H_1) There exist two integrable functions φ_1, φ_2 on $[1,\infty)$ such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \quad \forall t \in [1,\infty). \quad (7)$$

Then, for $t > 1$, $\alpha, \beta > 0$, one has

$$\begin{aligned} &{}_HJ^{\beta}\varphi_1(t)\,{}_HJ^{\alpha}f(t) + {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}f(t) \\ &\quad \geq {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}\varphi_1(t) + {}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}f(t). \end{aligned} \quad (8)$$

Proof. From (H_1), for all $\tau \geq 1$, $\rho \geq 1$, we have

$$(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \geq 0. \quad (9)$$

Therefore,

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \quad (10)$$

Multiplying both sides of (10) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (1,t)$, we get

$$\begin{aligned} &f(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + \varphi_1(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau) \\ &\quad \geq \varphi_1(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + f(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau). \end{aligned} \quad (11)$$

Integrating both sides of (11) with respect to τ on $(1,t)$, we obtain

$$\begin{aligned} &f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} \\ &\quad + \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau} \\ &\geq \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} \\ &\quad + f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}, \end{aligned} \quad (12)$$

which yields

$$\begin{aligned} &f(\rho)\,{}_HJ^{\alpha}\varphi_2(t) + \varphi_1(\rho)\,{}_HJ^{\alpha}f(t) \\ &\quad \geq \varphi_1(\rho)\,{}_HJ^{\alpha}\varphi_2(t) + f(\rho)\,{}_HJ^{\alpha}f(t). \end{aligned} \quad (13)$$

Multiplying both sides of (13) by $(\log(t/\rho))^{\beta-1}/\rho\Gamma(\beta)$, $\rho \in (1,t)$, we have

$$\begin{aligned} &{}_HJ^{\alpha}\varphi_2(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}f(\rho) \\ &\quad + {}_HJ^{\alpha}f(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\varphi_1(\rho) \\ &\geq {}_HJ^{\alpha}\varphi_2(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\varphi_1(\rho) \\ &\quad + {}_HJ^{\alpha}f(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}f(\rho). \end{aligned} \quad (14)$$

Integrating both sides of (14) with respect to ρ on $(1,t)$, we get

$$\begin{aligned} &{}_HJ^{\alpha}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho} \\ &\quad + {}_HJ^{\alpha}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho} \\ &\geq {}_HJ^{\alpha}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho} \\ &\quad + {}_HJ^{\alpha}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho}. \end{aligned} \quad (15)$$

Hence, we deduce inequality (8) as requested. This completes the proof. □

As special cases of Theorems 4, we obtain the following results.

Corollary 5. *Let f be an integrable function on $[1,\infty)$ satisfying $m \le f(t) \le M$, for all $t \in [1,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 1$ and $\alpha, \beta > 0$, one has*

$$\begin{aligned} &m\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha} f(t) + M\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta} f(t) \\ &\quad\ge mM\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + {}_HJ^{\alpha} f(t)\,{}_HJ^{\beta} f(t). \end{aligned} \tag{16}$$

Corollary 6. *Let f be an integrable function on $[1,\infty)$. Assume that there exists an integrable function $\varphi(t)$ on $[1,\infty)$ and a constant $M > 0$ such that*

$$\varphi(t) - M \le f(t) \le \varphi(t) + M, \tag{17}$$

for all $t \in [1,\infty)$. Then for $t > 1$ and $\alpha, \beta > 0$, one has

$$\begin{aligned} &{}_HJ^{\beta}\varphi(t)\,{}_HJ^{\alpha} f(t) + {}_HJ^{\alpha}\varphi(t)\,{}_HJ^{\beta} f(t) \\ &\quad+ \frac{M(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}\varphi(t) + \frac{M(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta} f(t) \\ &\quad+ \frac{M^2(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\ge {}_HJ^{\alpha}\varphi(t)\,{}_HJ^{\beta}\varphi(t) + {}_HJ^{\alpha} f(t)\,{}_HJ^{\beta} f(t) \\ &\quad+ \frac{M(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha} f(t) + \frac{M(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\varphi(t). \end{aligned} \tag{18}$$

Example 7. Let f be a function satisfying $\log t \le f(t) \le 1 + \log t$ for $t \in [1,\infty)$. Then for $t > 1$ and $\alpha > 0$, we have

$$\begin{aligned} &\left(\frac{2(\log t)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right){}_HJ^{\alpha} f(t) \\ &\quad\ge \left(\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right) \\ &\quad+ \left({}_HJ^{\alpha} f(t)\right)^2. \end{aligned} \tag{19}$$

Theorem 8. *Let f be an integrable function on $[1,\infty)$ and $\theta_1, \theta_2 > 0$ satisfying $1/\theta_1 + 1/\theta_2 = 1$. Suppose that (H_1) holds. Then, for $t > 1$, $\alpha, \beta > 0$, one has*

$$\begin{aligned} &\frac{1}{\theta_1}\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\,{}_HJ^{\alpha}\left((\varphi_2 - f)^{\theta_1}\right)(t) \\ &\quad+ \frac{1}{\theta_2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\left((f-\varphi_1)^{\theta_2}\right)(t) \\ &\quad+ {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}\varphi_1(t) + {}_HJ^{\alpha} f(t)\,{}_HJ^{\beta} f(t) \\ &\ge {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta} f(t) + {}_HJ^{\alpha} f(t)\,{}_HJ^{\beta}\varphi_1(t). \end{aligned} \tag{20}$$

Proof. According to the well-known Young's inequality

$$\begin{gathered} \frac{1}{\theta_1}x^{\theta_1} + \frac{1}{\theta_2}y^{\theta_2} \ge xy, \quad \forall x, y \ge 0,\ \theta_1, \theta_2 > 0, \\ \frac{1}{\theta_1} + \frac{1}{\theta_2} = 1, \end{gathered} \tag{21}$$

setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > 1$, we have

$$\begin{aligned} &\frac{1}{\theta_1}(\varphi_2(\tau) - f(\tau))^{\theta_1} + \frac{1}{\theta_2}(f(\rho) - \varphi_1(\rho))^{\theta_2} \\ &\quad\ge (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)). \end{aligned} \tag{22}$$

Multiplying both sides of (22) by $(\log(t/\tau))^{\alpha-1}(\log(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta)$, $\tau, \rho \in (1, t)$, we get

$$\begin{aligned} &\frac{1}{\theta_1}\frac{(\log(t/\tau))^{\alpha-1}(\log(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau) - f(\tau))^{\theta_1} \\ &\quad+ \frac{1}{\theta_2}\frac{(\log(t/\tau))^{\alpha-1}(\log(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(f(\rho) - \varphi_2(\rho))^{\theta_2} \\ &\quad\ge \frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau) - f(\tau))\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} \\ &\quad\times (f(\rho) - \varphi_1(\rho)). \end{aligned} \tag{23}$$

Integrating the above inequality with respect to τ and ρ from 1 to t, we have

$$\begin{aligned} &\frac{1}{\theta_1}{}_HJ^{\beta}(1)(t)\,{}_HJ^{\alpha}(\varphi_2 - f)^{\theta_1}(t) \\ &\quad+ \frac{1}{\theta_2}{}_HJ^{\alpha}(1)(t)\,{}_HJ^{\beta}(f - \varphi_1)^{\theta_2}(t) \\ &\quad\ge {}_HJ^{\alpha}(\varphi_2 - f)(t)\,{}_HJ^{\beta}(f - \varphi_1)(t), \end{aligned} \tag{24}$$

which implies (20). □

Corollary 9. *Let f be an integrable function on $[1,\infty)$ satisfying $m \le f(t) \le M$, for all $t \in [1,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 1$ and $\alpha, \beta > 0$, one has*

$$\begin{aligned} &(m+M)^2\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha} f^2(t) \\ &\quad+ \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta} f^2(t) + 2\,{}_HJ^{\alpha} f(t)\,{}_HJ^{\beta} f(t) \\ &\ge 2(m+M) \\ &\quad\times\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha} f(t) + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta} f(t)\right). \end{aligned} \tag{25}$$

Example 10. Let f be a function satisfying $\log t \le f(t) \le 1 + \log t$ for $t \in [1,\infty)$. Then for $t > 1$ and $\alpha > 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad\times\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{2(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{4(\log t)^{\alpha+2}}{\Gamma(\alpha+3)}+2{}_HJ^{\alpha}f^{2}(t)\right) \\
&\quad+\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}+\left({}_HJ^{\alpha}f(t)\right)^{2} \\
&\quad\ge 2\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right){}_HJ^{\alpha}f(t) \\
&\quad+\frac{2(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}(f\log t)(t).
\end{aligned} \tag{26}$$

Theorem 11. *Let f be an integrable function on $[1,\infty)$ and $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. In addition, suppose that (H_1) holds. Then, for $t>1$, $\alpha,\beta>0$, one has*

$$\begin{aligned}
&\theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}\varphi_2(t)+\theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t) \\
&\quad\ge {}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t)\,{}_HJ^{\beta}(f-\varphi_1)^{\theta_2}(t) \\
&\quad+\theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t)+\theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\varphi_1(t).
\end{aligned} \tag{27}$$

Proof. From the well-known weighted AM-GM inequality

$$\begin{gathered}
\theta_1x+\theta_2y\ge x^{\theta_1}y^{\theta_2},\quad \forall x,y\ge 0,\ \theta_1,\theta_2>0, \\
\theta_1+\theta_2=1,
\end{gathered} \tag{28}$$

by setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, $\tau,\rho>1$, we have

$$\begin{aligned}
&\theta_1(\varphi_2(\tau)-f(\tau))+\theta_2(f(\rho)-\varphi_1(\rho)) \\
&\quad\ge(\varphi_2(\tau)-f(\tau))^{\theta_1}(f(\rho)-\varphi_1(\rho))^{\theta_2}.
\end{aligned} \tag{29}$$

Multiplying both sides of (29) by $(\log(t/\tau))^{\alpha-1}(\log(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta)$, $\tau,\rho\in(1,t)$, we get

$$\begin{aligned}
&\theta_1\frac{(\log(t/\tau))^{\alpha-1}(\log(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau)-f(\tau)) \\
&\quad+\theta_2\frac{(\log(t/\tau))^{\alpha-1}(\log(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(f(\rho)-\varphi_2(\rho)) \\
&\ge\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau)-f(\tau))^{\theta_1} \\
&\quad\times\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}(f(\rho)-\varphi_1(\rho))^{\theta_2}.
\end{aligned} \tag{30}$$

Integrating the above inequality with respect to τ and ρ from 1 to t, we have

$$\begin{aligned}
&\theta_1\,{}_HJ^{\beta}(1)(t)\,{}_HJ^{\alpha}(\varphi_2-f)(t) \\
&\quad+\theta_2\,{}_HJ^{\alpha}(1)(t)\,{}_HJ^{\beta}(f-\varphi_1)(t) \\
&\quad\ge {}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t)\,{}_HJ^{\beta}(f-\varphi_1)^{\theta_2}(t).
\end{aligned} \tag{31}$$

Therefore, we deduce inequality (27). □

Corollary 12. *Let f be an integrable function on $[1,\infty)$ satisfying $m\le f(t)\le M$, for all $t\in[1,\infty)$ and $m,M\in\mathbb{R}$. Then for $t>1$ and $\alpha,\beta>0$, one has*

$$\begin{aligned}
&M\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t) \\
&\quad\ge m\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t) \\
&\quad+2\,{}_HJ^{\alpha}(M-f)^{1/2}(t)\,{}_HJ^{\beta}(f-m)^{1/2}(t).
\end{aligned} \tag{32}$$

Example 13. Let f be a function satisfying $\log t\le f(t)\le 1+\log t$ for $t\in[1,\infty)$. Then for $t>1$ and $\alpha>0$, we have

$$\begin{aligned}
&\frac{(\log t)^{2\alpha}}{\Gamma^2(\alpha+1)} \\
&\quad\ge 2\,{}_HJ^{\alpha}\left(\sqrt{1+\log t-f}\right)(t)\,{}_HJ^{\alpha}\left(\sqrt{f-\log t}\right)(t).
\end{aligned} \tag{33}$$

Lemma 14 (see [18]). *Assume that $a\ge 0$, $p\ge q\ge 0$, and $p\ne 0$. Then*

$$a^{q/p}\le\left(\frac{q}{p}k^{q-p/p}a+\frac{p-q}{p}k^{q/p}\right),\quad \text{for any } k>0. \tag{34}$$

Theorem 15. *Let f be an integrable function on $[1,\infty)$ and constants $p\ge q\ge 0$, $p\ne 0$. In addition, assume that (H_1) holds. Then for any $k>0$, $t>1$, α, $\beta>0$, the following two inequalities hold:*

$$\begin{aligned}
(A_1)\quad &{}_HJ^{\alpha}(\varphi_2-f)^{q/p}(t)+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^{\alpha}f(t) \\
&\le\frac{q}{p}k^{(q-p)/p}\,{}_HJ^{\alpha}\varphi_2(t)+\frac{p-q}{p}k^{q/p}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}, \\
(B_1)\quad &{}_HJ^{\alpha}(f-\varphi_1)^{q/p}(t)+\frac{q}{p}k^{q-p/p}\,{}_HJ^{\alpha}\varphi_1(t) \\
&\le\frac{q}{p}k^{(q-p)/p}\,{}_HJ^{\alpha}f(t)+\frac{p-q}{p}k^{q/p}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{35}$$

Proof. By condition (H_1) and Lemma 14, for $p\ge q\ge 0$, $p\ne 0$, it follows that

$$\begin{aligned}
&(\varphi_2(\tau)-f(\tau))^{q/p} \\
&\quad\le\frac{q}{p}k^{(q-p)/p}(\varphi_2(\tau)-f(\tau))+\frac{p-q}{p}k^{q/p},
\end{aligned} \tag{36}$$

for any $k > 0$. Multiplying both sides of (36) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (1,t)$, and integrating the resulting identity with respect to τ from 1 to t, one has

$$\begin{aligned}&{}_HJ^{\alpha}(f-\varphi_1)^{q/p}\\ &\leq \frac{q}{p}k^{(q-p)/p}\left({}_HJ^{\alpha}\varphi_2(t)-{}_HJ^{\alpha}f(t)\right)+\frac{p-q}{p}k^{q/p}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)},\end{aligned} \tag{37}$$

which leads to inequality (A_1). Inequality (B_1) is proved by similar arguments. □

Corollary 16. *Let f be an integrable function on $[1,\infty)$ satisfying $m \leq f(t) \leq M$, for all $t \in [1,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 1$ and $\alpha > 0$, one has*

$$\begin{aligned}&(A_2)\quad 2{}_HJ^{\alpha}(M-f)^{1/2}(t)+{}_HJ^{\alpha}f(t)\leq(M+1)\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)},\\ &(B_2)\quad 2{}_HJ^{\alpha}(f-m)^{1/2}(t)+m\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\\ &\qquad\leq {}_HJ^{\alpha}f(t)+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned} \tag{38}$$

Example 17. Let f be a function satisfying $\log t \leq f(t) \leq 1+\log t$ for $t \in [1,\infty)$. Then for $t > 1$ and $\alpha > 0$, we have

$$\begin{aligned}&2{}_HJ^{\alpha}\left(\sqrt{1+\log t-f}\right)(t)+{}_HJ^{\alpha}f(t)\\ &\qquad\leq\frac{2(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)},\\ &2{}_HJ^{\alpha}\left(\sqrt{f-\log t}\right)(t)+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\leq{}_HJ^{\alpha}f(t)+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned} \tag{39}$$

Theorem 18. *Let f and g be two integrable functions on $[1,\infty)$. Suppose that (H_1) holds and moreover one assumes the following.*

(H_2) There exist ψ_1 and ψ_2 integrable functions on $[1,\infty)$ such that

$$\psi_1(t)\leq g(t)\leq\psi_2(t)\quad\forall t\in[1,\infty). \tag{40}$$

Then, for $t > 0$, $\alpha, \beta > 0$, the following inequalities hold:

$$\begin{aligned}&(A_3)\quad {}_HJ^{\beta}\psi_1(t)\,{}_HJ^{\alpha}f(t)+{}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}g(t)\\ &\qquad\geq{}_HJ^{\beta}\psi_1(t)\,{}_HJ^{\alpha}\varphi_2(t)+{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t),\\ &(B_3)\quad {}_HJ^{\beta}\varphi_1(t)\,{}_HJ^{\alpha}g(t)+{}_HJ^{\alpha}\psi_2(t)\,{}_HJ^{\beta}f(t)\\ &\qquad\geq{}_HJ^{\beta}\varphi_1(t)\,{}_HJ^{\alpha}\psi_2(t)+{}_HJ^{\beta}f(t)\,{}_HJ^{\alpha}g(t),\\ &(C_3)\quad {}_HJ^{\beta}\psi_2(t)\,{}_HJ^{\alpha}\varphi_2(t)+{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t)\\ &\qquad\geq{}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}g(t)+{}_HJ^{\beta}\psi_2(t)\,{}_HJ^{\alpha}f(t),\\ &(D_3)\quad {}_HJ^{\alpha}\varphi_1(t)\,{}_HJ^{\beta}\psi_1(t)+{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t)\\ &\qquad\geq{}_HJ^{\alpha}\varphi_1(t)\,{}_HJ^{\beta}g(t)+{}_HJ^{\beta}\psi_1(t)\,{}_HJ^{\alpha}f(t).\end{aligned} \tag{41}$$

Proof. To prove (A_3), from (H_1) and (H_2), we have for $t \in [1,\infty)$ that

$$(\varphi_2(\tau)-f(\tau))(g(\rho)-\psi_1(\rho))\geq 0. \tag{42}$$

Therefore,

$$\varphi_2(\tau)g(\rho)+\psi_1(\rho)f(\tau)\geq\psi_1(\rho)\varphi_2(\tau)+f(\tau)g(\rho). \tag{43}$$

Multiplying both sides of (43) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (1,t)$, we get

$$\begin{aligned}&g(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau)+\psi_1(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau)\\ &\quad\geq\psi_1(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau)+g(\rho)\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau).\end{aligned} \tag{44}$$

Integrating both sides of (44) with respect to τ on $(1,t)$, we obtain

$$\begin{aligned}&g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau}\\ &\quad+\psi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}\\ &\geq\psi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau}\\ &\quad+g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}.\end{aligned} \tag{45}$$

Then we have

$$\begin{aligned}&g(\rho)\,{}_HJ^{\alpha}\varphi_2(t)+\psi_1(\rho)\,{}_HJ^{\alpha}f(t)\\ &\quad\geq\psi_1(\rho)\,{}_HJ^{\alpha}\varphi_2(t)+g(\rho)\,{}_HJ^{\alpha}f(t).\end{aligned} \tag{46}$$

Multiplying both sides of (46) by $(\log(t/\rho))^{\beta-1}/\rho\Gamma(\beta)$, $\rho \in (1,t)$, we have

$$\begin{aligned}
&{}_HJ^{\alpha}\varphi_2(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}g(\rho)\\
&\quad+{}_HJ^{\alpha}f(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\psi_1(\rho)\\
&\geq {}_HJ^{\alpha}\varphi_2(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\psi_1(\rho)\\
&\quad+{}_HJ^{\alpha}f(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}g(\rho).
\end{aligned}\tag{47}$$

Integrating both sides of (47) with respect to ρ on $(1,t)$, we get the desired inequality (A_3).

To prove (B_3)–(D_3), we use the following inequalities:

$$\begin{aligned}
&(B_3)\quad (\psi_2(\tau)-g(\tau))(f(\rho)-\varphi_1(\rho))\geq 0,\\
&(C_3)\quad (\varphi_2(\tau)-f(\tau))(g(\rho)-\psi_2(\rho))\leq 0,\\
&(D_3)\quad (\varphi_1(\tau)-f(\tau))(g(\rho)-\psi_1(\rho))\leq 0.
\end{aligned}\tag{48}$$

□

As a special case of Theorem 18, we have the following corollary.

Corollary 19. *Let f and g be two integrable functions on $[1,\infty)$. Assume the following.*

(H_3) *There exist real constants m, M, n, N such that*

$$m\leq f(t)\leq M,\quad n\leq g(t)\leq N\quad \forall t\in[1,\infty).\tag{49}$$

Then, for $t>1$, α, $\beta>0$, one has

$$\begin{aligned}
(A_4)\quad &\frac{n(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t)+\frac{M(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}g(t)\\
&\geq \frac{nM(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t),\\
(B_4)\quad &\frac{m(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}g(t)+\frac{N(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t)\\
&\geq \frac{mN(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+{}_HJ^{\beta}f(t)\,{}_HJ^{\alpha}g(t),\\
(C_4)\quad &\frac{MN(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t)\\
&\geq \frac{M(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}g(t)+\frac{N(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t),\\
(D_4)\quad &\frac{mn(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t)\\
&\geq \frac{m(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}g(t)+\frac{n(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t).
\end{aligned}\tag{50}$$

Theorem 20. *Let f and g be two integrable functions on $[1,\infty)$ and $\theta_1,\theta_2>0$ satisfying $1/\theta_1+1/\theta_2=1$. Suppose that (H_1) and (H_2) hold. Then, for $t>1$, $\alpha,\beta>0$, the following inequalities hold:*

$$\begin{aligned}
(A_5)\quad &\frac{1}{\theta_1}\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t)\\
&\quad+\frac{1}{\theta_2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}(\psi_2-g)^{\theta_2}(t)\\
&\geq {}_HJ^{\alpha}(\varphi_2-f)(t)\,{}_HJ^{\beta}(\psi_2-g)(t),\\
(B_5)\quad &\frac{1}{\theta_1}{}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t)\,{}_HJ^{\beta}(\psi_2-g)^{\theta_1}(t)\\
&\quad+\frac{1}{\theta_2}{}_HJ^{\alpha}(\psi_2-g)^{\theta_2}(t)\,{}_HJ^{\beta}(\varphi_2-f)^{\theta_2}(t)\\
&\geq {}_HJ^{\alpha}(\varphi_2-f)(\psi_2-g)(t)\,{}_HJ^{\beta}(\varphi_2-f)(\psi_2-g)(t),\\
(C_5)\quad &\frac{1}{\theta_1}\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}(f-\varphi_1)^{\theta_1}(t)\\
&\quad+\frac{1}{\theta_2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}(g-\psi_1)^{\theta_2}(t)\\
&\geq {}_HJ^{\alpha}(f-\varphi_1)(t)\,{}_HJ^{\beta}(g-\psi_1)(t),\\
(D_5)\quad &\frac{1}{\theta_1}{}_HJ^{\alpha}(f-\varphi_1)^{\theta_1}(t)\,{}_HJ^{\beta}(g-\psi_1)^{\theta_1}(t)\\
&\quad+\frac{1}{\theta_2}{}_HJ^{\alpha}(g-\psi_1)^{\theta_2}(t)\,{}_HJ^{\beta}(f-\varphi_1)^{\theta_2}(t)\\
&\geq {}_HJ^{\alpha}(f-\varphi_1)(g-\psi_1)(t)\,{}_HJ^{\beta}(f-\varphi_1)(g-\psi_1)(t).
\end{aligned}\tag{51}$$

Proof. The inequalities (A_5)–(D_5) can be proved by choosing of the parameters in the Young inequality

$$\begin{aligned}
(A_5)\quad &x=\varphi_2(\tau)-f(\tau),\qquad y=\psi_2(\rho)-g(\rho),\\
(B_5)\quad &x=(\varphi_2(\tau)-f(\tau))(\psi_2(\rho)-g(\rho)),\\
&y=(\psi_2(\tau)-g(\tau))(\varphi_2(\rho)-f(\rho)),\\
(C_5)\quad &x=f(\tau)-\varphi_1(\tau),\qquad y=g(\rho)-\psi_1(\rho),\\
(D_5)\quad &x=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho)),\\
&y=(g(\tau)-\psi_1(\tau))(f(\rho)-\varphi_1(\rho)).
\end{aligned}\tag{52}$$

□

Theorem 21. *Let f and g be two integrable functions on $[1,\infty)$ and $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. Suppose that (H_1) and (H_2) hold. Then, for $t>1$, $\alpha,\beta>0$, the following inequalities hold:*

$$\begin{aligned}
(A_6)\quad & \theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha\varphi_2(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta\psi_2(t)\\
&\geq {}_HJ^\alpha(\varphi_2-f)^{\theta_1}(t)\,{}_HJ^\beta(\psi_2-g)^{\theta_2}(t)\\
&\quad+\theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha f(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta g(t),\\
(B_6)\quad & \theta_1\,{}_HJ^\alpha\varphi_2(t)\,{}_HJ^\beta\psi_2(t)+\theta_1\,{}_HJ^\alpha f(t)\,{}_HJ^\beta g(t)\\
&\quad+\theta_2\,{}_HJ^\alpha\psi_2(t)\,{}_HJ^\beta\varphi_2(t)+\theta_2\,{}_HJ^\alpha g(t)\,{}_HJ^\beta f(t)\\
&\geq {}_HJ^\alpha(\varphi_2-f)^{\theta_1}(\psi_2-g)^{\theta_2}(t)\,{}_HJ^\beta(\psi_2-g)^{\theta_1}\\
&\quad\times(\varphi_2-f)^{\theta_2}(t)\\
&\quad+\theta_1\,{}_HJ^\alpha\varphi_2(t)\,{}_HJ^\beta g(t)+\theta_1\,{}_HJ^\alpha f(t)\,{}_HJ^\beta\psi_2(t)\\
&\quad+\theta_2\,{}_HJ^\alpha\psi_2(t)\,{}_HJ^\beta f(t)+\theta_2\,{}_HJ^\alpha g(t)\,{}_HJ^\beta\varphi_2(t),\\
(C_6)\quad & \theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha f(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta g(t)\\
&\geq {}_HJ^\alpha(f-\varphi_1)^{\theta_1}(t)\,{}_HJ^\beta(g-\psi_1)^{\theta_2}(t)\\
&\quad+\theta_1\frac{(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha\varphi_1(t)+\theta_2\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta\psi_1(t),\\
(D_6)\quad & \theta_1\,{}_HJ^\alpha f(t)\,{}_HJ^\beta g(t)+\theta_1\,{}_HJ^\alpha\varphi_1(t)\,{}_HJ^\beta\psi_1(t)\\
&\quad+\theta_2\,{}_HJ^\alpha g(t)\,{}_HJ^\beta f(t)+\theta_2\,{}_HJ^\alpha\psi_1(t)\,{}_HJ^\beta\varphi_1(t)\\
&\geq {}_HJ^\alpha(f-\varphi_1)^{\theta_1}(g-\psi_1)^{\theta_2}(t)\,{}_HJ^\beta(g-\psi_1)^{\theta_1}\\
&\quad\times(f-\varphi_1)^{\theta_2}(t)\\
&\quad+\theta_1\,{}_HJ^\alpha f(t)\,{}_HJ^\beta\psi_1(t)+\theta_1\,{}_HJ^\alpha\varphi_1(t)\,{}_HJ^\beta g(t)\\
&\quad+\theta_2\,{}_HJ^\alpha g(t)\,{}_HJ^\beta\varphi_1(t)+\theta_2\,{}_HJ^\alpha\psi_1(t)\,{}_HJ^\beta f(t).
\end{aligned}\tag{53}$$

Proof. The inequalities (A_6)–(D_6) can be proved by choosing of the parameters in the weighted AM-GM:

$$\begin{aligned}
(A_6)\quad & x=\varphi_2(\tau)-f(\tau),\qquad y=\psi_2(\rho)-g(\rho),\\
(B_6)\quad & x=(\varphi_2(\tau)-f(\tau))(\psi_2(\rho)-g(\rho)),\\
& y=(\psi_2(\tau)-g(\tau))(\varphi_2(\rho)-f(\rho)),\\
(C_6)\quad & x=f(\tau)-\varphi_1(\tau),\qquad y=g(\rho)-\psi_1(\rho),\\
(D_6)\quad & x=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho)),\\
& y=(g(\tau)-\psi_1(\tau))(f(\rho)-\varphi_1(\rho)).
\end{aligned}\tag{54}$$

□

Theorem 22. *Let f and g be two integrable functions on $[1,\infty)$ and constants $p\geq q\geq 0$, $p\neq 0$. Assume that (H_1) and (H_2) hold. Then, for any $k>0$, $t>1$, $\alpha,\beta>0$, the following inequalities hold:*

$$\begin{aligned}
(A_7)\quad & {}_HJ^\alpha(\varphi_2-f)^{q/p}(\psi_2-g)^{q/p}(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_2 g(t)+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha f\psi_2(t)\\
&\leq\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_2\psi_2(t)+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha fg(t)\\
&\quad+\frac{p-q}{p}k^{q/p}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)},\\
(B_7)\quad & {}_HJ^\alpha(\varphi_2-f)^{q/p}(t)\,{}_HJ^\beta(\psi_2-g)^{q/p}(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_2(t)\,{}_HJ^\beta g(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha f(t)\,{}_HJ^\beta\psi_2(t)\\
&\leq\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_2(t)\,{}_HJ^\beta\psi_2(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha f(t)\,{}_HJ^\beta g(t)\\
&\quad+\frac{p-q}{p}k^{q/p}\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)},\\
(C_7)\quad & {}_HJ^\alpha(f-\varphi_1)^{q/p}(g-\psi_1)^{q/p}(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha f\psi_1(t)+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_1 g(t)\\
&\leq\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha fg(t)+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_1\psi_1(t)\\
&\quad+\frac{p-q}{p}k^{q/p}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)},\\
(D_7)\quad & {}_HJ^\alpha(f-\varphi_1)^{q/p}(t)\,{}_HJ^\beta(g-\psi_1)^{q/p}(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha f(t)\,{}_HJ^\beta\psi_1(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_1(t)\,{}_HJ^\beta g(t)\\
&\leq\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha f(t)\,{}_HJ^\beta g(t)\\
&\quad+\frac{q}{p}k^{(q-p)/p}\,{}_HJ^\alpha\varphi_1(t)\,{}_HJ^\beta\psi_1(t)\\
&\quad+\frac{p-q}{p}k^{q/p}\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.
\end{aligned}\tag{55}$$

Proof. The inequalities (A_7)–(D_7) can be proved by choosing of the parameters in Lemma 14:

$$\begin{aligned}
&(A_7) \quad a = (\varphi_2(\tau) - f(\tau))(\psi_2(\tau) - g(\tau)),\\
&(B_7) \quad a = (\varphi_2(\tau) - f(\tau))(\psi_2(\rho) - g(\rho)),\\
&(C_7) \quad a = (f(\tau) - \varphi_1(\tau))(g(\tau) - \psi_1(\tau)),\\
&(D_7) \quad a = (f(\tau) - \varphi_1(\tau))(g(\rho) - \psi_1(\rho)).
\end{aligned} \tag{56}$$

□

Lemma 23. *Let f be an integrable function on $[1,\infty)$ and φ_1, φ_2 are two integrable functions on $[1,\infty)$. Assume that the condition (H_1) holds. Then, for $t > 1$, $\alpha > 0$, one has*

$$\begin{aligned}
&\frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha f^2(t) - ({}_HJ^\alpha f(t))^2\\
&\quad = ({}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha f(t))({}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t))\\
&\qquad - \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha ((\varphi_2 - f)(f - \varphi_1)(t))\\
&\qquad + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_1 f(t) - {}_HJ^\alpha \varphi_1(t)\, {}_HJ^\alpha f(t)\\
&\qquad + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_2 f(t) - {}_HJ^\alpha \varphi_2(t)\, {}_HJ^\alpha f(t)\\
&\qquad + {}_HJ^\alpha \varphi_1(t)\, {}_HJ^\alpha \varphi_2(t) - \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_1\varphi_2(t).
\end{aligned} \tag{57}$$

Proof. For any $\tau > 1$ and $\rho > 1$, we have

$$\begin{aligned}
&(\varphi_2(\rho) - f(\rho))(f(\tau) - \varphi_1(\tau))\\
&\qquad + (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho))\\
&\qquad - (\varphi_2(\tau) - f(\tau))(f(\tau) - \varphi_1(\tau))\\
&\qquad - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho))\\
&\quad = f^2(\tau) + f^2(\rho) - 2f(\tau) f(\rho) + \varphi_2(\rho) f(\tau)\\
&\qquad + \varphi_1(\tau) f(\rho) - \varphi_1(\tau)\varphi_2(\rho)\\
&\qquad + \varphi_2(\tau) f(\rho) + \varphi_1(\rho) f(\tau) - \varphi_1(\rho)\varphi_2(\tau)\\
&\qquad - \varphi_2(\tau) f(\tau) + \varphi_1(\tau)\varphi_2(\tau)\\
&\qquad - \varphi_1(\tau) f(\tau) - \varphi_2(\rho) f(\rho)\\
&\qquad + \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho) f(\rho).
\end{aligned} \tag{58}$$

Multiplying (58) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (1,t)$, $t > 1$, and integrating the resulting identity with respect to τ from 1 to t, we get

$$\begin{aligned}
&(\varphi_2(\rho) - f(\rho))({}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t))\\
&\qquad + ({}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha f(t))(f(\rho) - \varphi_1(\rho))\\
&\qquad - {}_HJ^\alpha ((\varphi_2 - f)(f - \varphi_1)(t)) - (\varphi_2(\rho) - f(\rho))\\
&\qquad \times (f(\rho) - \varphi_1(\rho)) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}\\
&\quad = {}_HJ^\alpha f^2(t) + f^2(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}\\
&\qquad - 2f(\rho)\, {}_HJ^\alpha f(t) + \varphi_2(\rho)\, {}_HJ^\alpha f(t)\\
&\qquad + f(\rho)\, {}_HJ^\alpha \varphi_1(t) - \varphi_2(\rho)\, {}_HJ^\alpha \varphi_1(t)\\
&\qquad + f(\rho)\, {}_HJ^\alpha \varphi_2(t) + \varphi_1(\rho)\, {}_HJ^\alpha f(t)\\
&\qquad - \varphi_1(\rho)\, {}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha \varphi_2 f(t) + {}_HJ^\alpha \varphi_1\varphi_2(t)\\
&\qquad - {}_HJ^\alpha \varphi_1 f(t) - \varphi_2(\rho) f(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}\\
&\qquad + \varphi_1(\rho)\varphi_2(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} - \varphi_1(\rho) f(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned} \tag{59}$$

Multiplying (59) by $(\log(t/\rho))^{\alpha-1}/\rho\Gamma(\alpha)$, $\rho \in (1,t)$, $t > 1$, and integrating the resulting identity with respect to ρ from 1 to t, we have

$$\begin{aligned}
&({}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha f(t))({}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t))\\
&\qquad + ({}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha f(t))({}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t))\\
&\qquad - {}_HJ^\alpha ((\varphi_2 - f)(f - \varphi_1)(t)) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}\\
&\qquad - {}_HJ^\alpha ((\varphi_2 - f)(f - \varphi_1)(t)) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}\\
&\quad = \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha f^2(t) + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha f^2(t)\\
&\qquad - 2\, {}_HJ^\alpha f(t)\, {}_HJ^\alpha f(t) + {}_HJ^\alpha \varphi_2(t)\, {}_HJ^\alpha f(t)\\
&\qquad + {}_HJ^\alpha \varphi_1(t)\, {}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t)\, {}_HJ^\alpha \varphi_2(t)\\
&\qquad + {}_HJ^\alpha \varphi_2(t)\, {}_HJ^\alpha f(t)\\
&\qquad + {}_HJ^\alpha \varphi_1(t)\, {}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t)\, {}_HJ^\alpha \varphi_2(t)\\
&\qquad - \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_2 f(t)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\varphi_1\varphi_2(t)-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\varphi_1 f(t)\\
&-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\varphi_2 f(t)\\
&+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\varphi_1\varphi_2(t)-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\varphi_1 f(t),
\end{aligned}
\tag{60}
$$

which implies (57). □

Corollary 24. *Let f be an integrable function on $[1,\infty)$ satisfying $m \le f(t) \le M$, for all $t \in [1,\infty)$. Then for all $t > 1$, $\alpha > 0$ one has*

$$
\begin{aligned}
&\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\,{}_{H}J^{\alpha}f^2(t)-\left({}_{H}J^{\alpha}f(t)\right)^2\\
&=\left(M\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}-{}_{H}J^{\alpha}f(t)\right)\left({}_{H}J^{\alpha}f(t)-m\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right)\\
&\quad-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\left(\left(M-f(t)\right)\left(f(t)-m\right)\right).
\end{aligned}
\tag{61}
$$

Theorem 25. *Let f and g be two integrable functions on $[1,\infty)$ and φ_1, φ_2, ψ_1, and ψ_2 are four integrable functions on $[1,\infty)$ satisfying the conditions (H_1) and (H_2) on $[1,\infty)$. Then for all $t > 1$, $\alpha > 0$, one has*

$$
\begin{aligned}
&\left|\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}fg(t)-{}_{H}J^{\alpha}f(t)\,{}_{H}J^{\alpha}g(t)\right|\\
&\qquad\le\left|T(f,\varphi_1,\varphi_2)\right|^{1/2}\left|T(g,\psi_1,\psi_2)\right|^{1/2},
\end{aligned}
\tag{62}
$$

where $T(u,v,w)$ is defined by

$$
\begin{aligned}
&T(u,v,w)\\
&=\left({}_{H}J^{\alpha}w(t)-{}_{H}J^{\alpha}u(t)\right)\left({}_{H}J^{\alpha}u(t)-{}_{H}J^{\alpha}v(t)\right)\\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}vu(t)-{}_{H}J^{\alpha}v(t)\,{}_{H}J^{\alpha}u(t)\\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}wu(t)-{}_{H}J^{\alpha}w(t)\,{}_{H}J^{\alpha}u(t)\\
&\quad+{}_{H}J^{\alpha}v(t)\,{}_{H}J^{\alpha}w(t)-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}vw(t).
\end{aligned}
\tag{63}
$$

Proof. Let f and g be two integrable functions defined on $[1,\infty)$ satisfying (H_1) and (H_2). Define

$$
\begin{aligned}
H(\tau,\rho) &:= \left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right),\\
&\qquad \tau,\rho\in(1,t),\ t>1.
\end{aligned}
\tag{64}
$$

Multiplying both sides of (64) by $(\log(t/\tau))^{\alpha-1}(\log(t/\rho))^{\alpha-1}/\tau\rho\Gamma^2(\alpha)$, $\tau,\rho \in (1,t)$, and integrating the resulting identity with respect to τ and ρ from 1 to t, we can state that

$$
\begin{aligned}
&\frac{1}{2\Gamma^2(\alpha)}\iint_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}H(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}\\
&\quad=\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}fg(t)-{}_{H}J^{\alpha}f(t)\,{}_{H}J^{\alpha}g(t).
\end{aligned}
\tag{65}
$$

Applying the Cauchy-Schwarz inequality to (65), we have

$$
\begin{aligned}
&\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}fg(t)-{}_{H}J^{\alpha}f(t)\,{}_{H}J^{\alpha}g(t)\right)^2\\
&=\left(\frac{1}{2\Gamma^2(\alpha)}\iint_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\right.\\
&\qquad\left.\times H(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right)^2\\
&\le\left(\frac{1}{2\Gamma^2(\alpha)}\iint_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\right.\\
&\qquad\left.\times\left(f(\tau)-f(\rho)\right)^2\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right)\\
&\quad\times\left(\frac{1}{2\Gamma^2(\alpha)}\iint_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\right.\\
&\qquad\left.\times\left(g(\tau)-g(\rho)\right)^2\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right)\\
&=\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}f^2(t)-\left({}_{H}J^{\alpha}f(t)\right)^2\right)\\
&\quad\times\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}g^2(t)-\left({}_{H}J^{\alpha}g(t)\right)^2\right).
\end{aligned}
\tag{66}
$$

Since $(\varphi_2(t)-f(t))(f(t)-\varphi_1(t)) \ge 0$ and $(\psi_2(t)-g(t))(g(t)-\psi_1(t)) \ge 0$ for $t \in [1,\infty)$, we have

$$
\begin{aligned}
&\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\left(\varphi_2-f\right)\left(f-\varphi_1\right)(t)\ge 0,\\
&\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\left(\psi_2-g\right)\left(g-\psi_1\right)(t)\ge 0.
\end{aligned}
\tag{67}
$$

Thus, from Lemma 23, we get

$$\begin{aligned}
&\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}f^2(t)-\left({}_HJ^{\alpha}f(t)\right)^2\\
&\le\left({}_HJ^{\alpha}\varphi_2(t)-{}_HJ^{\alpha}f(t)\right)\left({}_HJ^{\alpha}f(t)-{}_HJ^{\alpha}\varphi_1(t)\right)\\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_1f(t)-{}_HJ^{\alpha}\varphi_1(t)\,{}_HJ^{\alpha}f(t)\\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_2f(t)-{}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\alpha}f(t)\\
&\quad+{}_HJ^{\alpha}\varphi_1(t)\,{}_HJ^{\alpha}\varphi_2(t)-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_1\varphi_2(t)\\
&=T\left(f,\varphi_1,\varphi_2\right),\\
&\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}g^2(t)-\left({}_HJ^{\alpha}g(t)\right)^2\\
&\quad\le\left({}_HJ^{\alpha}\psi_2(t)-{}_HJ^{\alpha}g(t)\right)\\
&\quad\quad\times\left({}_HJ^{\alpha}g(t)-{}_HJ^{\alpha}\psi_1(t)\right)\\
&\quad\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\psi_1g(t)\\
&\quad\quad-{}_HJ^{\alpha}\psi_1(t)\,{}_HJ^{\alpha}g(t)\\
&\quad\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\psi_2g(t)\\
&\quad\quad-{}_HJ^{\alpha}\psi_2(t)\,{}_HJ^{\alpha}g(t)\\
&\quad\quad+{}_HJ^{\alpha}\psi_1(t)\,{}_HJ^{\alpha}\psi_2(t)\\
&\quad\quad-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\psi_1\psi_2(t)\\
&\quad=T\left(g,\psi_1,\psi_2\right).
\end{aligned}\tag{68}$$

From (66) and (68), we obtain (62). □

Remark 26. If $T(f,\varphi_1,\varphi_2)=T(f,m,M)$ and $T(g,\psi_1,\psi_2)=T(g,p,P)$, $m,M,p,P\in\mathbb{R}$, then inequality (62) reduces to the following Grüss type Hadamard fractional integral inequality:

$$\begin{aligned}
&\left|\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}fg(t)-{}_HJ^{\alpha}f(t)\,{}_HJ^{\alpha}g(t)\right|\\
&\quad\le\left(\frac{1}{2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right)^2(M-m)(P-p).
\end{aligned}\tag{69}$$

Example 27. Let f and g be two functions satisfying $\log t\le f(t)\le 1+\log t$ and $-1+\log t\le g(t)\le\log t$ for $t\in[1,\infty)$. Then for $t>1$ and $\alpha>0$, we have

$$\begin{aligned}
&\left|\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}fg(t)-{}_HJ^{\alpha}f(t)\,{}_HJ^{\alpha}g(t)\right|\\
&\quad\le\left|T\left(f,\log t,1+\log t\right)\right|^{1/2}\left|T\left(g,-1+\log t,\log t\right)\right|^{1/2},
\end{aligned}\tag{70}$$

where

$$\begin{aligned}
&T\left(f,\log t,1+\log t\right)\\
&\quad=\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}-{}_HJ^{\alpha}f(t)\right)\\
&\quad\times\left({}_HJ^{\alpha}f(t)-\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\left(f\log t\right)(t)\\
&\quad-\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}{}_HJ^{\alpha}f(t)\\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\left(\left(1+\log t\right)f\right)(t)\\
&\quad-\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right){}_HJ^{\alpha}f(t)\\
&\quad+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\\
&\quad-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{2(\log t)^{\alpha+2}}{\Gamma(\alpha+3)}\right),\\
&T\left(g,-1+\log t,\log t\right)\\
&\quad=\left(\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}-{}_HJ^{\alpha}g(t)\right)\\
&\quad\quad\times\left({}_HJ^{\alpha}g(t)+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}-\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\\
&\quad\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\left(\left(-1+\log t\right)g\right)(t)\\
&\quad\quad+\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}-\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right){}_HJ^{\alpha}g(t)\\
&\quad\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\left(g\log t\right)(t)-\left(\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\right){}_HJ^{\alpha}g(t)
\end{aligned}$$

$$- \left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)} \right) \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} \left(\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{2(\log t)^{\alpha+2}}{\Gamma(\alpha+3)} \right). \tag{71}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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