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Research Article

Constants within Error Estimates for Legendre-Galerkin Spectral Approximations of Control-Constrained Optimal Control Problems

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Explicit formulae of constants within the a *posteriori* error estimate for optimal control problems are investigated with Legendre-Galerkin spectral methods. The constrained set is put on the control variable. For simpleness, one-dimensional bounded domain is taken. Meanwhile, the corresponding a *posteriori* error indicator is established with explicit constants.

1. Introduction

Recently, spectral method has been extended to approximate the discretization of partial differential equations for design optimization, engineering design, and other engineering computations. It provides higher accurate approximations with a relatively small number of unknowns if the solution is smooth; see [1]. There have been extensive researches on finite element methods for optimal control problems, which focus on control-constrained problems; see [2–8]. The authors [9] studied state-constrained optimal control problems with finite element methods. However, there are few works on optimal control problems with spectral methods.

In order to get a numerical solution with acceptable accuracy, spectral methods only increase the degree of basis where the error indicator is larger than the *a posteriori* error indicator, while the finite element methods refine meshes (see [10]). There have been lots of papers concerning on *a posteriori* error estimates for h-version finite element methods, but not for spectral methods. Guo [11] got a reliable and efficient error indicator for p-version finite element method in one dimension with a certain weight. Zhou and Yang [12] deduced a simple error indicator for spectral Galerkin methods. In [13], the authors investigated Legendre-Galerkin spectral method for optimal control problems with integral constraint for state in one-dimensional bounded domain. It is difficult to obtain optimal *a posteriori* error estimates. Thus, if one gets the constants within upper bound *a posteriori* error estimates, it is easy to ensure the degree of polynomials to get an acceptable accuracy.

In this paper, the control-constrained optimal control problems are solved with Legendre-Galerkin spectral methods, and constants within upper bound of the *a posteriori* error indicator, which can be used to decide the least unknowns for acceptable accuracy, are proposed. By introducing auxiliary systems, explicit formulae of the constants within the *a posteriori* error estimates are obtained.

The outline of this paper is as follows. In Section 2, the model problem and its Legendre-Galerkin spectral approximations are listed. In Section 3, the constants within the *a posteriori* error estimates are investigated in details, and the explicit formulae are obtained. The conclusions are given in Section 4.

2. A Model Problem and Its Legendre-Galerkin Spectral Approximations

Throughout this paper, we focus on $I = (-1, 1)$ and adopt the standard notations $W^{m,p}$ for Sobolev spaces with the norm $\|\cdot\|_{W^{m,p}}$ and the seminorm $|\cdot|_{W^{m,p}}$; see [14]. Specially, we set $W_0^{m,p} = \{w \in W^{m,p} : w|_{\partial I} = 0\}$. If $p = 2$, we denote $W^{m,2}$ and $W_0^{m,2}$ by H^m and H_0^1, respectively.

The problem in which we are interest is the following distributed convex optimal control problem with integral constraint on the control variable:

$$\min_{u\in K} \quad J(u,y)=\frac{1}{2}\int_I (y-y_d)^2+\frac{\alpha}{2}\int_I u^2, \tag{1}$$

$$\text{subject to} \quad -y''=f+u \quad \text{in } I, \qquad y|_{\partial I}=0, \tag{2}$$

where $K=\{w\in L^2(I):\int_I w\geq 0\}$, and the control variable $u\in U=L^2(I)$, the state variable $y\in V=H_0^1(I)$, and $y_d\in L^2(I)$ is the observation.

In order to assure existence and regularity of the solution, we assume that f and y_d are infinitely smooth functions; α is a given positive constant, for simplicity, we set $\alpha=1$. It is well-known that (1) has a unique solution (see [5, 15]).

Now, we introduce the weak formula of (1). We give some basic notations which will be used in the sequel. Let

$$(v,w)=\int_I vw, \quad \forall v,w\in L^2(I), \qquad a(v,w)=\int_I v'w', \quad \forall v,w\in H_0^1(I). \tag{3}$$

Hence, the state equation (2) reduces to

$$a(y,w)=(f+u,w), \quad \forall w\in H_0^1(I). \tag{4}$$

Then, (1) can be rewritten as follows: find (u,y) such that

$$(\mathscr{P})\begin{cases}\min\limits_{u\in K} & J(u,y)=\frac{1}{2}\int_I (y-y_d)^2+\frac{1}{2}\int_I u^2,\\ \text{s.t.} & a(y(u),w)=(f+u,w), \quad \forall w\in V.\end{cases} \tag{5}$$

We recall following optimality conditions of the optimal control problem (for the details, please refer to [8, 15]): (1) has a unique solution (y,u). Meanwhile, (y,u) is the solution of (1) if and only if there is a costate $p\in V$ such that the triplet (y,p,u) satisfies the following optimal conditions:

$$\begin{aligned} a(y,w)&=(f+u,w), \quad \forall w\in V,\\ a(q,p)&=(y-y_d,q), \quad \forall q\in V,\\ (u+p,v-u)&\geq 0, \quad \forall v\in K\subset U. \end{aligned} \tag{6}$$

Let $\mathscr{P}_N(I)=\{\text{polynomials of degree} \leqslant N \text{ on } I\}$ and let $V_N=\mathscr{P}_N\cap H_0^1(I)$. One may expand the discrete polynomial spaces as

$$V_N=\operatorname{span}\{\phi_1(x),\phi_2(x),\ldots,\phi_N(x)\}\subset V, \qquad U_N=\mathscr{P}_N(I)\cap U, \qquad K_N=\mathscr{P}_N(I)\cap K. \tag{7}$$

One prefers to choose appropriate bases of V_N such that the resulting linear system is as simple as possible. Following [16], we choose the basis functions as

$$\phi_i(x)=c_i(L_{i-1}(x)-L_{i+1}(x)), \quad c_i=\frac{1}{\sqrt{4i+2}}, \qquad i=1,2,\ldots,N, \tag{8}$$

where $L_r(x)$ denotes the r-th degree Legendre polynomial. Then, Galerkin spectral approximations of (5) read as follows: find (u_N,y_N) such that

$$(\mathscr{P}^N)\begin{cases}\min\limits_{u_N\in K\subset U_N} & J(u_N,y_N)=\frac{1}{2}\int_I (y_N-y_d)^2+\frac{1}{2}\int_I u_N^2,\\ \text{s.t.} & a(y_N,w_N)=(f+u_N,w_N), \quad \forall w_N\in V_N.\end{cases} \tag{9}$$

It is obvious that (9) has a solution(y_N,u_N) and (y_N,u_N) is the solution if and only if there is a costate $p_N\in V_N$ satisfies the triplet (y_N,p_N,u_N) such that

$$\begin{aligned} a(y_N,w_N)&=(f+u_N,w_N), \quad \forall w_N\in V_N,\\ a(q_N,p_N)&=(y_N-y_d,q_N), \quad \forall q_N\in V_N,\\ (u_N+p_N,v_N-u_N)&\geq 0, \quad \forall v_N\in K_N. \end{aligned} \tag{10}$$

Now, we are at the point to analyse the relationship between the optimal control and costate, which reads as follows:

$$u=\max\{0,\overline{p}\}-p, \tag{11}$$

where $\overline{p}$ denotes the integral average on I of the costate p (see [2]). Thus, for Galerkin spectral approximations, it follows that there holds

$$u_N=\max\{0,\overline{p}_N\}-p_N. \tag{12}$$

Let

$$J(u)=\frac{1}{2}\int_I (y-y_d)^2+\frac{1}{2}\int_I u^2, \qquad J_N(u_N)=\frac{1}{2}\int_I (y_N-y_d)^2+\frac{1}{2}\int_I u_N^2. \tag{13}$$

It is clear that $J(\cdot)$ is uniformly convex. Then, there exits a $c_0>0$ independent of N, such that

$$\left(J'(u)-J'(u_N),u-u_N\right)\geq c_0\|u-u_N\|_{0,I}^2. \tag{14}$$

3. Constants within the *a Posteriori* Error Estimates

In this section, we calculate all constants within the *a posteriori* error estimates. Firstly, we analyze the constant in Poincaré inequality.

For $I=(-1,1)$, we recall the Poincaré inequality with L^2-norm as (see [17])

$$\|v\|_{0,I}\leq\frac{|I|}{2}\|v'\|_{0,I}. \tag{15}$$

Now, we are at the point to investigate all of constants in details. We introduce an auxiliary state $y(u_N)\in H_0^1(I)$, which satisfies

$$a(y(u_N),w)=(f+u_N,w), \quad \forall w\in H_0^1(I). \tag{16}$$

Subtracting (16) from (5), we get

$$a(y-y(u_N),w)=(u-u_N,w), \quad \forall w\in H_0^1(I). \tag{17}$$

Let $w = y(u_N) - y \in H_0^1(\Omega)$. It is clear that

$$a(y(u_N) - y, y(u_N) - y) = (u_N - u, y(u_N) - y), \quad (18)$$

and then there hold

$$\begin{aligned}\|(y(u_N) - y)'\|_{0,I}^2 &\le \|u_N - u\|_{0,I}\|(y(u_N) - y)'\|_{0,I} \\ &\le \frac{|I|}{2}\|u_N - u\|_{0,I}\|(y(u_N) - y)'\|_{0,I},\end{aligned} \quad (19)$$

which means that

$$\|(y(u_N) - y)'\|_{0,I} \le \frac{|I|}{2}\|u_N - u\|_{0,I}. \quad (20)$$

Hence,

$$\begin{aligned}&\|y(u_N) - y\|_{1,I} \\ &\le \left(\|(y(u_N) - y)'\|_{0,I}^2 + \left(\frac{|I|}{2}\right)^2\|(y(u_N) - y)'\|_{0,I}^2\right)^{1/2} \\ &= \left(1 + \left(\frac{|I|}{2}\right)^2\right)^{1/2}\|(y(u_N) - y)'\|_{0,I}.\end{aligned} \quad (21)$$

So, we can easily obtain that

$$\|y(u_N) - y\|_{1,I} \le \left(1 + \left(\frac{|I|}{2}\right)^2\right)^{1/2}\frac{|I|}{2}\|u_N - u\|_{0,I}. \quad (22)$$

We denote by c_1 the constant in (22), and then

$$c_1 = \left(1 + \left(\frac{|I|}{2}\right)^2\right)^{1/2}\frac{|I|}{2}. \quad (23)$$

Here, we recall the following orthogonal projection operator: for any $v \in L^2(I)$, $\mathbb{P}_N : L^2(I) \mapsto V_N$ satisfies:

$$(\mathbb{P}_N v - v, w_N) = 0 \quad \forall w_N \in V_N. \quad (24)$$

Lemma 1. *For all $v \in H^\sigma(I)$ $(\sigma \ge 0)$, one has*

$$\|\mathbb{P}_N v - v\|_{0,I} \le c_2 N^{-\sigma}\|v\|_{\sigma,I}, \quad (25)$$

where $c_2 = 2\sqrt{2}$.

We denote by $y(u_N)$ and $p(u_N)$ two intermediate variables, and there hold

$$\begin{aligned}(J'(u), v) &= (u + p, v), \\ (J_N'(u_N), v) &= (u_N + p_N, v), \\ (J'(u_N), v) &= (u_N + p(u_N), v).\end{aligned} \quad (26)$$

Using (6), (10) and (14), for $\forall v_N = \mathbb{P}_N v$, we have

$$\begin{aligned}&c_0\|u - u_N\|_{0,I} \\ &\le (J'(u) - J'(u_N), u - u_N) \\ &\le -(J'(u_N), u - u_N) \\ &= (J_N'(u_N), u_N - u) + (J_N'(u_N) - J'(u_N), u - u_N) \\ &\le (J_N'(u_N), v_N - u) + (J_N'(u_N) - J'(u_N), u - u_N) \\ &= (J_N'(u_N) - J'(u_N), u - u_N) = (p_N - p(u_N), u - u_N) \\ &\le \|p_N - p(u_N)\|_{0,I}\|u - u_N\|_{0,I},\end{aligned} \quad (27)$$

which means that

$$\|u - u_N\|_{0,I} \le \frac{1}{c_0}\|p_N - p(u_N)\|_{0,I}. \quad (28)$$

Now, we are at the point to derive the constant for $\|y_N - y(u_N)\|_{1,I}$. Let $E^y = y_N - y(u_N)$ and $E_I^y = \mathbb{P}_N E^y \in V_N$. Then

$$\begin{aligned}&\|y_N - y(u_N)\|_{1,I}^2 \\ &= \|E^y\|_{1,I}^2 \le \left(1 + \left(\frac{|I|}{2}\right)^2\right) a(E^y, E^y) \\ &= \left(1 + \left(\frac{|I|}{2}\right)^2\right) a(E^y - E_I^y, E^y) \\ &= \left(1 + \left(\frac{|I|}{2}\right)^2\right)(f + u_N + y_N'', E^y - E_I^y) \\ &\le \left(1 + \left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\|f + u_N + y_N''\|_{0,I} \cdot \|E^y\|_{1,I},\end{aligned} \quad (29)$$

which is equivalent to

$$\|y_N - y(u_N)\|_{1,I} \le \left(1 + \left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\|f + u_N + y_N''\|_{0,I}. \quad (30)$$

Hence,

$$\|y_N - y(u_N)\|_{1,I} \le c_3 N^{-1}\|f + u_N + y_N''\|_{0,I}, \quad (31)$$

where

$$c_3 = \left(1 + \left(\frac{|I|}{2}\right)^2\right) c_2. \quad (32)$$

Likewise, we derive the constant for $\|p_N - p(u_N)\|_{1,I}$. Similarly, let $E^p = p_N - p(u_N)$ and $E_I^p = \mathbb{P}_N E^p \in V_N$. Then

$$\begin{aligned}
\|p_N - p(u_N)\|_{1,I}^2 &= \|E^p\|_{1,I}^2 \leq \left(1 + \left(\frac{|I|}{2}\right)^2\right) a(E^p, E^p) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(a\left(E^p, E^p - E_I^p\right) + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(a\left(p(u_N) - p_N, E^p - E_I^p\right) + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(\left(-p''(u_N), E^p - E_I^p\right) + \left(p_N'', E^p - E_I^p\right) + \left(y(u_N) - y_N, E_I^p\right)\right) \\
&= \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left(\left(y_N - y_d + p_N'', E^p - E_I^p\right) + \left(y(u_N) - y_N, E^p\right)\right) \\
&\leq \left(1 + \left(\frac{|I|}{2}\right)^2\right)\|E^p\|_{1,I}\left\{c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I} + \|y_N - y(u_N)\|_{0,I}\right\}.
\end{aligned} \tag{33}$$

We deduce that

$$\|p_N - p(u_N)\|_{1,I} \leq \left(1 + \left(\frac{|I|}{2}\right)^2\right)\left\{c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I} + \|y_N - y(u_N)\|_{0,I}\right\}. \tag{34}$$

Combining all of the above analyses, we derive that

$$\begin{aligned}
&\|u - u_N\|_{0,I} + \|y - y_N\|_{1,I} + \|p - p_N\|_{1,I} \\
&\leq \|u - u_N\|_{0,I} + \|y - y(u_N)\|_{1,I} + \|y_N - y(u_N)\|_{1,I} + \|p - p(u_N)\|_{1,I} + \|p_N - p(u_N)\|_{1,I} \\
&= \|u - u_N\|_{0,I} + \|y_N - y(u_N)\|_{1,I} + \|p_N - p(u_N)\|_{1,I} + \|y - y(u_N)\|_{1,I} + \|p - p(u_N)\|_{1,I} \\
&\leq \|u - u_N\|_{0,I} + \|y_N - y(u_N)\|_{1,I} + \|p_N - p(u_N)\|_{1,I} + \|y - y(u_N)\|_{1,I} + c_1\|y - y(u_N)\|_{0,I} \\
&\leq \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I} \\
&\quad + \left(1 + \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right)\right) c_3 N^{-1} \times \|f + u_N + y_N''\|_{0,I},
\end{aligned} \tag{35}$$

which means that

$$\begin{aligned}
&\|u - u_N\|_{0,I} + \|p - p_N\|_{1,I} + \|y - y_N\|_{1,I} \\
&\leq \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right) c_2 N^{-1}\|y_N - y_d + p_N''\|_{0,I} \\
&\quad + \left(1 + \left(\frac{1 + c_1 + c_1^2}{c_0} + 1\right)\left(1 + \left(\frac{|I|}{2}\right)^2\right)\right) c_3 N^{-1} \times \|f + u_N + y_N''\|_{0,I}.
\end{aligned} \tag{36}$$

For $|I| = 2$, there holds

$$\|u - u_N\|_{0,I} + \|p - p_N\|_{1,I} + \|y - y_N\|_{1,I} \leq \eta, \tag{37}$$

where the *a posteriori* error indicator η is defined as

$$\begin{aligned}
\eta = {} & 4\sqrt{2}\left(1 + \frac{3 + \sqrt{2}}{c_0}\right) N^{-1}\|y_N - y_d + p_N''\|_{0,I} \\
& + 4\sqrt{2}\left(3 + \frac{6 + 2\sqrt{2}}{c_0}\right) N^{-1}\|f + u_N + y_N''\|_{0,I}.
\end{aligned} \tag{38}$$

4. Conclusion

This paper discussed the explicit formulae of constants in the upper bound of the *a posteriori* error estimate for optimal control problems with Legendre-Galerkin spectral methods in one-dimensional bounded domain. Thus, with those formulae, it is easy to choose a suitable degree of polynomials to obtain acceptable accuracy. In the future, we are going to discuss the corresponding constants in the lower bound of the *a posteriori* error indicator.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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