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Research Article

Stability and Hopf Bifurcation Analysis on a Bazykin Model with Delay

Jianming Zhang,[1] Lijun Zhang,[1,2] and Chaudry Masood Khalique[2]

[1] *Department of Mathematics, School of Science, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, China*
[2] *International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa*

Correspondence should be addressed to Lijun Zhang; li-jun0608@163.com

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The dynamics of a prey-predator system with a finite delay is investigated. We show that a sequence of Hopf bifurcations occurs at the positive equilibrium as the delay increases. By using the theory of normal form and center manifold, explicit expressions for determining the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions are derived.

1. Introduction

The theoretical study of predator-prey systems in mathematical ecology has a long history beginning with the famous Lotka-Volterra equations because of their universal existence and importance. One of the ecological models proposed and analyzed by Bazykin [1] is

$$\begin{aligned} \dot{x}_1 &= x_1 (1 - x_2) - \varepsilon x_1^2, \\ \dot{x}_2 &= -\gamma x_2 + \frac{x_2}{n + x_2} x_1 x_2, \end{aligned} \tag{1}$$

where ε, γ, and n are positive constants and x_1 and x_2 are functions of time representing population densities of prey and predator, respectively. This system can be used to describe the dynamics of the prey-predator system when the nonlinearity of predator reproduction and prey competitive are both taken into account. Bazykin [1] pointed out that for the system (1) the degenerate Bogdanov-Takens bifurcation exists when $\gamma = 4/3$, $n = 1/3$, and $\varepsilon = 1/4$ and conjectured that it is a nondegenerate codim 3 bifurcation. Kuznetsov [2] proved the conjecture is correct by using critical (generalized) eigenvectors of the linearized matrix and its transpose. However, time delays commonly exist in biological system, information transfer system, and so on. Therefore, time delays of one type or another have been incorporated into mathematical models of population dynamics due to maturation time, capturing time, or other reasons. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay may lead to changes of stability of equilibrium and the fluctuation of the populations. So far, a great deal of research has been devoted to the delayed predator-prey system. See, for example, the monographs of Cushing [3], Gopalsamy [4], and Kuang [5] for general delayed biological systems and Beretta and Kuang [6, 7], Faria [8], Gopalsamy [9, 10], May [11], Song et al. [12–14], Xiao and Ruan [15], and Liu and Yuan [16] and the references cited therein for studies on delayed prey-predator systems. In the above references, normal form and center manifold theory were one of important methods to study the stability and Hopf bifurcation of the delayed predator-prey systems. Considering the maturation time of the predator, Bazykin [1] becomes the following delayed model:

$$\begin{aligned} \dot{x}_1 &= x_1 (1 - x_2 (t - \tau)) - \varepsilon x_1^2, \\ \dot{x}_2 &= -\gamma x_2 + \frac{x_2}{n + x_2} x_1 x_2. \end{aligned} \tag{2}$$

In this paper, we first discuss the effect of the time τ on the stability of the positive equilibrium of the system (2). Then we investigate the existence of the Hopf bifurcation, the bifurcating direction, and the stability of the bifurcation periodic solutions by the theory of normal form and center manifold. Explicit expressions for determining the direction of the Hopf bifurcations and the stability of the bifurcation periodic solutions are derived.

2. The Existence of Hopf Bifurcations

In this section, we study the existence of the Hopf bifurcations of system (2). Clearly, when $-1 < n < 0$, system (2) has only one positive equilibrium, that is,

$$E\left(\frac{1+\gamma\varepsilon-\sqrt{(1-\gamma\varepsilon)^2-4\varepsilon\gamma n}}{2\varepsilon},\right.$$
$$\left.\frac{1-\gamma\varepsilon+\sqrt{(1-\gamma\varepsilon)^2-4\varepsilon\gamma n}}{2}\right). \tag{3}$$

Let

$$x_1^{(0)}=\frac{1+\gamma\varepsilon-\sqrt{(1-\gamma\varepsilon)^2-4\varepsilon\gamma n}}{2\varepsilon};$$
$$x_2^{(0)}=\frac{1-\gamma\varepsilon+\sqrt{(1-\gamma\varepsilon)^2-4\varepsilon\gamma n}}{2}, \tag{4}$$

then system (2) becomes

$$\begin{aligned}\dot{x}_1 &= -\varepsilon x_1^{(0)}\left(x_1-x_1^{(0)}\right)-x_1^{(0)}\left(x_2(t-\tau)-x_2^{(0)}\right)\\ &\quad -\left(x_1-x_1^{(0)}\right)\left(x_2(t-\tau)-x_2^{(0)}\right)-\varepsilon\left(x_1-x_1^{(0)}\right)^2,\\ \dot{x}_2 &= \frac{x_2^{(0)}}{n+x_2^{(0)}}\left(x_1-x_1^{(0)}\right)\\ &\quad +\left(-\gamma+\frac{2nx_1^{(0)}x_2^{(0)}+x_1^{(0)}\left(x_2^{(0)}\right)^2}{\left(n+x_2^{(0)}\right)^2}\right)\left(x_2-x_2^{(0)}\right)+\cdots.\end{aligned} \tag{5}$$

By introducing the new variables $z_1(t)=x_1(t)-x_1^{(0)}$, $z_2(t)=x_2(t)-x_2^{(0)}$ and denoting $f(x_1,x_2)=-\gamma x_2+(x_2/(n+x_2))x_1x_2$, system (5) can be rewritten in a simpler form as

$$\begin{aligned}\dot{z}_1(t) &= -\alpha_1 z_1(t)-\alpha_2 z_2(t-\tau)-z_1(t)z_2(t-\tau)-\varepsilon z_1^2(t),\\ \dot{z}_2(t) &= r_1 z_1(t)+r_2 z_2(t)+\sum\frac{1}{i!j!}c_{ij}z_1^i(t)z_2^j(t),\end{aligned} \tag{6}$$

where

$$\begin{aligned}&\alpha_1=\varepsilon x_1^{(0)},\qquad \alpha_2=x_1^{(0)},\qquad r_1=\frac{x_2^{(0)}}{n+x_2^{(0)}},\\ &r_2=-\gamma+\frac{2nx_1^{(0)}x_2^{(0)}+x_1^{(0)}\left(x_2^{(0)}\right)^2}{\left(n+x_2^{(0)}\right)^2},\end{aligned} \tag{7}$$

and $c_{ij}=\partial^{i+j}f(x_1^{(0)},x_2^{(0)})/\partial x_1^i\partial x_2^j$. Then the linearization of system (2) at E is

$$\begin{aligned}\dot{z}_1(t) &= -\alpha z_1(t)-\alpha_2 z_2(t-\tau),\\ \dot{z}_2(t) &= r_1 z_1(t)+r_2 z_2(t).\end{aligned} \tag{8}$$

The associated characteristic equation of (8) is given by

$$\begin{vmatrix}\lambda+\alpha_1 & \alpha_2 e^{-\lambda\tau}\\ -r_1 & \lambda-r_2\end{vmatrix}=0. \tag{9}$$

That is,

$$\lambda^2+(\alpha_1-r_2)\lambda-\alpha_1 r_2+\alpha_2 r_1 e^{-\lambda\tau}=0. \tag{10}$$

The equilibrium E is stable if all roots of (10) have negative real parts. Clearly, when $\tau=0$, the characteristic equation (10) becomes

$$\lambda^2+(\alpha_1-r_2)\lambda-\alpha_1 r_2+\alpha_2 r_1=0. \tag{11}$$

By directly computing, we known that $r_2<0$ when $-1<n<0$. Therefore all roots of (11) have negative real parts. Obviously, $\lambda=i\omega$ $(\omega>0)$ is a root of (10) if and only if ω satisfies

$$-\omega^2+i(\alpha_1-r_2)\omega-\alpha_1 r_2+\alpha_2 r_1 e^{-i\omega\tau}=0. \tag{12}$$

Separating the real and imaginary parts, we have

$$\begin{aligned}-\omega^2-\alpha_1 r_2+\alpha_2 r_1\cos\omega\tau &= 0,\\ (\alpha_1-r_2)\omega-\alpha_2 r_1\sin\omega\tau &= 0,\end{aligned} \tag{13}$$

which leads to

$$\omega^4+\left(\alpha_1^2+r_2^2\right)\omega^2+\alpha_1^2r_2^2-\alpha_2^2r_1^2=0. \tag{14}$$

When $\alpha_1^2r_2^2-\alpha_2^2r_1^2<0$, (14) has only one positive root

$$\omega_*=\sqrt{\frac{-\left(\alpha_1^2+r_2^2\right)+\sqrt{\left(\alpha_1^2+r_2^2\right)^2-4\left(\alpha_1^2r_2^2-\alpha_2^2r_1^2\right)}}{2}}. \tag{15}$$

Substituting (15) into (13), we obtain

$$\tau_j=\frac{1}{\omega_*}\arccos\frac{\omega_*^2+\alpha_1 r_2}{\alpha_2 r_1}+\frac{2j\pi}{\omega_*},\quad j=0,1,2,\ldots. \tag{16}$$

Thus, when $\tau=\tau_j$, the characteristic equation (10) has a pair of purely imaginary roots $\pm i\omega_*$.

Lemma 1. *Let $\lambda_j(\tau)=\eta_j(\tau)+i\omega_j(\tau)$ be the root of (10) satisfying*

$$\eta_j(\tau_j)=0,\quad \omega_j(\tau_j)=\omega_*,\quad j=0,1,2,\ldots, \tag{17}$$

and then

$$\eta_j'(\tau_j)>0. \tag{18}$$

Proof. Differentiating both sides of (10) with respect to τ, we obtain

$$\left(2\lambda + \alpha_1 - r_2 - \alpha_2 r_1 \tau e^{-\lambda\tau}\right)\frac{d\lambda}{d\tau} = \alpha_2 r_1 \lambda e^{-\lambda\tau}. \tag{19}$$

Therefore,

$$\begin{aligned}
&\operatorname{sign}\left\{\frac{d(\operatorname{Re}(\lambda))}{d\tau}\right\}_{\tau=\tau_j} \\
&\quad = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\tau=\tau_j} \\
&\quad = \operatorname{sign}\left\{\operatorname{Re}\frac{[2\omega_* + (\alpha_1 - r_2)](\cos\lambda\tau + i\sin\lambda\tau) - \tau_j\alpha_2 r_1}{i\omega_*\alpha_2 r_1}\right\} \\
&\quad = \operatorname{sign}\left\{\frac{[[2\omega_* + (\alpha_1 - r_2)][\alpha_1 - r_2]]}{\alpha_2^2 r_1^2}\right\} > 0.
\end{aligned} \tag{20}$$

Thus, the lemma follows. □

Therefore, from Lemma 1 and the relations between roots of (10) and (11) [17], we have the following conclusion.

Lemma 2. *When $\tau \in [0, \tau_0)$, all roots of (10) have negative real parts. When $\tau = \tau_0$, all roots of (10) have negative real parts except $\pm i\omega_*$. When $\tau \in (\tau_j, \tau_{j+1}]$, (10) has $2(j+1)$ roots with positive real parts.*

Furthermore, from Lemma 2, the following theorem holds.

Theorem 3. *If $\tau \in [0, \tau_0)$, then the positive equilibrium E is asymptotically stable and unstable if $\tau > \tau_0$. If $\tau = \tau_j$, (2) undergoes a Hopf bifurcation at E.*

3. Stability and Direction of the Hopf Bifurcation

Let $u_i = z_i(\tau t)$ and $\tau = \tau_j + \mu$, where $\mu \in R$. Then (2) can be written as a functional differential equation in $C = C([-1, 0], R^2)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{21}$$

where $u_t(\theta) = x(t+\theta) \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given, respectively, by

$$\begin{aligned}
L_\mu(\phi) &= \left(\tau^{(j)} + \mu\right)\begin{pmatrix} -\alpha_1 & 0 \\ r_1 & r_2 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} \\
&\quad + \left(\tau^{(j)} + \mu\right)\begin{pmatrix} 0 & -\alpha_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix},
\end{aligned} \tag{22}$$

$$F(\mu, \phi) = \left(\tau^{(j)} + \mu\right)\begin{pmatrix} -\phi_1(0)\phi_2(-1) - \varepsilon\phi_1^2(0) + \text{h.o.t} \\ \Sigma \dfrac{1}{i!\,j!} c_{ij}\phi_1^i(0)\phi_2^j(0) + \text{h.o.t} \end{pmatrix}, \tag{23}$$

where h.o.t denotes the higher order terms.

From the discussions above, we known that if $\mu = 0$, then system (21) undergoes a Hopf bifurcation at the zero equilibrium and the associated characteristic equation of system (21) has a pair of simple imaginary roots $\pm i\tau^j\omega_0$.

By the Reiz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, 0)\,\phi(\theta) \quad \text{for } \phi \in C. \tag{24}$$

In fact, we can choose

$$\begin{aligned}
\eta(\theta, \mu) &= \left(\tau^{(j)} + \mu\right)\begin{pmatrix} -\alpha_1 & 0 \\ r_1 & r_2 \end{pmatrix}\delta(\theta) \\
&\quad - \left(\tau^{(j)} + \mu\right)\begin{pmatrix} 0 & -\alpha_2 \\ 0 & 0 \end{pmatrix}\delta(\theta + 1),
\end{aligned} \tag{25}$$

where

$$\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0. \end{cases} \tag{26}$$

For $\phi \in C^1([-1, 0], R^2)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0], \\ \displaystyle\int_{-1}^{0} d\eta(s, \mu)\,\phi(s), & \theta = 0, \end{cases} \tag{27}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0], \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

Then we can rewrite (21) as

$$\dot{u}_t = A(\mu) u_t + R(\mu) x_t, \tag{28}$$

where $u_t(\theta) = u(t + \theta)$, $\theta \in [-1, 0]$. For $\phi \in C^1([0, 1], R^2)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in [0, 1], \\ \displaystyle\int_{-1}^{0} \psi(-t)\,d\eta(t, 0), & s = 0 \end{cases} \tag{29}$$

and a bilinear inner product

$$\begin{aligned}
&\langle \psi(s), \phi(\theta) \rangle \\
&\quad = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi,
\end{aligned} \tag{30}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then A^* and $A(0)$ are adjoint operators. By the discussion of Section 2, we known that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*.

Suppose that $q^*(s) = D(1,\alpha^*)e^{is\omega_0\tau^{(j)}}$ is the eigenvector of $A(0)$ corresponding to $i\tau^{(j)}\omega_0$. Then, $A(0)q(\theta) = i\tau^{(j)}\omega_0 q(\theta)$. From the definition of $A(0)$ and (25), we obtain

$$\tau^{(j)}\begin{pmatrix} i\omega+\alpha_1 & \alpha_2 e^{-i\omega_0\tau_j} \\ -r_1 & i\omega_0 - r_2 \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (31)$$

which yields

$$q(0) = (1,\alpha)^T = \left(1, \frac{r_1}{i\omega_0 - r_2}\right)^T. \qquad (32)$$

Similarly, it can be verified that $q^*(s) = D(1,\alpha^*)e^{is\omega_0\tau^{(j)}}$ is the eigenvector of A^* corresponding to $-i\omega_0\tau^{(j)}$, where

$$\alpha^* = \frac{\alpha_1 - i\omega_0}{r_1}. \qquad (33)$$

Let $\langle q^*(s), q(\theta)\rangle = 1$; that is,

$$\begin{aligned} &\langle q^*(s), q(\theta)\rangle \\ &\quad = \overline{D}\left(1,\overline{\alpha^*}\right)(1,\alpha)^T \\ &\quad\quad - \int_{-1}^{0}\int_{\xi}^{0} \overline{D}\left(1,\overline{\alpha^*}\right)e^{-i(\xi-\theta)\omega_0\tau^{(j)}} d\eta(\theta)(1,\alpha)^T e^{i\xi\omega_0\tau^{(j)}} d\xi \\ &\quad = \overline{D}\left\{1+\alpha\overline{\alpha^*} - \int_{-1}^{0}\left(1,\overline{\alpha^*}\right)\theta e^{i\theta\omega_0\tau^{(j)}} d\eta(\theta)(1,\alpha)^T\right\} \\ &\quad = \overline{D}\left\{1+\alpha\overline{\alpha^*} - \alpha_2\alpha\tau^{(j)}e^{-i\omega_0\tau^{(j)}}\right\} \\ &\quad = 1. \end{aligned} \qquad (34)$$

Thus, we can choose

$$D = \frac{1}{1+\overline{\alpha}\alpha^* - \alpha_2\overline{\alpha}\tau^{(j)}e^{i\omega_0\tau^{(j)}}} \qquad (35)$$

such that $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

Using the same notations as in Hassard et al. [18] and Song et al. [19], we first compute the center manifold C_0 at $\mu = 0$. Let x_t be the solution of (21) when $\mu = 0$. Define

$$\begin{aligned} z(t) &= \langle q^*, x_t\rangle, \\ W(t,\theta) &= x_t(\theta) - \left(x(t)q(\theta) + \overline{z}(t)\overline{q}(\theta)\right) \\ &= x_t(\theta) - 2\,\mathrm{Re}\left\{z(t)q(\theta)\right\}. \end{aligned} \qquad (36)$$

On the center manifold C_0, we have

$$W(t,\theta) = W(z(t),\overline{z}(t),\theta), \qquad (37)$$

where

$$\begin{aligned} W(z,\overline{z},\theta) &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} \\ &\quad + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \end{aligned} \qquad (38)$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q^*}$. Note that W is real if x_t is real. Here we consider only real solutions. For the solution $x_t \in C_0$ of (24), since $\mu = 0$, we have

$$\begin{aligned} \dot{z} &= i\tau^{(j)}\omega_0 z + \langle q^*(\theta), F(0, W(z,\overline{z},\theta) + 2\,\mathrm{Re}\{zq(\theta)\})\rangle \\ &= i\tau^{(j)}\omega_0 z + \overline{q^*}(0)F(0, W(z,\overline{z},0) + 2\,\mathrm{Re}\{zq(0)\}) \\ &= i\tau^{(j)}\omega_0 z + \overline{q^*}(0)F_0(z,\overline{z}). \end{aligned} \qquad (39)$$

We rewrite this equation as

$$\dot{z}(t) = i\tau^{(j)}\omega_0 z(t) + g(z,\overline{z}) \qquad (40)$$

with

$$\begin{aligned} g(z,\overline{z}) &= \overline{q^*}(0)F_0(z,\overline{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \qquad (41)$$

By (36), we have $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta)) = W(t,\theta) + zq(\theta) + \overline{z}\,\overline{q}(\theta)$ and $q(\theta) = (1,\alpha)^T e^{i\theta\omega_0\tau^{(j)}}$, and then

$$\begin{aligned} x_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} \\ &\quad + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + o\left(|(z,\overline{z})|^3\right), \\ x_{2t}(0) &= z\alpha e^{-i\omega_0\tau^{(j)}} + \overline{z}\,\overline{\alpha}e^{i\omega_0\tau^{(j)}} + W_{20}^{(2)}(-1)\frac{z^2}{2} \\ &\quad + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + o\left(|(z,\overline{z})|^3\right). \end{aligned} \qquad (42)$$

It follows, together with (23), that

$$\begin{aligned} &x_{2t}(0) = \alpha z + \overline{\alpha}\,\overline{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots, \\ &g(z,\overline{z}) \\ &\quad = \overline{q^*}(0)F_0(z,\overline{z}) \\ &\quad = \overline{D}\tau^{(j)}\left(1,\overline{\alpha^*}\right)\begin{pmatrix} -x_{1t}(0)x_{2t}(-1) - \varepsilon x_{1t}^2(0) + \text{h.o.t} \\ \Sigma\frac{1}{i!j!}c_{ij}x_{1t}^2(0)x_{2t}^j(0) + \text{h.o.t} \end{pmatrix} \\ &\quad = \overline{D}\tau^{(j)}\Big(-x_{1t}(0)x_{2t}(-1) - \varepsilon x_{1t}^2(0) \\ &\quad\quad + \overline{\alpha^*}\sum\frac{1}{i!j!}c_{ij}x_{1t}^2(0)x_{2t}^j(0) + \text{h.o.t}\Big) \end{aligned}$$

$$= \overline{D}\tau^{(j)} \Bigg\{ \left(-\alpha e^{-i\omega_0\tau^{(j)}} - \varepsilon + \frac{\overline{\alpha^*}}{2!}\left(c_{20} + c_{02}\alpha^2\right) \right) z^2$$
$$+ \Bigg(-\alpha e^{-i\omega_0\tau^{(j)}} - \overline{\alpha} e^{i\omega_0\tau^{(j)}} - 2\varepsilon$$
$$+ \overline{\alpha^*}\left(c_{20} + \frac{c_{02}\alpha\overline{\alpha}}{2} \right) \Bigg) z\overline{z}$$
$$+ \left(-\overline{\alpha} e^{i\omega_0\tau^{(j)}} - \varepsilon + \frac{\overline{\alpha^*} c_{20}}{2!} + \frac{\overline{\alpha}^2\overline{\alpha^*} c_{02}}{2!} \right) \overline{z}^2$$
$$+ \Bigg(-\frac{\overline{\alpha}}{2} e^{i\omega_0\tau^{(j)}} W_{20}^{(1)}(0) - \frac{1}{2} W_{20}^{(2)}(-1)$$
$$- \varepsilon W_{20}^{(1)}(0) + \overline{\alpha^*}\Bigg(\frac{1}{2} c_{20} W_{20}^{(1)}(0)$$
$$+ \frac{1}{2}\overline{\alpha} c_{02} W_{20}^{(2)} \Bigg)$$
$$\times c_{20} W_{11}^{(1)}(0)$$
$$+ \alpha c_{02} W(2)_{11}(0) \Bigg) z^2\overline{z} + \cdots \Bigg\}. \tag{43}$$

Comparing the coefficients with (41), we have

$$g_{20} = \overline{D}\tau^{(j)} \left(-2\alpha e^{-i\omega_0\tau^{(j)}} - 2\varepsilon + \overline{\alpha^*}\left(c_{20} + c_{02}\alpha^2\right) \right),$$
$$g_{11} = \overline{D}\tau^{(j)} \Bigg(-\alpha e^{-i\omega_0\tau^{(j)}} - \overline{\alpha} e^{i\omega_0\tau^{(j)}} - 2\varepsilon + \overline{\alpha^*}\left(c_{20} + \frac{c_{02}\alpha\overline{\alpha}}{2} \right) \Bigg),$$
$$g_{02} = \overline{D}\tau^{(j)} \left(-2\overline{\alpha} e^{i\omega_0\tau^{(j)}} - 2\varepsilon + \overline{\alpha}\left(c_{20} + \overline{\alpha} c_{02}\right) \right),$$
$$g_{21} = -\overline{\alpha} e^{i\omega_0\tau^{(j)}} W_{20}^{(1)}(0) - W_{20}^{(2)}(-1) - 2\varepsilon W_{20}^{(1)}(0) + \overline{\alpha^*}\left(c_{20} W_{20}^{(1)}(0) + c_{02}\overline{\alpha} W_{20}^{(2)}(0) \right) + 2c_{20} W_{11}^{(1)}(0) + 2c_{02}\alpha W_{11}^{(2)}(0). \tag{44}$$

In order to determine g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (28) and (36), we have

$$\dot{W} = \dot{x}_t - \dot{z} q - \dot{\overline{z}}\,\overline{q}$$
$$= \begin{cases} AW - 2R\left\{\overline{q*(0)} F_0 q(\theta)\right\}, & \theta \in [-1, 0], \\ AW - 2R\left\{\overline{q*(0)} F_0 q(\theta)\right\} + F_0, & \theta = 0 \end{cases} \tag{45}$$
$$\equiv AW + H(z, \overline{z}, \theta),$$

where

$$H(z, \overline{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{46}$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$\left(A - 2i\tau^{(j)}\omega_0\right) W_{20}(\theta) = -H_{20}(\theta),$$
$$AW_{11}(\theta) = -H_{11}(\theta), \ldots. \tag{47}$$

Following (45), we know that for $\theta \in [-1, 0]$,

$$H(z, \overline{z}, \theta) = -\overline{q^*(0)} F_0 q(\theta) - q^*(0)\overline{F_0}\,\overline{q(\theta)}$$
$$= -g q(\theta) - \overline{g}\,\overline{q(\theta)}. \tag{48}$$

Comparing the coefficients with (46), we get

$$H_{20}(\theta) = -g_{20} q(\theta) - \overline{g_{02}}\,\overline{q(\theta)},$$
$$H_{11}(\theta) = -g_{11} q(\theta) - \overline{g_{11}}\,\overline{q(\theta)}. \tag{49}$$

Substituting these relations into (47), we obtain

$$\dot{W}_{20}(\theta) = 2i\tau^{(j)}\omega_0 W_{20}(\theta) + g_{20} q(\theta) + \overline{g_{02}}\,\overline{q(\theta)}. \tag{50}$$

Solving $W_{20}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{i g_{20} q(0)}{\tau^{(j)}\omega_0} e^{i\tau^{(j)}\omega_0\theta} + \frac{i\overline{g_{02}}\,\overline{q(0)}}{3\tau^{(j)}\omega_0} e^{-i\tau^{(j)}\omega_0\theta} + E_1 e^{2i\tau^{(j)}\omega_0\theta}, \tag{51}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in R^2$ is a constant vector.

Similarly, we can obtain

$$W_{11}(\theta) = \frac{-i g_{11} q(0)}{\tau^{(j)}\omega_0} e^{i\tau^{(j)}\omega_0\theta} + \frac{i\overline{g_{11}}\,\overline{q(0)}}{\tau^{(j)}\omega_0} e^{-i\tau^{(j)}\omega_0\theta} + E_2, \tag{52}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in R^2$ is also a constant vector.

In what follows, we determine the constant vectors E_1 and E_2. From (47) and the definition of A, we obtain

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\tau^{(j)}\omega_0 W_{20}(0) - H_{20}(0), \tag{53}$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(\theta), \tag{54}$$

where $\eta(\theta) = \eta(0, \theta)$. From (45) and (46), we have

$$H_{20}(0) = -g_{20} q(0) - \overline{g_{02}}\,\overline{q(0)} + 2\tau^{(j)} \begin{pmatrix} -2\alpha e^{-i\omega_0\tau^{(j)}} - 2\varepsilon \\ c_{20} + c_{02}\alpha^2 \end{pmatrix}, \tag{55}$$

$$H_{11}(0) = -g_{11} q(0) - \overline{g_{11}}\,\overline{q(0)} + 2\tau^{(j)} \begin{pmatrix} -2\alpha e^{-i\omega_0\tau^{(j)}} - \overline{\alpha} e^{i\omega_0\tau^{(j)}} - 2\varepsilon \\ c_{20} + c_{02}\alpha\overline{\alpha} \end{pmatrix}. \tag{56}$$

Substituting (51) and (55) into (53) and noticing that

$$\left(i\tau^{(j)}\omega_0 I - \int_{-1}^{0} e^{i\theta\omega_0\tau^{(j)}} d\eta(\theta)\right) q(0) = 0,$$
$$\left(-i\tau^{(j)}\omega_0 I - \int_{-1}^{0} e^{-i\theta\omega_0\tau^{(j)}} d\eta(\theta)\right) \overline{q}(0) = 0, \quad (57)$$

we get

$$\left(2i\tau^{(j)}\omega_0 I - \int_{-1}^{0} e^{2i\theta\omega_0\tau^{(j)}} d\eta(\theta)\right) E_1 = 2\tau^{(j)} \begin{pmatrix} -2\alpha e^{-i\omega_0\tau^{(j)}} - 2\varepsilon \\ c_{20} + c_{02}\alpha^2 \end{pmatrix}; \quad (58)$$

that is,

$$\begin{pmatrix} 2i\omega_0 + \alpha_1 & -\alpha_2 e^{-2i\omega_0\tau^{(j)}} \\ -r_1 & 2i\omega_0 \end{pmatrix} E_1 = 2 \begin{pmatrix} -2\alpha e^{-i\omega_0\tau^{(j)}} - 2\varepsilon \\ c_{20} + c_{02}\alpha^2 \end{pmatrix}. \quad (59)$$

It follows that

$$E_1^{(1)} = \frac{2}{A} \begin{vmatrix} -2\alpha e^{-i\omega_0\tau^{(j)}} - 2\varepsilon & -\alpha_2 e^{-2i\omega_0\tau^{(j)}} \\ c_{20} + c_{02}\alpha^2 & 2i\omega_0 \end{vmatrix},$$
$$E_1^{(2)} = \frac{2}{A} \begin{vmatrix} 2i\omega_0 + \alpha_1 & -2\alpha e^{-i\omega_0\tau^{(j)}} - 2\varepsilon \\ -r_1 & c_{20} + c_{02}\alpha^2 \end{vmatrix}, \quad (60)$$

where $A = \begin{vmatrix} 2i\omega_0+\alpha_1 & -\alpha_2 e^{-2i\omega_0\tau^{(j)}} \\ -r_1 & 2i\omega_0 \end{vmatrix}$.

Similarly, substituting (52) and (56) into (54), we have

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ -r_1 & -r_2 \end{pmatrix} E_2 = 2 \begin{pmatrix} -2\alpha e^{-i\omega_0\tau^{(j)}} - \overline{\alpha} e^{i\omega_0\tau^{(j)}} - 2\varepsilon \\ c_{20} + c_{02}\alpha\overline{\alpha} \end{pmatrix}. \quad (61)$$

Then we obtain

$$E_2^{(1)} = \frac{2}{B} \begin{vmatrix} -2\alpha e^{-i\omega_0\tau^{(j)}} - \overline{\alpha} e^{i\omega_0\tau^{(j)}} - 2\varepsilon & -\alpha_2 \\ c_{20} + c_{02}\alpha\overline{\alpha} & -r_2 \end{vmatrix},$$
$$E_2^{(2)} = \frac{2}{B} \begin{vmatrix} \alpha_1 & -2\alpha e^{-i\omega_0\tau^{(j)}} - \overline{\alpha} e^{i\omega_0\tau^{(j)}} - 2\varepsilon \\ -r_1 & c_{20} + c_{02}\alpha\overline{\alpha} \end{vmatrix}, \quad (62)$$

where $B = \begin{vmatrix} \alpha_1 & \alpha_2 \\ -r_1 & -r_2 \end{vmatrix}$.

Therefore, all g_{ij} in (41) have been expressed in terms of the parameters and the delay given in (2). Substituting expressions of g_{02}, g_{11}, g_{20}, and g_{21} into the following relations,

$$C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 + \frac{g_{21}}{2}\right), \quad (63)$$

we obtain

$$K_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau)\}}, \qquad \beta_2 = 2\operatorname{Re}\{C_1(0)\},$$
$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + K_2 \operatorname{Im}\{\lambda'(\tau^{(j)})\}}{\tau^{(j)}\omega_0}. \quad (64)$$

We follow the idea in Hassard et al. [18] and Song et al. [19], which implies that the direction of the Hopf bifurcation is determined by the sign of β_2, and the stability of the bifurcating periodic solutions is determined by the sign of K_2 and T_2 determines the period of the bifurcating periodic solution. Thus we have the following.

Theorem 4. *(1) If $K_2 > 0$ ($K_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau^{(j)}$ ($\tau < \tau^{(j)}$).*

(2) If $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable).

(3) If $T_2 > 0$ ($T_2 < 0$), then the periodic of the bifurcating periodic solutions increase (decrease).

4. Conclusions

In the paper, we focused on the effect of the maturation time of the predator in Bazykin [1]. We first discussed the effect of the time τ on the stability of the positive equilibrium of the system (2), and then we investigated the existence of the Hopf bifurcation, the bifurcating direction, and the stability of the bifurcating periodic solutions by the normal form and center manifold. In fact, we can also incorporate other time delays such as capturing time into the mathematical model and look at their dynamics by other methods. In this regard, we can obtain other complicated and interesting results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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