Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 537971, 9 pages
http://dx.doi.org/10.1155/2014/537971

Research Article

Existence and Global Asymptotic Behavior of Positive Solutions for Nonlinear Fractional Dirichlet Problems on the Half-Line

Imed Bachar[1] and Habib Mâagli[2]

[1] *College of Sciences, Mathematics Department, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *College of Sciences and Arts, Department of Mathematics, King Abdulaziz University, Ragigh Campus, P.O. Box 344, Rabigh 21911, Saudi Arabia*

Correspondence should be addressed to Imed Bachar; abachar@ksu.edu.sa

Received 4 April 2014; Accepted 17 June 2014; Published 16 July 2014

Academic Editor: Mohamed Boussairi Jleli

We are interested in the following fractional boundary value problem: $D^\alpha u(t) + a(t)u^\sigma = 0$, $t \in (0,\infty)$, $\lim_{t\to 0} t^{2-\alpha}u(t) = 0$, $\lim_{t\to\infty} t^{1-\alpha}u(t) = 0$, where $1 < \alpha < 2$, $\sigma \in (-1,1)$, D^α is the standard Riemann-Liouville fractional derivative, and a is a nonnegative continuous function on $(0,\infty)$ satisfying some appropriate assumptions related to Karamata regular variation theory. Using the Schauder fixed point theorem, we prove the existence and the uniqueness of a positive solution. We also give a global behavior of such solution.

1. Introduction

Fractional differential equations arise in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetic. They also serve as an excellent tool for the description of hereditary properties of various materials and processes (see [1–3]). In consequence, the subject of fractional differential equations has been gaining much importance and attention. Most of the related results focused on developing the global existence and uniqueness of solutions on finite intervals (see [4–12]) and the references therein). However, to the best of our knowledge, there exist few articles dealing with the existence of solutions to fractional differential equations on the half-line; see, for instance, [13–21]. In [17], by using the recent Leggett-Williams norm-type theorem due to O'Regan and Zima, the author established the existence of positive solutions for fractional boundary value problems of resonance on infinite intervals. On the other hand, in [20], Su and Zhang studied the following fractional differential problem on the half-line by using Schauder's fixed point theorem:

$$\begin{gathered} D^\alpha u(t) = f\left(t, u, D^{\alpha-1}u\right), \quad t \in (0,\infty),\ 1 < \alpha \le 2, \\ u(0) = 0, \qquad \lim_{t\to\infty} D^{\alpha-1}u(t) = u_\infty, \quad u_\infty \in \mathbb{R}, \end{gathered} \tag{1}$$

where D^α is the standard Riemann-Liouville fractional derivative (see Definition 8 below).

In [21], by means of the Leray-Schauder alternative theorem, Zhao and Ge proved the existence of solutions to the following boundary value problem:

$$\begin{gathered} D^\alpha u(t) + f(t,u) = 0, \quad t \in (0,\infty),\ 1 < \alpha < 2, \\ u(0) = 0, \qquad \lim_{t\to\infty} D^{\alpha-1}u(t) = \beta u(\xi), \end{gathered} \tag{2}$$

where $\beta \in \mathbb{R}$ and $0 < \xi < \infty$.

In this paper, we aim at studying the existence, uniqueness, and the exact asymptotic behavior of a positive solution to the following fractional boundary value problem:

$$\begin{gathered} D^\alpha u(t) + a(t)u^\sigma = 0, \quad t \in (0,\infty), \\ \lim_{t\to 0} t^{2-\alpha}u(t) = 0, \qquad \lim_{t\to\infty} t^{1-\alpha}u(t) = 0, \end{gathered} \tag{3}$$

where $1 < \alpha < 2$, $\sigma \in (-1,1)$, and a is a nonnegative continuous function on $(0,\infty)$ that may be singular at 0.

To state our result, we need some notations. We first introduce the following Karamata classes.

Definition 1. The class $\mathscr{K}$ is the set of all Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c\exp\left(\int_t^\eta \frac{z(s)}{s}ds\right), \quad (4)$$

for some $\eta > 1$ and where $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$.

Definition 2. The class $\mathscr{K}^\infty$ is the set of all Karamata functions L defined on $[1,\infty)$ by

$$L(t) := c\exp\left(\int_1^t \frac{z(s)}{s}ds\right), \quad (5)$$

where $c > 0$ and $z \in C([1,\infty))$ such that $\lim_{t\to\infty} z(t) = 0$.

It is easy to verify the following.

Remark 3. (i) A function L is in $\mathscr{K}$ if and only if L is a positive function in $C^1((0,\eta])$, for some $\eta > 1$, such that $\lim_{t\to 0^+}(tL'(t)/L(t)) = 0$.

(ii) A function L is in $\mathscr{K}^\infty$ if and only if L is a positive function in $C^1([1,\infty))$ such that $\lim_{t\to\infty}(tL'(t)/L(t)) = 0$.

Remark 4 (see [22]). Let L be a function in $\mathscr{K}^\infty$, and then there exists $m \geq 0$ such that for every $\beta > 0$ and $t \geq 1$ we have

$$(1+\beta)^{-m}L(t) \leq L(\beta+t) \leq (1+\beta)^m L(t). \quad (6)$$

As a typical example of function belonging to the class $\mathscr{K}$, we quote

$$L(t) = \prod_{k=1}^m \left(\log_k\left(\frac{\omega}{t}\right)\right)^{\xi_k}, \quad (7)$$

where ξ_k are real numbers, $\log_k x = \log\circ\log\circ\cdots\log x$ (k times), and ω is a sufficiently large positive real number such that L is defined and positive on $(0,\eta]$, for some $\eta > 1$.

In the sequel, we denote by $B^+((0,\infty))$ the set of nonnegative Borel measurable functions in $(0,\infty)$ and by $C_{2-\alpha}([0,\infty))$ the set of all functions f such that $t \to t^{2-\alpha}f(t)$ is continuous on $[0,\infty)$.

We also denote by $C_0([0,\infty))$ the set of continuous functions v on $[0,\infty)$ such that $\lim_{t\to\infty} v(t) = 0$. It is easy to see that $C_0([0,\infty))$ is a Banach space with the uniform norm $\|v\|_\infty = \sup_{t>0}|v(t)|$.

For two nonnegative functions f and g defined on a set S, the notation $f(t) \approx g(t)$, $t \in S$ means that there exists $c > 0$ such that $(1/c)f(t) \leq g(t) \leq cf(t)$, for all $t \in S$.

Finally, for $\lambda \in \mathbb{R}$, we put $\lambda^+ = \max(\lambda, 0)$.

Throughout this paper we assume that the function a is nonnegative on $(0,\infty)$ and satisfies the following condition:

(H) $a \in C((0,\infty))$ such that

$$a(t) \approx t^{-\lambda}(1+t)^{\lambda-\mu}L_1(\min(t,1))\,L_2(\max(t,1)), \quad t > 0, \quad (8)$$

where $\lambda \leq 2 + (\alpha-2)\sigma$, $\mu \geq 1 + (\alpha-1)\sigma$, $L_1 \in \mathscr{K}$ defined on $(0,\eta]$, for some $\eta > 1$ and $L_2 \in \mathscr{K}^\infty$ satisfying

$$\int_0^\eta \frac{L_1(s)}{s^{\lambda-(\alpha-2)\sigma-1}}ds < \infty, \qquad \int_1^\infty \frac{L_2(s)}{s^{\mu-(\alpha-1)\sigma}}ds < \infty. \quad (9)$$

In what follows, we put

$$\nu = \min\left(1, \frac{2-\lambda+(\alpha-2)\sigma}{1-\sigma}\right),$$
$$\zeta = \max\left(0, \frac{2-\mu+(\alpha-2)\sigma}{1-\sigma}\right), \quad (10)$$

and we define the function θ on $(0,\infty)$ by

$$\theta(t) = t^\nu(1+t)^{\zeta-\nu}\left(\widetilde{L}_1(\min(t,1))\right)^{1/(1-\sigma)} \times\left(\widetilde{L}_2(\max(t,1))\right)^{1/(1-\sigma)}, \quad (11)$$

where, for $t \in (0,\eta)$,

$$\widetilde{L}_1(t) = \begin{cases} \int_0^t \frac{L_1(s)}{s}ds & \text{if } \lambda = 2+(\alpha-2)\sigma, \\ L_1(t) & \text{if } 1+(\alpha-1)\sigma < \lambda < 2+(\alpha-2)\sigma, \\ \int_t^\eta \frac{L_1(s)}{s}ds & \text{if } \lambda = 1+(\alpha-1)\sigma, \\ 1 & \text{if } \lambda < 1+(\alpha-1)\sigma \end{cases} \quad (12)$$

and, for $t \geq 1$,

$$\widetilde{L}_2(t) = \begin{cases} \int_t^\infty \frac{L_2(s)}{s}ds & \text{if } \mu = 1+(\alpha-1)\sigma, \\ L_2(t) & \text{if } 1+(\alpha-1)\sigma < \mu < 2+(\alpha-2)\sigma, \\ \int_1^{t+1} \frac{L_2(s)}{s}ds & \text{if } \mu = 2+(\alpha-2)\sigma, \\ 1 & \text{if } \mu > 2+(\alpha-2)\sigma. \end{cases} \quad (13)$$

Our main result is the following.

Theorem 5. *Let $1 < \alpha < 2$, $\sigma \in (-1,1)$ and assume (H). Then problem* (3) *has a unique positive solution $u \in C_{2-\alpha}([0,\infty))$ satisfying, for $t \in (0,\infty)$,*

$$u(t) \approx t^{\alpha-2}\theta(t). \quad (14)$$

Remark 6. The conclusion of Theorem 5 remains valid for the case $\alpha = 2$ and $\sigma < 1$ (see [23]).

The content of this paper is organized as follows. In Section 2, we present some properties of the Green function $G_\alpha(t,s)$ of the operator $u \to -D^\alpha u$ on $(0,\infty)$ with Dirichlet conditions $\lim_{t\to 0}t^{2-\alpha}u(t) = 0$ and $\lim_{t\to\infty}t^{1-\alpha}u(t) = 0$. Next, we give some fundamental properties of the two Karamata classes $\mathscr{K}$ and $\mathscr{K}^\infty$ and we establish sharp estimates on some potential functions. In Section 3, exploiting the results of the previous section and using the Schauder fixed point theorem, we prove Theorem 5.

2. Preliminaries

2.1. Fractional Calculus and Green Function. For the convenience of the reader, we recall in this section some basic definitions on fractional calculus (see [2, 24, 25]) and we give some properties of the Green function $G_\alpha(t,s)$.

Definition 7. The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\beta} h(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)\, ds, \quad t > 0, \tag{15}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 8. The Riemann-Liouville fractional derivative of order $\beta > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\beta} h(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1} h(s)\, ds, \quad t > 0, \tag{16}$$

where $n = [\beta] + 1$ provided that the right-hand side is pointwise defined on $(0, \infty)$. Here $[\beta]$ means the integer part of the number β.

So we have the following properties (see [2]).

Proposition 9. *(1) Let $\beta > 0$ and let $h \in L^1(0, \infty)$, and then one has*

$$D^{\beta} I^{\beta} h(t) = h(t), \quad \text{for } t > 0. \tag{17}$$

(2) Let $\beta > 0$, and then

$$D^{\beta} h(t) = 0 \quad \text{iff } h(t) = \sum_{j=1}^{m} c_j t^{\beta-j}, \tag{18}$$

where m is the smallest integer greater than or equal to β and $(c_1, c_2, \ldots, c_m) \in \mathbb{R}^m$.

Corollary 10. *Let $\beta > 0$ and assume that $D^{\beta} h \in L^1(0, \infty)$. Then,*

$$I^{\beta} D^{\beta} h(t) = h(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_m t^{\beta-m}, \quad t > 0, \tag{19}$$

where m is the smallest integer greater than or equal to β and $(c_1, c_2, \ldots, c_m) \in \mathbb{R}^m$.

Lemma 11. *Let $1 < \alpha < 2$ and $h \in L^1(0, \infty)$. The unique solution of*

$$\begin{gathered} D^{\alpha} u(t) + h(t) = 0, \quad t > 0 \\ \lim_{t \to 0} t^{2-\alpha} u(t) = 0, \qquad \lim_{t \to \infty} t^{1-\alpha} u(t) = 0 \end{gathered} \tag{20}$$

is given by

$$u(t) = \int_0^{\infty} G_{\alpha}(t,s) h(s)\, ds, \tag{21}$$

where

$$G_{\alpha}(t,s) = \frac{1}{\Gamma(\alpha)} \left[t^{\alpha-1} - \left((t-s)^{+}\right)^{\alpha-1}\right] \tag{22}$$

is Green's function for the boundary value problem (20).

Proof. We may apply Corollary 10 and Proposition 9 to reduce equation $D^{\alpha} u(t) + h(t) = 0$ to an equivalent integral equation

$$u(t) = -I^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{23}$$

where $(c_1, c_2) \in \mathbb{R}^2$. Hence the general solution of $D^{\alpha} u(t) + h(t) = 0$ is

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds. \tag{24}$$

By using $\lim_{t \to 0} t^{2-\alpha} u(t) = 0$ and $\lim_{t \to \infty} t^{1-\alpha} u(t) = 0$, we get

$$c_2 = 0, \qquad c_1 = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} h(s)\, ds. \tag{25}$$

Therefore, the unique solution of problem (20) is

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} h(s)\, ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \\ &= \int_0^{\infty} G_{\alpha}(t,s) h(s)\, ds. \end{aligned} \tag{26}$$

The proof is complete. □

Next we give sharp estimates on the Green function $G_{\alpha}(t,s)$. To this end, we need the following lemma.

Lemma 12. *(i) For $\lambda, \mu \in (0, \infty)$ and $x \in [0,1]$, one has*

$$\min\left(1, \frac{\mu}{\lambda}\right)\left(1 - x^{\lambda}\right) \le 1 - x^{\mu} \le \max\left(1, \frac{\mu}{\lambda}\right)\left(1 - x^{\lambda}\right). \tag{27}$$

(ii) For $(t,s) \in (0, \infty) \times (0, \infty)$ one has

$$\min(1,t) \min(1,s) \le \min(t,s) \le \max(1,t) \min(1,s). \tag{28}$$

Proposition 13. *The Green function $G_{\alpha}(t,s)$ defined by* (22) *satisfies*

$$G_{\alpha}(t,s) \approx t^{\alpha-2} \min(t,s), \quad \text{for } (t,s) \in (0, \infty) \times (0, \infty). \tag{29}$$

Proof. For $(t,s) \in (0, \infty) \times (0, \infty)$ we have

$$G_{\alpha}(t,s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left[1 - \left(\frac{(t-s)^{+}}{t}\right)^{\alpha-1}\right]. \tag{30}$$

Since $((t-s)^{+}/t) \in [0,1)$ for $(t,s) \in (0, \infty) \times (0, \infty)$, then by applying Lemma 12 (i) with $\mu = \alpha - 1$ and $\lambda = 1$, we obtain

$$G_{\alpha}(t,s) \approx t^{\alpha-1} \left[1 - \left(\frac{(t-s)^{+}}{t}\right)\right] = t^{\alpha-2} \min(t,s). \tag{31}$$

□

From here on, we define the potential kernel G_{α} on $B^{+}((0, \infty))$ by

$$G_{\alpha} h(t) := \int_0^{\infty} G_{\alpha}(t,s) h(s)\, ds, \quad \text{for } t > 0. \tag{32}$$

Using Proposition 13 and Lemma 12 (ii), we deduce the following.

Corollary 14. *Let $1 < \alpha < 2$ and $h \in B^+((0,\infty))$, and then the function $t \to G_\alpha h(t)$ belongs to $C_{2-\alpha}([0,\infty))$ if and only if the integral $\int_0^\infty \min(1,s)h(s)ds$ converges.*

Proposition 15. *Let $1 < \alpha < 2$ and f be a function such that the map $s \to \min(1,s)f(s)$ is continuous and integrable on $(0,\infty)$. Then $G_\alpha f$ is the unique solution in $C_{2-\alpha}([0,\infty))$ of the boundary value problem*

$$\begin{gathered} D^\alpha u(t) + f(t) = 0, \quad t > 0 \\ \lim_{t\to 0} t^{2-\alpha} u(t) = 0, \qquad \lim_{t\to\infty} t^{1-\alpha} u(t) = 0. \end{gathered} \tag{33}$$

Proof. From Corollary 14, the function $G_\alpha f$ is well defined in $(0,\infty)$. Using Proposition 13 and Lemma 12 (ii), we get

$$G_\alpha |f|(t) \le c\left(t^{\alpha-2}(t+1)\int_0^\infty \min(1,s)|f(s)|ds\right). \tag{34}$$

This implies that $I^{2-\alpha}(G_\alpha|f|)$ is finite on $(0,\infty)$. So by using Fubini's theorem, we obtain

$$\begin{aligned} I^{2-\alpha}(G_\alpha f)(t) &= \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha} G_\alpha f(s)\,ds \\ &= \frac{1}{\Gamma(2-\alpha)}\int_0^\infty\left(\int_0^t (t-s)^{1-\alpha} G_\alpha(s,r)\,ds\right) f(r)\,dr. \end{aligned} \tag{35}$$

Observe that by considering the substitution $s = r + (t-r)\theta$, we obtain

$$\int_r^t (t-s)^{1-\alpha}(s-r)^{\alpha-1}ds = \Gamma(\alpha)\Gamma(2-\alpha)(t-r). \tag{36}$$

Using this fact and (22) we deduce that

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha} G_\alpha(s,r)\,ds \\ &= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)} \\ &\quad\times\left[\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}ds - \int_0^t (t-s)^{1-\alpha}\left((s-r)^+\right)^{\alpha-1}ds\right] \\ &= t - \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{1-\alpha}\left((s-r)^+\right)^{\alpha-1}ds. \end{aligned} \tag{37}$$

Now, assume that $r \le t$, and then by (36) we have

$$\begin{aligned} &\int_0^t (t-s)^{1-\alpha}\left((s-r)^+\right)^{\alpha-1}ds \\ &\quad= \int_r^t (t-s)^{1-\alpha}(s-r)^{\alpha-1}ds = \Gamma(\alpha)\Gamma(2-\alpha)(t-r). \end{aligned} \tag{38}$$

On the other hand, if $t \le r$ and $s \in (0,t)$, we have

$$\int_0^t (t-s)^{1-\alpha}\left((s-r)^+\right)^{\alpha-1}ds = 0. \tag{39}$$

So combining (38) and (39), we obtain

$$\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha} G_\alpha(s,r)\,ds = t - (t-r)^+ = \min(t,r). \tag{40}$$

This implies that

$$\begin{aligned} I^{2-\alpha}(G_\alpha f)(t) &= \int_0^\infty \min(t,r) f(r)\,dr \\ &= \int_0^t r f(r)\,dr + t\int_t^\infty f(r)\,dr, \end{aligned} \tag{41}$$

and $D^\alpha(G_\alpha f)(t) = (d^2/dt^2)(I^{2-\alpha}(G_\alpha f))(t) = -f(t)$, for $t > 0$.

Moreover, using Proposition 13 and the dominated convergence theorem, we deduce that

$$\lim_{t\to 0} t^{2-\alpha} G_\alpha f(t) = 0, \qquad \lim_{t\to\infty} t^{1-\alpha} G_\alpha f(t) = 0. \tag{42}$$

Finally, we need to prove the uniqueness. Let $u, v \in C_{2-\alpha}([0,\infty))$ be two solutions of (33) and put $\omega = u - v$. Then $\omega \in C_{2-\alpha}([0,\infty))$ and $D^\alpha\omega = 0$. Hence, it follows from Corollary 10 that $\omega(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$. Using the fact that $\lim_{t\to 0} t^{2-\alpha}\omega(t) = \lim_{t\to\infty} t^{1-\alpha}\omega(t) = 0$, we deduce that $\omega = 0$ and therefore $u = v$. The proof is complete. □

2.2. Sharp Estimates on the Potential of Some Karamata Functions. We collect in this paragraph some properties of functions belonging to the Karamata class $\mathcal{K}$ (resp., $\mathcal{K}^\infty$) and we give estimates on some potential functions.

Proposition 16 (see [26, 27]). *(i) Let $L_1, L_2 \in \mathcal{K}$ (resp, $\mathcal{K}^\infty$) and $p \in \mathbb{R}$. Then the functions*

$$L_1 + L_2, \quad L_1 L_2 \text{ and } L_1^p \text{ belong to the class } \mathcal{K}\ (\text{resp, } \mathcal{K}^\infty). \tag{43}$$

(ii) Let L be a function in $\mathcal{K}$ (resp., $\mathcal{K}^\infty$) and $\varepsilon > 0$. Then one has

$$\lim_{t\to 0^+} t^\varepsilon L(t) = 0, \quad \left(\text{resp. } \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0\right). \tag{44}$$

Theorem 17 (see [26, 27]). *(a) Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. One has the following.*

(i) *If $\gamma < -1$, then $\int_0^\eta s^\gamma L(s)ds$ diverges and $\int_t^\eta s^\gamma L(s)ds \underset{t\to 0^+}{\sim} -t^{\gamma+1}L(t)/(\gamma+1)$.*

(ii) *If $\gamma > -1$, then $\int_0^\eta s^\gamma L(s)ds$ converges and $\int_0^t s^\gamma L(s)ds \underset{t\to 0^+}{\sim} t^{\gamma+1}L(t)/(\gamma+1)$.*

(b) Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}^\infty$. One has the following.

(i) If $\gamma > -1$, then $\int_1^\infty s^\gamma L(s)ds$ diverges and $\int_1^t s^\gamma L(s) ds \underset{t\to\infty}{\sim} t^{\gamma+1}L(t)/(\gamma+1)$.

(ii) If $\gamma < -1$, then $\int_1^\infty s^\gamma L(s)ds$ converges and $\int_t^\infty s^\gamma L(s) ds \underset{t\to\infty}{\sim} -t^{\gamma+1}L(t)/(\gamma+1)$.

The proof of the next lemma can be found in [11].

Lemma 18. *Let L be a function in $\mathcal{K}$ defined on $(0,\eta]$. Then one has*

$$\lim_{t\to 0^+}\frac{L(t)}{\int_t^\eta (L(s)/s)\,ds} = 0. \tag{45}$$

In particular,

$$t \longrightarrow \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}. \tag{46}$$

If further $\int_0^\eta (L(s)/s)ds$ converges, then one has $\lim_{t\to 0^+}(L(t)/\int_0^t (L(s)/s)ds) = 0$.

In particular,

$$t \longrightarrow \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}. \tag{47}$$

In the next lemma, we have the following properties related to the class $\mathcal{K}^\infty$. For the proof we refer to [22].

Lemma 19. *Let L be a function in $\mathcal{K}^\infty$. Then one has*

$$\lim_{t\to \infty}\frac{L(t)}{\int_1^t (L(s)/s)\,ds} = 0. \tag{48}$$

In particular,

$$t \longrightarrow \int_1^{t+1} \frac{L(s)}{s}ds \in \mathcal{K}^\infty. \tag{49}$$

If further $\int_1^\infty (L(s)/s)ds$ converges, then one has

$$\lim_{t\to \infty}\frac{L(t)}{\int_t^\infty (L(s)/s)\,ds} = 0. \tag{50}$$

In particular,

$$t \longrightarrow \int_t^{\infty} \frac{L(s)}{s}ds \in \mathcal{K}^\infty. \tag{51}$$

Now, we put

$$b(t) = t^{-\beta}(1+t)^{\beta-\gamma}L_3(\min(t,1))\,L_4(\max(t,1)), \quad t>0, \tag{52}$$

where $L_3 \in \mathcal{K}$ and $L_4 \in \mathcal{K}^\infty$. We aim at giving sharp estimates on the potential function $G_\alpha b(t)$.

Proposition 20. *Assume that $L_3 \in \mathcal{K}$ defined on $(0,\eta]$, for some $\eta>1$ and $L_4 \in \mathcal{K}^\infty$. Let $\beta \le 2$ and $\gamma \ge 1$ such that*

$$\int_0^\eta s^{1-\beta}L_3(s)\,ds < \infty, \qquad \int_1^\infty s^{-\gamma}L_4(s)\,ds < \infty. \tag{53}$$

Then for $t>0$

$$G_\alpha b(t) \approx t^{\alpha-2}\psi_\beta(\min(t,1))\,\phi_\gamma(\max(t,1)), \tag{54}$$

where, for $t\in(0,1]$,

$$\psi_\beta(t) = \begin{cases} \int_0^t \frac{L_3(s)}{s}ds & \text{if } \beta = 2,\\ t^{2-\beta}L_3(t) & \text{if } 1<\beta<2,\\ t\int_t^\eta \frac{L_3(s)}{s}ds & \text{if } \beta=1,\\ t & \text{if } \beta<1 \end{cases} \tag{55}$$

and, for $t\ge 1$,

$$\phi_\gamma(t) = \begin{cases} t\int_t^\infty \frac{L_4(s)}{s}ds & \text{if } \gamma = 1,\\ t^{2-\gamma}L_4(t) & \text{if } 1<\gamma<2,\\ \int_1^{t+1} \frac{L_4(s)}{s}ds & \text{if } \gamma=2,\\ 1 & \text{if } \gamma>2. \end{cases} \tag{56}$$

Proof. Using Proposition 13 and Remark 4, we have

$$\begin{aligned} t^{2-\alpha}G_\alpha b(t) &\approx \int_0^\eta \min(t,s)\,s^{-\beta}L_3(s)\,ds \\ &\quad + \int_\eta^\infty \min(t,s)\,s^{-\gamma}L_4(s)\,ds \\ &= I(t) + J(t). \end{aligned} \tag{57}$$

Case 1. Assume that $0<t\le 1$.

By using (53), we deduce that

$$J(t) \approx t. \tag{58}$$

On the other hand,

$$\begin{aligned} I(t) &= \int_0^t s^{1-\beta}L_3(s)\,ds + t\int_t^\eta s^{-\beta}L_3(s)\,ds \\ &= I_1(t) + I_2(t). \end{aligned} \tag{59}$$

Using Theorem 17 and hypothesis (53), we deduce that

$$\begin{aligned} I_1(t) &\approx \begin{cases} t^{2-\beta}L_3(t) & \text{if } \beta<2,\\ \int_0^t \frac{L_3(s)}{s}ds & \text{if } \beta=2, \end{cases} \\ I_2(t) &\approx \begin{cases} t^{2-\beta}L_3(t) & \text{if } 1<\beta\le 2,\\ t\int_t^\eta s^{-\beta}L_3(s)\,ds & \text{if } \beta\le 1. \end{cases} \end{aligned} \tag{60}$$

Hence, it follows by Lemma 18, Proposition 16, and hypothesis (53) that

$$I(t) \approx \begin{cases} \displaystyle\int_0^t \frac{L_3(s)}{s} ds & \text{if } \beta = 2, \\ t^{2-\beta} L_3(t) & \text{if } 1 < \beta < 2, \\ \displaystyle t\int_t^\eta \frac{L_3(s)}{s} ds & \text{if } \beta = 1, \\ t & \text{if } \beta < 1. \end{cases} \tag{61}$$

Combining (58) and (61) and using Proposition 16 and hypothesis (53), we deduce that, for $0 < t \le 1$,

$$t^{2-\alpha} G_\alpha b(t) \approx \psi_\beta(t). \tag{62}$$

Case 2. Assume that $t > \eta + 1$.

By using (53), we deduce that

$$I(t) \approx 1. \tag{63}$$

On the other hand,

$$\begin{aligned} J(t) &= \int_t^\eta s^{1-\gamma} L_4(s)\, ds + t\int_t^\infty s^{-\gamma} L_4(s)\, ds \\ &= J_1(t) + J_2(t). \end{aligned} \tag{64}$$

Using again Theorem 17 and hypothesis (53), we deduce that

$$\begin{gathered} J_1(t) \approx \begin{cases} t^{2-\gamma} L_4(t) & \text{if } 1 \le \gamma < 2, \\ \displaystyle\int_\eta^t s^{1-\gamma} L_4(s)\, ds & \text{if } \gamma \ge 2, \end{cases} \\ J_2(t) \approx \begin{cases} t^{2-\gamma} L_4(t) & \text{if } \gamma > 1, \\ \displaystyle t\int_t^\infty \frac{L_4(s)}{s} ds & \text{if } \gamma = 1. \end{cases} \end{gathered} \tag{65}$$

Hence, it follows from Lemma 19 and hypothesis (53) that

$$J(t) \approx \begin{cases} \displaystyle t\int_t^\infty \frac{L_4(s)}{s} ds & \text{if } \gamma = 1, \\ t^{2-\gamma} L_4(t) & \text{if } 1 < \gamma < 2, \\ \displaystyle\int_\eta^t \frac{L_4(s)}{s} ds & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2. \end{cases} \tag{66}$$

Combining (63) and (66) and using Proposition 16, hypothesis (53), and Remark 4, we deduce that, for $t > \eta + 1$,

$$\begin{aligned} t^{2-\alpha} G_\alpha b(t) &\approx \begin{cases} \displaystyle t\int_t^\infty \frac{L_4(s)}{s} ds & \text{if } \gamma = 1, \\ t^{2-\gamma} L_4(t) & \text{if } 1 < \gamma < 2, \\ \displaystyle\int_\eta^t \frac{L_4(s)}{s} ds & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2. \end{cases} \\ &\approx \phi_\gamma(t). \end{aligned} \tag{67}$$

Now since the functions $t \to t^{2-\alpha} G_\alpha b(t)$ and $t \to \phi_\gamma(t)$ are positive and continuous on $[1, \eta + 1]$, we deduce that, for $t \in [1, \eta + 1]$,

$$t^{2-\alpha} G_\alpha b(t) \approx \phi_\gamma(t). \tag{68}$$

Finally, using (62), (67), and (68), we obtain the required result. □

3. Proof of the Main Result

The next lemma will play a crucial role in the proof of Theorem 5

Lemma 21. *Assume that the function a satisfies (H) and put $\omega(t) = a(t) t^{(\alpha-2)\sigma} \theta^\sigma(t)$ for $t > 0$. Then one has for $t \in (0, \infty)$*

$$t^{2-\alpha} G_\alpha \omega(t) \approx \theta(t). \tag{69}$$

Proof. We recall that

$$\begin{aligned} \nu &= \min\left(1, \frac{2 - \lambda + (\alpha - 2)\sigma}{1 - \sigma}\right), \\ \zeta &= \max\left(0, \frac{2 - \mu + (\alpha - 2)\sigma}{1 - \sigma}\right), \\ \theta(t) &= t^\nu (1 + t)^{\zeta - \nu} \left(\widetilde{L}_1(\min(t, 1))\right)^{1/(1-\sigma)} \\ &\quad \times \left(\widetilde{L}_2(\max(t, 1))\right)^{1/(1-\sigma)}, \end{aligned} \tag{70}$$

where, for $t \in (0, 1]$,

$$\widetilde{L}_1(t) = \begin{cases} \displaystyle\int_0^t \frac{L_1(s)}{s} ds & \text{if } \lambda = 2 + (\alpha - 2)\sigma, \\ L_1(t) & \text{if } 1 + (\alpha - 1)\sigma < \lambda < 2 + (\alpha - 2)\sigma, \\ \displaystyle\int_t^\eta \frac{L_1(s)}{s} ds & \text{if } \lambda = 1 + (\alpha - 1)\sigma, \\ 1 & \text{if } \lambda < 1 + (\alpha - 1)\sigma \end{cases} \tag{71}$$

and, for $t \ge 1$,

$$\widetilde{L}_2(t) = \begin{cases} \displaystyle\int_t^\infty \frac{L_2(s)}{s} ds & \text{if } \mu = 1 + (\alpha - 1)\sigma, \\ L_2(t) & \text{if } 1 + (\alpha - 1)\sigma < \mu < 2 + (\alpha - 2)\sigma, \\ \displaystyle\int_1^{t+1} \frac{L_2(s)}{s} ds & \text{if } \mu = 2 + (\alpha - 2)\sigma, \\ 1 & \text{if } \mu > 2 + (\alpha - 2)\sigma. \end{cases} \tag{72}$$

For $t > 0$, we have

$$\begin{aligned} &\omega(t) \\ &\quad \approx t^{-\lambda + \nu\sigma + (\alpha-2)\sigma} (1 + t)^{\lambda - \mu + (\zeta - \nu)\sigma} \\ &\quad \times L_1(\min(t, 1)) \left(\widetilde{L}_1(\min(t, 1))\right)^{\sigma/(1-\sigma)} \\ &\quad \times L_2(\max(t, 1)) \left(\widetilde{L}_2(\max(t, 1))\right)^{\sigma/(1-\sigma)}. \end{aligned} \tag{73}$$

Using Proposition 20 with $\beta = \lambda - \nu\sigma - (\alpha - 2)\sigma$ and $\gamma = \mu - \zeta\sigma - (\alpha - 2)\sigma$, $L_3(t) = L_1(t)(\widetilde{L}_1(t))^{\sigma/(1-\sigma)}$ and $L_4(t) = L_2(t)(\widetilde{L}_2(t))^{\sigma/(1-\sigma)}$, we obtain for $t \in (0, 1]$

$$t^{2-\alpha}G_\alpha\omega(t)\approx\begin{cases}\displaystyle\int_0^t\frac{L_1(s)}{s}\left(\int_0^s\frac{L_1(r)}{r}dr\right)^{\sigma/(1-\sigma)}ds & \text{if } \lambda=2+(\alpha-2)\sigma,\\ t^{(2-\lambda+(\alpha-2)\sigma)/(1-\sigma)}L_1(t)\left(L_1(t)\right)^{\sigma/(1-\sigma)} & \text{if } 1+(\alpha-1)\sigma<\lambda<2+(\alpha-2)\sigma,\\ \displaystyle t\int_t^\eta\frac{L_1(s)}{s}\left(\int_s^\eta\frac{L_1(r)}{r}dr\right)^{\sigma/(1-\sigma)}ds & \text{if } \lambda=1+(\alpha-1)\sigma,\\ t & \text{if } \lambda<1+(\alpha-1)\sigma,\end{cases}$$
$$\approx\begin{cases}\displaystyle\left(\int_0^t\frac{L_1(s)}{s}ds\right)^{1/(1-\sigma)} & \text{if } \lambda=2+(\alpha-2)\sigma,\\ t^{(2-\lambda+(\alpha-2)\sigma)/(1-\sigma)}\left(L_1(t)\right)^{1/(1-\sigma)} & \text{if } 1+(\alpha-1)\sigma<\lambda<2+(\alpha-2)\sigma,\\ \displaystyle t\left(\int_t^\eta\frac{L_1(s)}{s}ds\right)^{1/(1-\sigma)} & \text{if } \lambda=1+(\alpha-1)\sigma,\\ t & \text{if } \lambda<1+(\alpha-1)\sigma,\end{cases}$$
$$\approx\theta(t). \tag{74}$$

On the other hand, using again Proposition 20 and Remark 4, we get, for $t\geq 1$,

$$t^{2-\alpha}G_\alpha\omega(t)\approx\begin{cases}\displaystyle t\int_t^\infty\frac{L_2(s)}{s}\left(\int_s^\infty\frac{L_2(r)}{r}dr\right)^{\sigma/(1-\sigma)}ds & \text{if } \mu=1+(\alpha-1)\sigma,\\ t^\zeta L_2(t)\left(L_2(t)\right)^{\sigma/(1-\sigma)} & \text{if } 1+(\alpha-1)\sigma<\mu<2+(\alpha-2)\sigma,\\ \displaystyle\int_1^{t+1}\frac{L_2(s)}{s}\left(\int_1^{s+1}\frac{L_2(r)}{r}dr\right)^{\sigma/(1-\sigma)}ds & \text{if } \mu=2+(\alpha-2)\sigma,\\ 1 & \text{if } \mu>2+(\alpha-2)\sigma.\end{cases}$$
$$\approx\begin{cases}\displaystyle t\left(\int_t^\infty\frac{L_2(s)}{s}ds\right)^{1/(1-\sigma)} & \text{if } \mu=1+(\alpha-1)\sigma,\\ t^\zeta\left(L_2(t)\right)^{1/(1-\sigma)} & \text{if } 1+(\alpha-1)\sigma<\mu<2+(\alpha-2)\sigma,\\ \displaystyle\left(\int_1^{t+1}\frac{L_2(s)}{s}ds\right)^{1/(1-\sigma)} & \text{if } \mu=2+(\alpha-2)\sigma,\\ 1 & \text{if } \mu>2+(\alpha-2)\sigma.\end{cases}$$
$$\approx\theta(t). \tag{75}$$

This completes the proof. □

Proof of Theorem 5. From Lemma 21, there exists $M>1$ such that for each $t>0$

$$\frac{1}{M}\theta(t)\leq t^{2-\alpha}G_\alpha\omega(t)\leq M\theta(t), \tag{76}$$

where $\omega(t)=a(t)t^{(\alpha-2)\sigma}\theta^\sigma(t)$.

Put $c_0=M^{1/(1-|\sigma|)}$ and let

$$\Lambda=\left\{v\in C_0([0,\infty)):\frac{\theta(t)}{c_0(1+t)}\leq v(t)\leq\frac{c_0\theta(t)}{1+t},\ t>0\right\}. \tag{77}$$

In order to use a fixed point theorem, we define the operator T on Λ by

$$Tv(t)=\frac{t^{2-\alpha}}{1+t}\int_0^\infty G_\alpha(t,s)\,a(s)\,s^{(\alpha-2)\sigma}(1+s)^\sigma v^\sigma(s)\,ds. \tag{78}$$

For this choice of c_0 and using (76), we easily prove that for all $v\in\Lambda$ and $t>0$

$$Tv(t)\leq\frac{c_0\theta(t)}{1+t},\qquad Tv(t)\geq\frac{\theta(t)}{c_0(1+t)}. \tag{79}$$

On the other hand, using Proposition 13 and Lemma 12 (ii), there exists $c > 0$ such that for all $t, s > 0$, we have

$$\frac{t^{2-\alpha} G_\alpha(t,s)}{1+t} \le c \min(1,s). \quad (80)$$

This implies that there exists $c > 0$ such that, for each $v \in \Lambda$ and $t > 0$,

$$|Tv(t)| \le c \int_0^\infty \min(1,s)\,\omega(s)\,ds. \quad (81)$$

Now by hypothesis (H) and Theorem 17, the function $t \to \min(1,t)\omega(t)$ is in $L^1(0,\infty)$, which implies that the family $\{Tv(t), v \in \Lambda\}$ is uniformly bounded.

Using (80) and the fact that, for each $s > 0$, the function $t \to (t^{2-\alpha}G_\alpha(t,s))/(1+t)$ is in $C_0([0,\infty))$, we deduce that the family $\{Tv(t), v \in \Lambda\}$ is equicontinuous in $[0,\infty]$.

Hence, it follows by Ascoli's theorem that $T(\Lambda)$ is relatively compact in $C_0([0,\infty))$ and therefore $T(\Lambda) \subset \Lambda$.

Next, we will prove the continuity of T in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges to v in Λ. Using again (80) and Lebesgue's theorem, we deduce that $Tv_k(t) \to Tv(t)$ as $k \to \infty$, for $t > 0$.

Since $T(\Lambda)$ is relatively compact in $C_0([0,\infty))$, then the pointwise convergence implies the uniform convergence. Thus we have proved that T is a compact mapping from Λ to itself.

Now, the Schauder fixed point theorem implies the existence of $v \in \Lambda$ such that

$$v(t) = \frac{t^{2-\alpha}}{1+t} \int_0^\infty G_\alpha(t,s)\,a(s)\,s^{(\alpha-2)\sigma}(1+s)^\sigma v^\sigma(s)\,ds. \quad (82)$$

Put $u(t) = t^{\alpha-2}(1+t)v(t)$. Then $u \in C_{2-\alpha}([0,\infty))$ and u satisfies the equation

$$u(t) = G_\alpha(au^\sigma)(t). \quad (83)$$

Since the function $s \to \min(1,s)a(s)u^\sigma(s)$ is continuous and integrable on $(0,\infty)$, then by Proposition 15, the function u is a positive solution in $C_{2-\alpha}([0,\infty))$ of problem (3).

Finally, it remains to prove that u is the unique positive solution in $C_{2-\alpha}([0,\infty))$ satisfying (14). To this end, assume that problem (3) has two positive solutions $u, v \in C_{2-\alpha}([0,\infty))$ satisfying (14). Then there exists a constant $m > 1$ such that

$$\frac{1}{m} \le \frac{u}{v} \le m. \quad (84)$$

This implies that the set

$$J = \left\{ m \ge 1 : \frac{1}{m} \le \frac{u}{v} \le m \right\} \quad (85)$$

is not empty. Let $c = \inf J$. Then $c \ge 1$ and we have $(1/c)v \le u \le cv$. It follows that $u^\sigma \le c^{|\sigma|} v^\sigma$ and consequently

$$-D^\alpha\left(c^{|\sigma|}v - u\right) = a\left(c^{|\sigma|}v^\sigma - u^\sigma\right) \ge 0, \quad (86)$$

$$\lim_{t\to 0} t^{2-\alpha}\left(c^{|\sigma|}v - u\right)(t) = 0, \qquad \lim_{t\to\infty} t^{1-\alpha}\left(c^{|\sigma|}v - u\right)(t) = 0. \quad (87)$$

This implies by Proposition 15 that $c^{|\sigma|}v - u = G_\alpha(a(c^{|\sigma|}v^\sigma - u^\sigma)) \ge 0$. By symmetry, we obtain also $v \le c^{|\sigma|}u$. Hence $c^{|\sigma|} \in J$ and $c \le c^{|\sigma|}$. Since $|\sigma| < 1$, then $c = 1$ and therefore $u = v$. □

Example 22. Let $1 < \alpha < 2$, $\sigma \in (-1,1)$, and a be a positive continuous function on $(0,\infty)$ such that

$$a(t) \approx t^{-\lambda}(1+t)^{\lambda-\mu} \log\left(\frac{2}{\min(t,1)}\right), \quad t > 0, \quad (88)$$

where $\lambda < 2+(\alpha-2)\sigma$ and $\mu > 1+(\alpha-1)\sigma$. Then by Theorem 5, problem (3) has a unique positive solution $u \in C_{2-\alpha}([0,\infty))$ satisfying, for $t > 0$,

$$u(t) \approx t^{\alpha-2+\nu}(1+t)^{\zeta-\nu}\left(\tilde{L}_1(\min(t,1))\right)^{1/(1-\sigma)} \times \left(\tilde{L}_2(\max(t,1))\right)^{1/(1-\sigma)}, \quad (89)$$

where $\nu = \min(1, (2-\lambda+(\alpha-2)\sigma)/(1-\sigma))$, $\zeta = \max(0, (2-\mu+(\alpha-2)\sigma)/(1-\sigma))$,

$$\tilde{L}_1(t) = \begin{cases} \log\left(\dfrac{2}{t}\right) & \text{if } 1+(\alpha-1)\sigma < \lambda < 2+(\alpha-2)\sigma, \\ \left(\log\left(\dfrac{2}{t}\right)\right)^2 & \text{if } \lambda = 1+(\alpha-1)\sigma, \\ 1 & \text{if } \lambda < 1+(\alpha-1)\sigma, \end{cases}$$

$$\tilde{L}_2(t) = \begin{cases} \log(1+t) & \text{if } \mu = 2+(\alpha-2)\sigma, \\ 1 & \text{if } \mu \ne 2+(\alpha-2)\sigma. \end{cases} \quad (90)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors thank the referees for a careful reading of the paper. This project was supported by King Saud University, Deanship of Scientic Research, College of Science Research Center.

References

[1] K. Diethelm and A. D. Freed, "On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity," in *Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics and Molecular Properties*, F. Keil, W. Mackens, and H. Voss, Eds., pp. 217–224, Springer, Heidelberg, Germany, 1999.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[3] W. Lin, "Global existence theory and chaos control of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 1, pp. 709–726, 2007.

[4] R. P. Agarwal, D. O'Regan, and S. StanĚk, "Positive solutions for Dirichlet problems of singular nonlinear fractional differential

equations," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 1, pp. 57–68, 2010.

[5] B. Ahmad, "Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 390–394, 2010.

[6] B. Ahmad and J. J. Nieto, "Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions," *Boundary Value Problems*, vol. 2011, 36 pages, 2011.

[7] Z. Bai and H. Lu, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[8] J. Caballero, J. Harjani, and K. Sadarangani, "Positive solutions for a class of singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1325–1332, 2011.

[9] N. Kosmatov, "A singular boundary value problem for nonlinear differential equations of fractional order," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 125–135, 2009.

[10] Y. Liu, W. Zhang, and X. Liu, "A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann-Liouville derivative," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1986–1992, 2012.

[11] H. Mâagli, N. Mhadhebi, and N. Zeddini, "Existence and exact asymptotic behavior of positive solutions for a fractional boundary value problem," *Abstract and Applied Analysis*, vol. 2013, Article ID 420514, 6 pages, 2013.

[12] Y. Zhao, S. Sun, Z. Han, and Q. Li, "Positive solutions to boundary value problems of nonlinear fractional differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 390543, 16 pages, 2011.

[13] R. P. Agarwal, M. Benchohra, S. Hamani, and S. Pinelas, "Boundary value problems for differential equations involving Riemann-Liouville fractional derivative on the half-line," *Dynamics of Continuous, Discrete & Impulsive Systems A: Mathematical Analysis*, vol. 18, no. 2, pp. 235–244, 2011.

[14] R. P. Agarwal and D. O. Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[15] R. P. Agarwal and D. O'Regan, "Boundary value problems of nonsingular type on the semi-infinite interval," *The Tohoku Mathematical Journal*, vol. 51, no. 3, pp. 391–397, 1999.

[16] A. Arara, M. Benchohra, N. Hamidi, and J. J. Nieto, "Fractional order differential equations on an unbounded domain," *Nonlinear Analysis. Theory: Methods & Applications*, vol. 72, no. 2, pp. 580–586, 2010.

[17] Y. Chen and X. Tang, "Positive solutions of fractional differential equations at resonance on the half-line," *Boundary Value Problems*, vol. 2012, article 64, 2012.

[18] C. Kou, H. Zhou, and Y. Yan, "Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 17, pp. 5975–5986, 2011.

[19] H. Maagli, "Existence of positive solutions for a nonlinear fractional differential equation," *Electronic Journal of Differential Equations*, vol. 2013, no. 29, pp. 1–5, 2013.

[20] X. Su and S. Zhang, "Unbounded solutions to a boundary value problem of fractional order on the half-line," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1079–1087, 2011.

[21] X. Zhao and W. Ge, "Unbounded solutions for a fractional boundary value problems on the infinite interval," *Acta Applicandae Mathematicae*, vol. 109, no. 2, pp. 495–505, 2010.

[22] R. Chemmam, A. Dhifli, and H. Mâagli, "Asymptotic behavior of ground state solutions for sublinear and singular nonlinear Dirichlet problems," *Electronic Journal of Differential Equations*, vol. 88, pp. 1–12, 2011.

[23] I. Bachar and H. Mâagli, "Existence and global asymptotic behavior of positive solutions for nonlinear problems on the half-line," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 181–194, 2014.

[24] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[25] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[26] V. Marić, *Regular Variation and Differential Equations*, vol. 1726 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[27] R. Seneta, *Regularly Varying Functions*, vol. 508 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.