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Research Article

On Some Classes of Linear Volterra Integral Equations

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The sufficient conditions are obtained for the existence and uniqueness of continuous solution to the linear nonclassical Volterra equation that appears in the integral models of developing systems. The Volterra integral equations of the first kind with piecewise smooth kernels are considered. Illustrative examples are presented.

1. Introduction

Volterra integral equations of the first kind with variable upper and lower limits of integration were studied by Volterra himself [1]. The publications on this topic in the first half of the 20th century were reviewed in [2] and later studies were discussed in [3–5].

A noticeable impetus to the development of this area is related to the research [6] which suggested a macroeconomic two-sector integral model. The Glushkov's models of developing systems were further extended in [7, 8] and used in many applications (see [9] and references therein). In particular, a one-sector version of the Glushkov's model applied to the power engineering problems was considered in [10–12]. In the recent years the researchers have got attracted by the equation (see [13] and references therein) that in a general case has the following form:

$$\sum_{i=1}^{n} \int_{a_i(t)}^{a_{i-1}(t)} K_i(t,s)\, x(s)\, ds = y(t), \quad t \in [0,T], \tag{1}$$

where

$$\begin{gathered} 0 \le a_n(t) < a_{n-1}(t) < \cdots < a_0(t) \equiv t, \\ a_i(0) = 0, \quad i = \overline{0,n}; \end{gathered} \tag{2}$$

kernels K_i and right-hand side $y(t)$ are given, and $x(t)$ is an unknown desired solution.

At $n = 1$ the problems of the existence and uniqueness of solution to (1) in the space $C_{[0,T]}$, as well as the numerical methods, are studied in detail in [5]. In this paper we will be interested in the same problems for (1) at $n > 1$. Further, for simplicity, we will consider only the case $n = 2$, since many results are easily generalized for the case $n > 2$.

2. Sufficient Conditions for the Correctness of (1) at n=2 in Pair $(C_{[0,T]}, \overset{\circ}{C}{}^{(1)}_{[0,T]})$

For convenience, present (1) with $n = 2$ in operator form

$$V_1 x + V_2 x \triangleq \int_{a_1(t)}^{t} K_1(t,s)\, x(s)\, ds + \int_{0}^{a_1(t)} K_2(t,s)\, x(s)\, ds = y(t), \quad t \in [0,T] \tag{3}$$

(in (3) $a_2(t) = 0$ is assumed with no loss of generality).

Let kernels K_1 and K_2 be continuous in arguments and continuously differentiable with respect to t in regions $\Delta_1 = \{(t,s) : 0 \le a_1(t) \le s \le t \le T\}$ and $\Delta_2 = \{(t,s) : 0 \le s \le a_1(t)\}$, respectively, so that $\Delta_1 \cup \Delta_2 = \Delta$, $\Delta = \{(t,s) : 0 \le s \le t \le T\}$, $\Delta_1 \cap \Delta_2 = l$, $l = \{(t,s) : s = a_1(t)\}$. We will assume that

$$a_1'(t) \in C_{+[0,T]}, \quad a_1'(0) < 1. \tag{4}$$

In particular, (4) holds true for $a_1(t) = \alpha t$, $\alpha \in (0,1)$. $\overset{\circ}{C}{}^{(1)}_{[0,T]}$ is further taken to mean the space of continuously differentiable functions $y(t)$ on $[0,T]$ with the norm

$\|y(t)\|_{\overset{\circ}{C}{}^{(1)}_{[0,T]}} = \max_{0\le t\le T}\{|y(t)|+|y'(t)|\}$ and additional condition $y(0)=0$. If

$$\min_{t\in[0,T]}\left|K_1(t,t)\right| = k > 0, \tag{5}$$

then, as established in [5, page 106], the following estimate is true:

$$\left\|V_1^{-1}\right\|_{\overset{\circ}{C}{}^{(1)}_{[0,T]}\to C_{[0,T]}} \le Sk^{-1}e^{k^{-1}L_1T}, \tag{6}$$

where

$$\begin{gathered} L_1 = \max_{(t,s)\in\Delta_1}\left|K'_{1_t}(t,s)\right|, \\ S = \sum_{j=0}^{\infty}\prod_{i=1}^{j}\gamma_i \ge 1, \\ \gamma_i = \beta_i + \left(z_i - z_{i+1}\right)L_1k^{-1}, \\ z_i = a_1^i(T) = a_1\left(a_1\left(\cdots a_1(T)\right)\right), \quad a_1^0(T) = T, \\ \beta_i = \max_{t\in[z_i,z_{i-1}]}\frac{a_1'(t)\left|K_1(t,a(t))\right|}{\left|K_1(t,t)\right|}. \end{gathered} \tag{7}$$

Estimating (6) makes it possible to obtain the sufficient condition for the existence, uniqueness, and stability of the solution to (3) in pair $(C_{[0,T]}, \overset{\circ}{C}{}^{(1)}_{[0,T]})$.

Theorem 1. *Let the following inequality hold true:*

$$a_1(T)\left(M_2+L_2\right) + A_1M_2 < kS^{-1}e^{-k^{-1}L_1T}, \tag{8}$$

where

$$\begin{gathered} A_1 = \max_{t\in[0,T]} a_1'(t); \\ M_2 = \max_{(t,s)\in\Delta_2}\left|K_2(t,s)\right|; \\ L_2 = \max_{(t,s)\in\Delta_2}\left|K'_{2_t}(t,s)\right|, \end{gathered} \tag{9}$$

Then (3) *is correct in the sense of Hadamard in pair* $(C_{[0,T]}, \overset{\circ}{C}{}^{(1)}_{[0,T]})$.

Proof. By virtue of a well-known theorem of functional analysis (see, e.g., [14, page 212]), if

$$\left\|V_2\right\|_{C_{[0,T]}\to \overset{\circ}{C}{}^{(1)}_{[0,T]}} < \frac{1}{\left\|V_1^{-1}\right\|_{\overset{\circ}{C}{}^{(1)}_{[0,T]}\to C_{[0,T]}}}, \tag{10}$$

then the operator $V = V_1 + V_2$ has a bounded inverse, and, consequently, (3) is correct in the sense of Hadamard in pair $(C_{[0,T]}, \overset{\circ}{C}{}^{(1)}_{[0,T]})$. We show that under (8)-(9) inequality (10) holds true.

As

$$\begin{aligned} \left\|V_2x\right\|_{\overset{\circ}{C}{}^{(1)}_{[0,T]}} &= \max_{0\le t\le T}\left\{\left|\int_0^{a_1(t)}K_2(t,s)\,x(s)\,ds\right|\right. \\ &\quad + \left|a_1'(t)\,K_2\left(t,a_1(t)\right)\right. \\ &\quad \left.\left. + \int_0^{a_1(t)}K'_{2_t}(t,s)\,x(s)\,ds\right|\right\} \\ &\le \left\{a_1(T)\left(M_2+L_2\right) + A_1M_2\right\}\|x(t)\|_{C_{[0,T]}}, \end{aligned} \tag{11}$$

then

$$\left\|V_2\right\|_{C_{[0,T]}\to \overset{\circ}{C}{}^{(1)}_{[0,T]}} \le a_1(T)\left(M_2+L_2\right) + A_1M_2 \tag{12}$$

and (10) follows from (6) and (12). □

Condition (8) was obtained in the assumption that kernel K_1 is defined on Δ_1. If it is possible to expand the domain of definition K_1 to Δ, so that $\Delta_1\cap\Delta_2 = \Delta\cap\Delta_2 = \Delta_2$, then the sufficient condition for the correctness of (3) is modified in the following way. Represent the first term in (3) in the form

$$\begin{aligned} \int_{a_1(t)}^{t}K_1(t,s)\,x(s)\,ds = &\int_0^{t}K_1(t,s)\,x(s)\,ds \\ &- \int_0^{a_1(t)}K_1(t,s)\,x(s)\,ds. \end{aligned} \tag{13}$$

Then (3) can be represented as

$$\begin{aligned} \widehat{V}_1x + \widehat{V}_2x \triangleq &\int_0^{t}K_1(t,s)\,x(s)\,ds \\ &+ \int_0^{a_1(t)}\left(K_2(t,s) - K_1(t,s)\right)x(s)\,ds = y(t), \\ &t\in[0,T]. \end{aligned} \tag{14}$$

Since (see [5, page 12])

$$\left\|\widehat{V}_1^{-1}\right\|_{\overset{\circ}{C}{}^{(1)}_{[0,T]}\to C_{[0,T]}} \le k^{-1}e^{k^{-1}\widehat{L}_1T}, \tag{15}$$

where

$$\widehat{L}_1 = \max_{(t,s)\in\Delta}\left|K'_{1_t}(t,s)\right|, \tag{16}$$

then sufficient conditions for the correctness of (14) give the following theorem.

Theorem 2. *Let inequality*

$$a_1(T)\left(\widehat{M}_2 + \widehat{L}_2\right) + A_1\widehat{M}_2 < ke^{-1}e^{-k^{-1}\widehat{L}_1T}, \tag{17}$$

where

$$\widehat{M}_2 = \max_{(t,s)\in\Delta_2}\left|K_2(t,s) - K_1(t,s)\right|, \tag{18}$$

$$\widehat{L}_2 = \max_{(t,s)\in\Delta_2}\left|K'_{2_t}(t,s) - K'_{1_t}(t,s)\right|, \tag{19}$$

hold true. Then (14) *is correct in the sense of Hadamard in pair* $(C_{[0,T]}, \overset{\circ}{C}{}^{(1)}_{[0,T]})$.

Proof. With obvious changes, repeat the proof of Theorem 1. □

Let us illustrate the obtained results with the following example.

Consider the equation

$$\int_{\alpha t}^{t} x(s)\,ds + \epsilon \int_{0}^{\alpha t} x(s)\,ds = y(t), \quad t \in [0,T]. \tag{20}$$

Here by (5)–(7) $k = 1$, $M_2 = |\epsilon|$, $\widehat{M_2} = |1-\epsilon|$, $L_1 = \widehat{L_1} = L_2 = 0$, $a_1(T) = \alpha T$, $A_1 = \alpha, \gamma_i = \beta_i = \alpha$, and $S = 1/(1-\alpha)$; therefore based on (8) inequality

$$\alpha T|\epsilon| + \alpha|\epsilon| < 1-\alpha \tag{21}$$

and based on (17) inequality

$$\alpha T|1-\epsilon| + \alpha|1-\epsilon| < 1 \tag{22}$$

give the following estimates ϵ, which guarantee the existence, uniqueness, and stability of solution to (20) in the space $C_{[0,T]}$:

$$|\epsilon| < \frac{1-\alpha}{\alpha(1+T)}; \qquad |1-\epsilon| < \frac{1}{\alpha(1+T)}. \tag{23}$$

It is useful to compare (23) with the estimate obtained by shifting from (20) to the equivalent functional equation. Differentiation of (20) gives

$$x(t) = \alpha(1-\epsilon)x(\alpha t) + y'(t), \tag{24}$$

whence

$$x(t) = \lim_{n\to\infty}\left[\alpha^n(1-\epsilon)^n x(\alpha^n t) + \sum_{j=0}^{n-1}\alpha^j(1-\epsilon)^j y'(\alpha^j t)\right] \tag{25}$$

and condition

$$|1-\epsilon| < \frac{1}{\alpha} \tag{26}$$

provides convergence of series (25) to continuous function $x(t)$ on $[0,T]$.

If in (20)

$$\epsilon = 1 - \frac{1}{\alpha}, \tag{27}$$

then condition (26) is violated. Then it is easy to see that the homogeneous equation

$$\int_{\alpha t}^{t} x(s)\,ds + \left(1-\frac{1}{\alpha}\right)\int_{0}^{\alpha t} x(s)\,ds = 0 \tag{28}$$

has a nontrivial solution $x(t) = \text{const}$, and if, for example, $y(t) = t$, the solution to the nonhomogeneous equation

$$\int_{\alpha t}^{t} x(s)\,ds + \left(1-\frac{1}{\alpha}\right)\int_{0}^{\alpha t} x(s)\,ds = t, \quad t \in [0,T] \tag{29}$$

is a one-parameter family:

$$x(t) = -\frac{\ln t}{\ln \alpha} + x(1). \tag{30}$$

Let now

$$\epsilon = 1 + \frac{1}{\alpha}. \tag{31}$$

Then, according to (24),

$$x(t) = -x(\alpha t) + y'(t), \tag{32}$$

whence

$$x(t) = \lim_{n\to\infty}\left[(-1)^n x(\alpha^n t) + \sum_{j=0}^{n-1}(-1)^j y'(\alpha^j t)\right] \tag{33}$$

so that for the right-hand side of (20) $y(t) = y(t) = t^k/k$, $k = 1, 2, 3, \ldots$, from (33) we obtain

$$x(t) = \frac{t^{k-1}}{1+\alpha^{k-1}}, \quad k = 1, 2, \ldots. \tag{34}$$

In conclusion of this section it should be noted that inequalities (8) and (17) can be interpreted as constraints on the value T, which guarantee at given $K_1(t,s), K_2(t,s)$, and $a_1(t)$ the correct solvability of (3) in $C_{[0,T]}$. Since all parameters in the left-hand side of (8) and (17) are nondecreasing functions of T and the right-hand side of (8) and (17) at $L_1 \neq 0$ ($\widehat{L}_1 \neq 0$), on the contrary, monotonously decreases, then the real positive root of corresponding nonlinear equation that gives a guaranteed lower-bound estimate of T exists and is unique if $a'(0)$ is sufficiently small. In some special cases this root can be found analytically in terms of the Lambert function W [15, 16].

In [17–22] the authors studied the characteristic of continuous solution locality and the role of the Lambert function as applied to the polynomial (multilinear) Volterra equations of the first kind. The calculations of the test examples show that the locality feature of the solution to the linear equation (3) is not the result of the inaccuracy of estimates (8) and (17) and reflects the specifics of the considered class of problems. In this paper we do not dwell on the problem of numerically solving (3). It is of independent interest and deserves special consideration.

3. The Volterra Integral Equations of the First Kind with Discontinuous Kernels

Equation (2) can be written in the form of Volterra integral equation of the first kind:

$$\int_{0}^{t} K(t,s)\,x(s)\,ds = y(t), \quad t \in [0,T], \tag{35}$$

with discontinuous kernel

$$K(t,s) = \begin{cases} K_1(t,s), & a_1(t) < s \le t; \\ K_i(t,s), & a_i(t) < s < a_{i-1}(t), \\ & i = \overline{2, n-1}; \\ \dfrac{(K_i(t,s) + K_{i+1}(t,s))}{2}, & s = a_i(t),\ i = \overline{1, n-1}; \\ K_n(t,s), & 0 \le s < a_{n-1}(t). \end{cases} \quad (36)$$

To illustrate the fundamental difference between (35), (36), and classical Volterra equation of the first kind with smooth kernel, we confine ourselves to (20) that has the form of (35) at

$$K(t,s) = \begin{cases} 1, & \alpha t < s \le t; \\ \dfrac{1+\epsilon}{2}, & s = \alpha t; \\ \epsilon, & 0 \le s < \alpha t, \end{cases} \quad (37)$$

where $\epsilon \neq 0, 1$, and $\alpha \in (0,1)$. In particular, at $\alpha = 1/2$, $\epsilon = -1$,

$$K(t,s) = \operatorname{sign}\left(s - \frac{t}{2}\right) = \begin{cases} 1, & s > \dfrac{t}{2}; \\ 0, & s = \dfrac{t}{2}; \\ -1, & s < \dfrac{t}{2}. \end{cases} \quad (38)$$

For this case the solution to (35) with $y(t) = t$ given in [23] is

$$x(t) = \frac{\ln t}{\ln 2} + x(s). \quad (39)$$

For kernel (38)

$$K(0,0) = 0, \qquad K(t,t) \neq 0, \quad t > 0. \quad (40)$$

If $K(t,s)$ is continuous in arguments and continuously differentiable with respect to t in Δ, then condition (40) means that (35) is Volterra integral equation of the third kind.

The theory (whose foundation was laid by Volterra (see [24, pages 104–106])) of such equations is developed in the research done by Magnitsky [25–28].

In particular, the author of [25–28] studies the structure of one- or many-parameter family of solutions to (35).

If $K(t,s)$ is discontinuous, then the solution to (35) may be nonunique, even if $K(t,t) \neq 0$ $\forall t \ge 0$.

For example, if $\alpha \neq 1/2$ and $\epsilon = 1 - (1/\alpha) \neq -1$, the solution to equation

$$\int_{\alpha t}^{t} x(s)\, ds + \epsilon \int_0^{\alpha t} x(s)\, ds = t, \quad t \in [0,T], \quad (41)$$

is a one-parameter family:

$$x(t) = -\frac{\ln t}{\ln \alpha} + x(1), \quad (42)$$

but, by (37) $K(0,0) = (1+\epsilon)/2 \neq 0$, $K(t,t) = 1$, $t > 0$.

Now we show that there can be a nonunique solution to (35) and (36) even in the case $K(t,t) \equiv 1$. Let

$$K(t,s) = \begin{cases} 1, & s \ge \alpha t, \\ \epsilon, & s < \alpha t, \end{cases} \quad (43)$$

so that condition $K(t,t) \equiv 1$ is true.

We prove that solutions to (35), (37) and (35), (43) coincide. It suffices to show that the equivalent functional equations for (35), (37) and (35), (43) coincide. Recall that for (35), (37) the equivalent functional equation is (24).

Theorem 3. *The equivalent functional equations for* (35), (37) *and* (35), (43) *coincide.*

Proof. Let us represent (43) by

$$K(t,s) \equiv 1 + (\epsilon - 1)\, e(\alpha t - s), \quad (44)$$

where $e(\cdot)$– is a Heaviside function:

$$e(\nu) = \begin{cases} 1, & \nu \ge 0, \\ 0, & \nu < 0. \end{cases} \quad (45)$$

Substitution of (44) in (35) gives

$$\int_0^t x(s)\, ds + (\epsilon - 1) \int_0^t e(\alpha t - s)\, x(s)\, ds = y(t), \quad t \in [0,T]. \quad (46)$$

Transform the second integral. Let $\nu = \alpha t - s$. Then

$$\int_0^t e(\alpha t - s)\, x(s)\, ds = \int_{(\alpha-1)t}^{\alpha t} e(\nu)\, x(\alpha t - \nu)\, d\nu = \int_0^{\alpha t} x(\alpha t - \nu)\, d\nu. \quad (47)$$

By virtue of (47), differentiation of (46) results in

$$x(t) + (\epsilon - 1)\alpha x(0) + (\epsilon - 1) \int_0^{\alpha t} x_t'(\alpha t - \nu)\, d\nu = y'(t). \quad (48)$$

But

$$x_t'(\alpha t - \nu) = -\alpha x_\nu'(\alpha t - \nu). \quad (49)$$

By virtue of (49) we have

$$x(t) + (\epsilon - 1)\alpha x(0) - (\epsilon - 1)\alpha \left[x(\alpha t - \nu)\big|_0^{\alpha t} \right] = y'(t), \quad (50)$$

from (48), whence finally

$$x(t) + \alpha(\epsilon - 1)\, x(\alpha t) = y'(t), \quad (51)$$

and (51) coincides with (24). □

The solution to (35), (43) in the class of piecewise continuous functions with a jump on line $s = \alpha t$ is interesting from the application perspective.

It is easy to see that this solution is

$$\hat{x}(t,s) = \begin{cases} y'(s), & s \geq \alpha t, \\ \frac{1}{\epsilon} y'(s), & s < \alpha t. \end{cases} \tag{52}$$

At last consider the concept of α-convolution. Volterra integral equations of convolution type

$$K(t) * x(t) \stackrel{\Delta}{=} \int_0^t K(t-s)\,x(s)\,ds = y(t), \quad t \in [0,T] \tag{53}$$

are important for application.

Examples (38) and (44) show the usefulness of the α-convolution concept:

$$\begin{aligned} K(t) \overset{\alpha}{*} x(t) &\stackrel{\Delta}{=} \int_0^t K(\alpha t - s)\,x(s)\,ds \\ &= y(t), \quad \alpha \in (0,1],\ t \in [0,T]. \end{aligned} \tag{54}$$

Give some inversion formulas of the integral equation

$$K(t) \overset{\alpha}{*} x(t) = y(t), \quad t \in [0,T]. \tag{55}$$

(1) If $K(t) = \delta(t)$, $y(t) \in C_{[0,T]}$, and $\alpha \in (0,1]$, then

$$x(\alpha t) = y(t). \tag{56}$$

(2) If $K(t) = e(t)$, $y(t) \in \overset{\circ}{C}{}^{(1)}_{[0,T]}$, and $\alpha \in (0,1]$, then

$$x(\alpha t) = \frac{1}{\alpha} y'(t). \tag{57}$$

(3) If $K(t) = \operatorname{sign} t$, $y(t) = t$, and $\alpha \in (0,1)$, then

$$x(t) = \frac{\ln t}{\ln \alpha} + x(1). \tag{58}$$

At $K(t) = t^n, n \geq 1$, (55) is Volterra integral equation of the third kind.

(4) If $K(t) = t$, $y(t) = t^2/2$, and $\alpha = 1/2$, then

$$x(t) = -2\ln t + x(1). \tag{59}$$

(5) If $K(t) = t$, $y(t) = t^2/2$, and $\alpha \in (0,1), \alpha \neq 1/2$, then

$$x(t) = \frac{x(1)}{t^{(2\alpha-1)/(\alpha-1)}}. \tag{60}$$

4. Conclusion

As is mentioned in the introduction, the main results of this study can be easily applied to the case $n > 2$ in (1). The equations of type (1) not only are of theoretical interest, but also play an important role in the mathematical modeling of developing dynamic systems. Moreover, by $y(t)$, we can mean some criterion that characterizes the level of development of the system as a whole, and the ith term in (1) represents a contribution of the system components $x(s)$ of the ith age group, whose operation is reflected by the efficiency coefficient $K_i(t-s)$. As a rule, $K_1 \geq \cdots \geq K_n \geq 0$. Such an approach is implemented, for instance, in [29, 30], in the problem of the analysis of strategies for the long-term expansion of the Russian electric power system, with the consideration of aging of the power plants equipment.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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