

Research Article

Geodesic B -Preinvex Functions and Multiobjective Optimization Problems on Riemannian Manifolds

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We introduce a class of functions called geodesic B -preinvex and geodesic B -invex functions on Riemannian manifolds and generalize the notions to the so-called geodesic quasi/pseudo B -preinvex and geodesic quasi/pseudo B -invex functions. We discuss the links among these functions under appropriate conditions and obtain results concerning extremum points of a nonsmooth geodesic B -preinvex function by using the proximal subdifferential. Moreover, we study a differentiable multiobjective optimization problem involving new classes of generalized geodesic B -invex functions and derive Kuhn-Tucker-type sufficient conditions for a feasible point to be an efficient or properly efficient solution. Finally, a Mond-Weir type duality is formulated and some duality results are given for the pair of primal and dual programming.

1. Introduction

Convex functions play an important role in optimization theory and there are several classes of functions given in the literature with the goal to weaken the limitations of convexity in mathematical programming. Generalized convex functions, labelled as B -vex functions, were introduced by Bector and Singh [1]. In 1981, Hanson [2] introduced the concept of invexity and proved that the Kuhn-Tucker conditions are sufficient for optimality of a nonlinear programming problem under invexity conditions. Preinvex functions were defined by Ben-Israel and Mond [3], and, in [4], Weir and Mond showed how and where preinvex functions could replace convex functions in multiple objective optimization problem. These functions were further generalized to pseudo/quasi B -vex, B -invex, and pseudo/quasi B -invex functions by Bector et al. [5] and to B -preinvex by Suneja et al. [6]. In [5], Bector et al. obtained sufficient optimality criteria and duality results for a nonlinear programming problem involving B -vex and B -invex functions. There are also many papers in the literature concerning the generalization of convexity in connection with sufficiency and duality in optimization problems (see, e.g., [7–12] and the references therein).

A manifold is not a linear space and extensions of concepts and techniques from linear spaces to Riemannian manifolds are natural. In the literature many authors studied generalized convex functions and many results in convex analysis and optimization theory were extended to Riemannian manifolds (see [13–28] and the references therein). Rapcsák [27] and Udriste [28] considered a generalization of convexity called geodesic convexity. In this setting the linear space is replaced by a Riemannian manifold and the line segment by a geodesic. Pini [22] introduced the notion of invex function on Riemannian manifolds, while Mititelu [24] investigated its generalization. The concepts of geodesic invex sets, geodesic invex, and preinvex functions on Riemannian manifolds were defined by Barani and Pouryayevali [17]. They established the relation between geodesic invexity and preinvexity of functions, and they also obtained results concerning extremum points of a nonsmooth geodesic preinvex function by using the proximal subdifferential. Subsequently, Agarwal et al. [20] proposed and discussed geodesic α -preinvexity on Riemannian manifolds, which generalized the corresponding results studied by Barani and Pouryayevali [17]. A new concept of geodesic roughly B -invexity and its generalization on Hadamard manifolds were introduced

by Zhou and Huang [26]. They studied the properties of these functions and they established sufficient optimality conditions and duality in nonlinear programming problems.

In this paper, we introduce a class of geodesic B -preinvex and B -invex functions on Riemannian manifolds and extend them to geodesic quasi/pseudo B -preinvex and geodesic quasi/pseudo B -invex functions. We discuss the links among these functions under suitable assumptions. By applying the proximal subdifferential, we relax the smoothness condition and study the question of global minima for geodesic B -preinvex functions on Riemannian manifolds. As applications, we investigate a multiobjective programming problem involving generalized geodesic B -invex functions and derive the Kuhn-Tucker-type sufficient optimality conditions for a feasible point to be an efficient or properly efficient solution. Finally, a Mond-Weir type duality is formulated and some duality results are obtained for the pair of primal and dual programming. The results presented in this paper extend some known results due to Barani and Pouryayevali [17, 23].

2. Preliminaries

In this section, we recall some definitions and known results about Riemannian manifolds which will be used throughout the paper. These can be found in many introductory books on Riemannian geometry, such as in [29–32].

Let M be a C^∞ smooth manifold modelled on a Hilbert space H , either finite dimensional or infinite dimensional, endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_p M \cong H$. The corresponding norm is denoted by $\| \cdot \|_p$. The tangent bundle of M is denoted by $TM = \bigcup_{x \in M} T_x M$, which is naturally a manifold. Given a piecewise C^1 path $\gamma : [a, b] \rightarrow M$ joining p to q , that is, $\gamma(a) = p$ and $\gamma(b) = q$, we can define the length of γ by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt. \tag{1}$$

For any two points $p, q \in M$, we define

$$d(p, q) = \inf \{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ path joining } p \text{ to } q\}. \tag{2}$$

Then d is a metric on M which defines the same topology as the one M naturally has as a manifold. For this metric we define the open ball centered at the point p with radius $r > 0$; that is,

$$B(p, r) = \{q \in M : d(p, q) < r\}. \tag{3}$$

Let us recall that in every Riemannian manifold there exists exactly one covariant derivation called the Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M . We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ ; that is, γ satisfies the equation $\nabla_{d\gamma(t)/dt} d\gamma(t)/dt = 0$. Any path γ joins p and q in M such that $L(\gamma) = d(p, q)$ is a geodesic and is called a minimal

geodesic. The existence theorem for ordinary differential equation implies that for every $v \in TM$, there exists an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma(v) : J(v) \rightarrow M$ with $\gamma'_v(0) = v$. This implies that there is an open neighborhood $\tilde{T}M$ of the submanifold M of TM such that for every $v \in \tilde{T}M$, the geodesic $\gamma_v(t)$ is defined for $|t| < 2$ (see, e.g., [25]). The exponential map $\exp_p : T_p M \subseteq \tilde{T}M \rightarrow M$ is then defined as $\exp_p(v) = \gamma_v(1)$, where γ_v is the geodesic defined by its position p and velocity $\gamma'_v(0) = v$ at p .

If γ is a geodesic, then for each $t_1, t_2 \in [a, b]$, the Levi-Civita connection ∇ induces an isometry $P_{t_1, \gamma}^{t_2} : T_{\gamma(t_1)} M \rightarrow T_{\gamma(t_2)} M$, the so-called parallel translation from $T_{\gamma(t_1)} M$ to $T_{\gamma(t_2)} M$ along γ , which is defined by

$$P_{t_1, \gamma}^{t_2}(v) = V(\gamma(t_2)) \quad \forall a, b \in R, v \in T_{\gamma(t_1)} M, \tag{4}$$

where V is the unique vector field satisfying $\nabla_{d\gamma(t)/dt} V = 0$ for all t and $V(\gamma(t_1)) = v$.

Let $f : M \rightarrow R$ be a differentiable function. We will denote by

$$df_x : T_p M \rightarrow T_{f(x)} R \cong R \tag{5}$$

the differential f at x .

We also recall that a simply connected complete Riemannian manifold of nonpositive curvature is called a Cartan-Hadamard manifold.

Barani and Pouryayevali [17] first defined geodesic invex sets and introduced geodesic preinvex functions on Riemannian manifolds.

Definition 1. Let M be a Riemannian manifold and $\eta : M \times M \rightarrow TM$ be a function such that $\eta(x, y) \in T_x M$ for every $x, y \in M$. A nonempty subset S of M is said to be geodesic invex with respect to η , if for every $x, y \in S$, there exists exactly one geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\begin{aligned} \gamma_{x,y}(0) &= y, & \gamma'_{x,y}(0) &= \eta(x, y), \\ \gamma_{x,y}(t) &\in S, & \forall t &\in [0, 1]. \end{aligned} \tag{6}$$

Definition 2. Let M be a Riemannian manifold and $S \subseteq M$ a geodesic invex set with respect to $\eta : M \times M \rightarrow TM$. We say that a function $f : S \rightarrow R$ is geodesic preinvex if

$$\begin{aligned} f(\gamma_{x,y}(t)) &\leq tf(x) + (1-t)f(y), \\ \forall x, y \in S, \quad t &\in [0, 1]. \end{aligned} \tag{7}$$

In 1993, Suneja et al. [6] introduced the generalization of preinvex functions on R^n , and we now improve and extend the definition of B -preinvex functions to Riemannian manifolds.

Definition 3. Let M be a Riemannian manifold and $S \subseteq M$ an open invex set with respect to $\eta : M \times M \rightarrow TM$. A function $f : S \rightarrow R$ is said to be

- (i) geodesic B -preinvex (GBPIX) at $y \in S$ with respect to η and b if

$$f(\gamma_{x,y}(t)) \leq tb(x, y, t) f(x) + (1 - tb(x, y, t)) f(y), \quad (8)$$

$$\forall x \in S, \quad t \in [0, 1],$$

where $b : S \times S \times [0, 1] \rightarrow R_+$ with $tb(x, y, t) \in [0, 1]$ for all $x, y \in S$ and $t \in [0, 1]$, and $\gamma_{x,y}$ is the unique geodesic defined in Definition 1;

- (ii) GBPIX on S with respect to η and b if it is GBPIX at each $y \in S$ with respect to the same η and b ;
- (iii) strictly GBPIX (SGBPIX) on S with respect to η and b if inequality (8) is strict for all $x, y \in S$ with $x \neq y$;
- (iv) geodesic quasi B -preinvex (GQBPIX) with respect to η and b at $y \in S$ if

$$f(x) \leq f(y) \implies b(x, y, t) (f(\gamma_{x,y}(t)) - f(y)) \leq 0, \quad (9)$$

$$\forall x \in S, \quad t \in [0, 1];$$

- (v) GQBPIX on S with respect to η and b if it is GQBPIX at each $y \in S$ with respect to the same η and b ;
- (vi) geodesic pseudo B -preinvex (GPBPIX) with respect to η and b if, there exists a strictly positive function $a : S \times S \rightarrow R$ such that

$$b(x, y, t) f(x) \leq b(x, y, t) f(y) \implies f(\gamma_{x,y}(t)) - f(y) \leq t(t - 1) a(x, y), \quad \forall x \in S, \quad t \in [0, 1]; \quad (10)$$

- (vii) GPBPIX on S with respect to η and b , if it is GPBPIX at each $y \in S$ with respect to the same η and b .

Remark 4. By Definition 1, it is clear that SGBPIX \implies GBPIX, GBPIX \implies GQBPIX, and GPBPIX \implies GQBPIX.

Remark 5. If $b > 0$, then GBPIX \implies GPBPIX.

Remark 6. The above definition of geodesic B -preinvexity on Riemannian manifolds is also a generalization of geodesic preinvexity discussed in [17]. It is easy to see that every geodesic preinvex function f with respect to η is a geodesic B -preinvex function with respect to η and b , where $b(x, y, t) = 1$, but the converse is not true, as illustrated in the following example.

Example 7. Let $M = \{e^{i\theta} \mid 0 \leq \theta \leq \pi/2\}$. For any $x, y \in M$ with $x = e^{i\theta}$ and $y = e^{i\phi}$, let $f : M \rightarrow R$ be defined by $f(x) = \sin \theta$, let $\eta : M \times M \rightarrow TM$ be defined by

$$\eta(e^{i\theta}, e^{i\phi}) = (\phi - \theta) (\sin \phi, -\cos \phi), \quad (11)$$

and let $b : M \times M \times [0, 1] \rightarrow R_+$ be defined by

$$b(e^{i\theta}, e^{i\phi}, t) = \begin{cases} \frac{1}{t}, & \theta \geq \phi, \\ 0, & \theta < \phi. \end{cases} \quad (12)$$

Define a geodesic $\gamma_{x,y}(t)$ on M as follows:

$$\gamma_{x,y}(t) = (\cos((1-t)\phi + t\theta), \sin((1-t)\phi + t\theta)). \quad (13)$$

Then, we can easily verify that f is geodesic B -preinvex with respect to η and b , but f is not geodesic preinvex with respect to η .

In [22], Pini introduced the concept of geodesic invexity on Riemannian manifolds. Motivated by the definitions of (pseudo/quasi) B -invex functions on R^n given in [5], we present the following definition.

Definition 8. Let M be a Riemannian manifold and $b : M \times M \rightarrow R_+$ a nonnegative real function. A differentiable function $f : M \rightarrow R$ is said to be

- (i) geodesic B -invex (GBIX) at $y \in M$ with respect to η and b if

$$df_y(\eta(x, y)) \leq b(x, y) (f(x) - f(y)), \quad \forall x \in M; \quad (14)$$

- (ii) GBIX on M with respect to η and b if it is GBIX at each $y \in M$ with respect to the same η and b ;
- (iii) strictly geodesic B -invex on M with respect to η and b if inequality (14) is strict for all $x, y \in M$ with $x \neq y$;
- (iv) geodesic quasi B -invex (GQBIX) at $y \in M$ with respect to η and b if

$$f(x) \leq f(y) \implies b(x, y) df_y(\eta(x, y)) \leq 0, \quad \forall x \in M; \quad (15)$$

- (v) GQBIX on M with respect to η and b if it is GQBIX at each $y \in M$ with respect to the same η and b ;
- (vi) geodesic pseudo B -invex (GPBIX) at $y \in M$ with respect to η and b if

$$df_y(\eta(x, y)) \geq 0 \implies b(x, y) f(x) \geq b(x, y) f(y), \quad \forall x \in M, \quad (16)$$

or equivalently,

$$b(x, y) f(x) < b(x, y) f(y) \implies df_y(\eta(x, y)) < 0, \quad \forall x \in M; \quad (17)$$

- (vii) strictly geodesic pseudo B -invex (SGPBIX) at $y \in M$ with respect to η and b if

$$df_y(\eta(x, y)) \geq 0 \implies b(x, y) f(x) > b(x, y) f(y), \quad (18)$$

$$\forall x \in M, \quad x \neq y,$$

or equivalently,

$$b(x, y) f(x) \leq b(x, y) f(y) \implies df_y(\eta(x, y)) < 0, \quad (19)$$

$$\forall x \in M, \quad x \neq y;$$

- (viii) GPBIX/SGBPIX on M with respect to η and b if it is GPBIX/SGBPIX at each $y \in M$ with respect to the same η and b .

Remark 9. An invex function f with respect to η discussed on Riemannian manifolds in [17] is also a GBIX function with respect to η and b with $b = 1$, but the converse is not true.

Example 10. Let M be a Riemannian manifold and $f : M \rightarrow R$ a differentiable function such that for every $y \in M, df_y \neq 0$. Let $b : M \times M \rightarrow R_+$ be a bifunction and let $\eta : M \times M \rightarrow TM$ be defined by

$$\eta(x, y) = \frac{b(x, y)(f(x) - f(y))}{\|df_y\|_y^2} df_y. \tag{20}$$

Then, for every $x, y \in M$, one has

$$\begin{aligned} \langle df_y, \eta(x, y) \rangle_y &= b(x, y) \left\langle df_y, \frac{f(x) - f(y)}{\|df_y\|_y^2} df_y \right\rangle_y \\ &= b(x, y) \frac{f(x) - f(y)}{\|df_y\|_y^2} \langle df_y, df_y \rangle_y \\ &= b(x, y) [f(x) - f(y)]. \end{aligned} \tag{21}$$

Therefore, f is geodesic B -invex with respect to η and b , but f is not geodesic invex with respect to the same η whenever $b(x, y) \neq 1$.

Remark 11. Every geodesic B -invex function f with respect to η and b , where $b(x, y) > 0, \forall x, y \in M$, is geodesic invex with respect to some $\bar{\eta}$, where

$$\bar{\eta}(x, y) = \frac{\eta(x, y)}{b(x, y)}. \tag{22}$$

Remark 12. Every geodesic pseudoinvex function with respect to η in [18] is geodesic pseudo B -invex with respect to the same η . However, the converse is not necessarily true when $b(x, y) = 0$, for some $x, y \in M$.

Remark 13. From Definition 8, we have

$$\begin{aligned} \text{SGBIX} &\implies \text{GBIX} \implies \text{GBPIX} \\ &\Downarrow \\ &\text{GBQIX}, \end{aligned} \tag{23}$$

Finally we present the following definitions which will be useful in the sequel.

Definition 14 (see [17]). Let M be a Riemannian manifold. We say that the function $\eta : M \times M \rightarrow TM$ satisfies the condition (C), if for each $x, y \in M$, and for the geodesic $\gamma : [0, 1] \rightarrow M$ satisfying $\gamma_{x,y}(0) = y, \gamma'(0) = \eta(x, y)$, we have

- (i) $P_{t,y}^0[\eta(y, \gamma_{x,y}(t))] = -t\eta(x, y)$;
- (ii) $P_{t,y}^0[\eta(x, \gamma_{x,y}(t))] = (1 - t)\eta(x, y)$.

Definition 15 (see [33]). Let M be a Riemannian manifold and $f : M \rightarrow (-\infty, +\infty]$ a lower semicontinuous function. A point $\zeta \in T_y M$ is a proximal subgradient of f at $y \in \text{dom}(f)$ if there exist positive numbers δ and σ such that

$$f(x) \geq f(y) + \langle \zeta, \exp_y^{-1} x \rangle_y - \sigma d(y, x)^2, \quad \forall x \in B(y, \delta), \tag{24}$$

where $\text{dom}(f) = \{x \in M : f(x) < \infty\}$.

The set of all proximal subgradients of f at $y \in M$ is denoted by $\partial_p f(y)$ and is called the proximal subdifferential of f at y .

3. GBPIX (GBIX) Functions and Their Generalization

Theorem 16. Let M be a Riemannian manifold, let S be an open invex subset of M with respect to $\eta : M \times M \rightarrow TM$, and let $b : S \times S \times [0, 1] \rightarrow R_+$ be a nonnegative real function. Assume that $f : S \rightarrow R$ is a differentiable GBPIX function with respect to η and b . Then f is a GBIX function with respect to η and \bar{b} , where $\bar{b}(x, y) = \lim_{t \rightarrow 0^+} b(x, y, t)$.

Proof. Let $x, y \in S$. Since S is a geodesic invex set with respect to η , there exists exactly one geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\begin{aligned} \gamma_{x,y}(0) &= y, & \gamma'_{x,y}(0) &= \eta(x, y), \\ \gamma_{x,y}(t) &\in S, & \forall t &\in [0, 1]. \end{aligned} \tag{25}$$

Noting that f is GBPIX with respect to η and b , we have

$$f(\gamma_{x,y}(t)) \leq tb f(x) + (1 - tb) f(y), \quad \forall t \in (0, 1), \tag{26}$$

which implies

$$f(\gamma_{x,y}(t)) - f(y) \leq tb(x, y, t)(f(x) - f(y)). \tag{27}$$

Dividing by t and taking the limit as $t \rightarrow 0$, we obtain

$$df_{\gamma_{x,y}(0)}(\gamma'_{x,y}(0)) \leq \bar{b}(x, y)(f(x) - f(y)). \tag{28}$$

Therefore, $df_{\gamma_{x,y}(0)}(\eta(x, y)) \leq \bar{b}(x, y)(f(x) - f(y))$. This completes the proof. \square

Theorem 17. Let M be a Riemannian manifold and $S \subseteq M$ an open geodesic invex set with respect to $\eta : M \times M \rightarrow TM$. Suppose that $f : S \rightarrow R$ is differentiable GBIX with respect to η and b , where $b : S \times S \rightarrow R_+$. If η satisfies condition (C) and $b(x, y) > 0$, for all $x, y \in S$, then f is GBPIX with respect to η and \bar{b} , where

$$\bar{b}(x, y, t) = \frac{b(x, \bar{x})}{(1 - t)b(y, \bar{x}) + tb(x, \bar{x})} \tag{29}$$

for all $x, y \in S$ and $\bar{x} = \gamma_{x,y}(t)$, for some $t \in [0, 1]$.

Proof. Since S is geodesic invex with respect to η , there exists exactly one geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\begin{aligned} \gamma_{x,y}(0) &= y, & \gamma'_{x,y}(0) &= \eta(x, y), & \gamma_{x,y}(t) &\in S, \\ & & & & \forall x, y \in S, & \forall t \in [0, 1]. \end{aligned} \tag{30}$$

Fix $t \in [0, 1]$ and set $\bar{x} = \gamma_{x,y}(t)$. Then we have

$$b(x, \bar{x})(f(x) - f(\bar{x})) \geq df_{\bar{x}}(\eta(x, \bar{x})), \tag{31}$$

$$b(y, \bar{x})(f(y) - f(\bar{x})) \geq df_{\bar{x}}(\eta(y, \bar{x})). \tag{32}$$

Now multiplying (31) and (32) by t and $(1-t)$, respectively, and adding, then we have

$$\begin{aligned} &tb(x, \bar{x})f(x) + (1-t)b(y, \bar{x})f(y) \\ &- (tb(x, \bar{x}) + (1-t)b(y, \bar{x}))f(\bar{x}) \\ &\geq df_{\bar{x}}(t\eta(x, \bar{x}) + (1-t)\eta(y, \bar{x})). \end{aligned} \tag{33}$$

It follows from condition (C) that

$$\begin{aligned} &t\eta(x, \bar{x}) + (1-t)\eta(y, \bar{x}) \\ &= t(1-t)P_{0,\gamma_{x,y}}^t[\eta(x, y)] \\ &+ (1-t)(-t)P_{0,\gamma_{x,y}}^t[\eta(x, y)] = 0. \end{aligned} \tag{34}$$

Therefore,

$$f(\bar{x}) \leq \bar{t}b(x, y, t)f(x) + (1-\bar{t}b(x, y, t))f(y), \tag{35}$$

where

$$\bar{b}(x, y, t) = \frac{b(x, \bar{x})}{(1-t)b(y, \bar{x}) + tb(x, \bar{x})}. \tag{36}$$

This completes the proof. \square

Theorem 18. Let M be a Riemannian manifold, let $S \subseteq M$ be an open geodesic invex set with respect to η , and let $f : S \rightarrow R$ be GBPIX with respect to η and b .

(i) Every lower section of f defined by

$$L(f, \lambda) = \{x \in S \mid f(x) \leq \lambda\}, \quad \lambda \in R, \tag{37}$$

is a geodesic invex set with respect to η .

(ii) The set K of solutions for problem

$$\begin{aligned} (P) \quad &\min f(x) \\ &\text{s.t. } x \in S \end{aligned} \tag{38}$$

is a geodesic invex set with respect to η .

(iii) If $\bar{x} \in S$ is a local optimal solution to the problem (P) and $b > 0$, then \bar{x} is a global minimum for (P). Moreover, if f is strictly GBPIX, then the global optimal solution of problem (P) is unique.

Proof. (i) Let $x, y \in L_f(\lambda)$. Since S is a geodesic invex set with respect to η , there exists exactly one geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\begin{aligned} \gamma_{x,y}(0) &= y, & \gamma'_{x,y}(0) &= \eta(x, y), \\ \gamma_{x,y}(t) &\in S, & \forall t &\in [0, 1]. \end{aligned} \tag{39}$$

The GBPIX of f gives

$$f(\gamma_{x,y}(t)) \leq tbf(x) + (1-tb)f(y) \leq \lambda, \quad \forall t \in [0, 1], \tag{40}$$

which implies that $\gamma_{x,y}(t) \in L(f, \lambda)$ for all $t \in [0, 1]$.

(ii) If f has no optimal solution in S , then $K = \emptyset$, which is obviously a geodesic invex set. If $K \neq \emptyset$ and \bar{x} is an arbitrary optimal point for (P), then $K = S \cap L_f(f(\bar{x}))$, which is also a geodesic invex set with respect to η by (i).

(iii) Suppose that $\bar{x} \in S$ is a local minimum. Then there is a neighborhood $N_\varepsilon(\bar{x})$ such that

$$f(\bar{x}) \leq f(x), \quad x \in S \cap N_\varepsilon(\bar{x}). \tag{41}$$

If \bar{x} is not a global minimum of f , then there exists a point $x^* \in S$ such that

$$f(x^*) < f(\bar{x}). \tag{42}$$

Since S is a geodesic invex set with respect to η , there exists exactly one geodesic $\gamma_{x^*,\bar{x}}$ such that

$$\begin{aligned} \gamma_{x^*,\bar{x}}(0) &= \bar{x}, & \gamma'_{x^*,\bar{x}}(0) &= \eta(x^*, \bar{x}), \\ \gamma_{x^*,\bar{x}}(t) &\in S, & \forall t &\in [0, 1]. \end{aligned} \tag{43}$$

By the continuity of the distance function d and the geodesic $\gamma_{x^*,\bar{x}}$, there exists a number $\delta > 0$ such that $d(\gamma_{x^*,\bar{x}}(t), \bar{x}) < \varepsilon$ for all $t \in (0, \delta)$. Hence, $\gamma_{x^*,\bar{x}}(t) \in N_\varepsilon(\bar{x})$. It follows from the GBPIX of f that

$$\begin{aligned} f(\gamma_{x^*,\bar{x}}(t)) &\leq tbf(x^*) + (1-tb)f(\bar{x}) < f(\bar{x}), \\ &\forall t \in (0, 1). \end{aligned} \tag{44}$$

Hence, for each $\gamma_{x^*,\bar{x}}(t) \in S \cap N_\varepsilon(\bar{x})$, $f(\gamma_{x^*,\bar{x}}(t)) < f(\bar{x})$, which is a contradiction to (41).

If x^0 is another global optimal solution for (P) and $x^0 \neq \bar{x}$, then $f(x^0) = f(\bar{x})$. It follows from the strict GBPIX of f that

$$f(\gamma_{x^0,\bar{x}}(t)) < tbf(\bar{x}) + (1-tb)f(x^0) = f(\bar{x}), \tag{45}$$

which contradicts the optimality of \bar{x} for (P). Therefore, the solution of (P) is unique. This completes the proof. \square

Similar reasoning to that in the proof of Theorem 5.2 in [17] yields the following result.

Theorem 19. Let M be a Cartan-Hadamard manifold and $S \subseteq M$ be an open geodesic invex set with respect to $\eta : M \times M \rightarrow TM$ with $\eta(x, y) \neq 0$ for $x \neq y$. Suppose that $f : S \rightarrow (-\infty, +\infty]$ is a lower semicontinuous GBPIX function

with respect to η and b . Let $y \in \text{dom}(f)$ and $\zeta \in \partial_p f(y)$. Then there exists a number $\delta > 0$ such that

$$\bar{b}(x, y) [f(x) - f(y)] \geq \langle \zeta, \eta(x, y) \rangle_y, \quad \forall x \in B(y, \delta), \quad (46)$$

where $\bar{b}(x, y) = \lim_{t \rightarrow 0^+} b(x, y, t)$.

From Theorem 19, we can obtain the following corollary.

Corollary 20. *Let M be a Cartan-Hadamard manifold and $S \subseteq M$ be an open geodesic invex set with respect to $\eta : M \times M \rightarrow TM$ with $\eta(x, y) \neq 0$ for $x \neq y$. Suppose that $f : S \rightarrow (-\infty, +\infty]$ is a lower semicontinuous geodesic B -preinvex function with respect to η and b and $\lim_{t \rightarrow 0^+} b(x, y, t) = \bar{b}(x, y) > 0$. Let $y \in \text{dom}(f)$ and $0 \in \partial_p f(y)$. Then y is a global minimum of f .*

Remark 21. It should be noted that if S is a subset of a Riemannian manifold M and $f : S \rightarrow (-\infty, +\infty]$ is a lower semicontinuous function which has a local minimum at $y \in S$, then $0 \in \partial_p f(y)$ (see [33]).

Remark 22. Theorems 16, 17, 18, and 19 extend not only the corresponding results from R^n to Riemannian manifolds, but also Theorems 4.1, 4.2, 5.1, and 5.2 of [17], respectively.

The following theorems reveal the relations among geodesic quasi B -preinvexity, geodesic quasi B -invexity, geodesic pseudo B -preinvexity, and geodesic pseudo B -invexity for a differentiable function f .

Theorem 23. *Let M be a Riemannian manifold and $S \subseteq M$ an open geodesic invex set with respect to η . Assume that $f : S \rightarrow (-\infty, +\infty]$ is differentiable GQBPIX with respect to η and b . Then f is GQBIX with respect to η and \bar{b} , where $\bar{b}(x, y) = \lim_{t \rightarrow 0^+} b(x, y, t)$.*

Proof. Let $x, y \in S$ and $f(x) \leq f(y)$. Since f is GQBPIX with respect to η and b , we have

$$b(x, y, t) (f(\gamma_{x,y}(t)) - f(y)) \leq 0, \quad \forall t \in (0, 1]. \quad (47)$$

Dividing the above inequality by t and letting $t \rightarrow 0$, we get

$$\bar{b}(x, y) df_y(\eta(x, y)) \leq 0 \quad \forall x, y \in S, \quad (48)$$

which shows that f is GQBIX with respect to η and \bar{b} . This completes the proof. \square

Theorem 24. *Let M be a Riemannian manifold and $S \subseteq M$ an open geodesic invex set with respect to η . Suppose that η satisfies condition (C) and $b : M \times M \rightarrow R_+$ is continuous with respect to the second argument. If f is differentiable GQBIX with respect to η and b , then f is GQBPIX with respect to η and \bar{b} , where $\bar{b}(x, y, t) = b(x, \hat{x})b(y, \hat{x})$ for all $x, y \in S$ and $\hat{x} = \gamma_{x,y}(t)$ for some $t \in [0, 1]$.*

Proof. Let $x, y \in S$. Since S is a geodesic invex set with respect to η , there exists exactly one geodesic $\gamma_{x,y} : [0, 1] \rightarrow M$ such that

$$\begin{aligned} \gamma_{x,y}(0) &= y, & \gamma'_{x,y}(0) &= \eta(x, y), \\ \gamma_{x,y}(t) &\in S, & \forall t &\in [0, 1]. \end{aligned} \quad (49)$$

Let $f(x) \leq f(y)$. Consider the set

$$\begin{aligned} \Omega &= \{z \mid z = \gamma_{x,y}(t), \\ &\bar{b}(x, y, t) f(\gamma_{x,y}(t)) \\ &> \bar{b}(x, y, t) f(y), t \in [0, 1]\}. \end{aligned} \quad (50)$$

In order to show that f is GQBPIX, we have to show that $\Omega = \emptyset$. It is evident that Ω is equivalent to the set

$$\begin{aligned} \Omega' &= \{z \mid z = \gamma_{x,y}(t), f(\gamma_{x,y}(t)) > f(y), \\ &\bar{b}(x, y, t) > 0, t \in [0, 1]\}. \end{aligned} \quad (51)$$

If $\Omega' \neq \emptyset$, then, by the continuity of f and b , the set

$$\begin{aligned} \Omega'' &= \{z \mid z = \gamma_{x,y}(t), f(\gamma_{x,y}(t)) > f(y), \\ &\bar{b}(x, y, t) > 0, t \in (0, 1)\} \end{aligned} \quad (52)$$

is also nonempty. Hence, it is sufficient to show that $\Omega'' = \emptyset$, to complete the proof.

Suppose now that $\bar{x} \in \Omega''$. We then have $\bar{x} = \gamma_{x,y}(\bar{t})$, for some $0 < \bar{t} < 1$, $f(\bar{x}) > f(y) \geq f(x)$, and $\bar{b}(x, y, \bar{t}) > 0$. By the definition of GQBIX of, it follows, considering the pair \bar{x} and x , that

$$b(x, \bar{x}) df_{\bar{x}}(\eta(x, \bar{x})) \leq 0. \quad (53)$$

Similarly, considering the pair \bar{x} and y , it follows that

$$b(y, \bar{x}) df_{\bar{x}}(\eta(y, \bar{x})) \leq 0. \quad (54)$$

Hence by condition (C), we have

$$\begin{aligned} (1 - \bar{t}) b(x, \bar{x}) df_{\bar{x}} P_{0,y}^{\bar{t}}[\eta(x, y)] &\leq 0, \\ -\bar{t} b(y, \bar{x}) df_{\bar{x}} P_{0,y}^{\bar{t}}[\eta(x, y)] &\leq 0. \end{aligned} \quad (55)$$

Now (55), together with the fact that $0 < \bar{t} < 1$, imply that

$$\begin{aligned} b(x, \bar{x}) b(y, \bar{x}) df_{\bar{x}} P_{0,y}^{\bar{t}}[\eta(x, y)] \\ = b(x, \bar{x}) b(y, \bar{x}) df_{\bar{x}} \gamma'_{x,y}(\bar{t}) = 0. \end{aligned} \quad (56)$$

Note that (56) holds for any $\bar{x} \in \Omega''$. Now suppose that $\Omega'' \neq \emptyset$. Let $\bar{x} \in \Omega''$ and let $\bar{x} = \gamma_{x,y}(\bar{t})$. By the continuity of f , γ , and b , we can find $0 \leq t^* < \bar{t} < 1$ such that for all $t \in (t^*, \bar{t})$, we have

$$\begin{aligned} f(\gamma_{x,y}(t)) &> f(y), & \bar{b}(x, y, t) &> 0, \\ f(\gamma_{x,y}(t^*)) &= f(y). \end{aligned} \quad (57)$$

Now, by the Mean Value Theorem, there exists $t_0 \in (t^*, \bar{t})$ such that

$$\begin{aligned} f(\gamma_{x,y}(\bar{t})) - f(y) &= f(\gamma_{x,y}(\bar{t})) - f(\gamma_{x,y}(t^*)) \\ &= (\bar{t} - t^*) df_{\hat{x}} \gamma'_{x,y}(t_0), \end{aligned} \tag{58}$$

where $\hat{x} = \gamma_{x,y}(t_0)$. The left-hand side is positive by our hypothesis, but the right-hand side is zero by (56), as $\hat{x} \in \Omega^n$, and hence, we have a contradiction. This completes the proof. \square

We can also easily obtain the following results.

Theorem 25. *Let M be a Riemannian manifold, let $S \subseteq M$ be an open geodesic invex set with respect to η , and let $f : S \rightarrow R$ be differentiable GPBPIX with respect to η and b . Then f is GPBIX with respect to η and \bar{b} , where $\bar{b}(x, y) = \lim_{t \rightarrow 0^+} b(x, y, t)$.*

Theorem 26. *Let M be a Riemannian manifold, let $S \subseteq M$ be an open geodesic invex set with respect to η , and let $f : S \rightarrow R$ be differentiable GPBPIX with respect to η and b . Then f is QGBIX with respect to η and \bar{b} , where $\bar{b}(x, y) = \lim_{t \rightarrow 0^+} b(x, y, t)$.*

Remark 27. Theorems 25 and 26 generalize the known results from R^n to Riemannian manifolds.

4. Optimality Conditions and Duality

In this section, we discuss a multiobjective optimization problem (VOP) involving generalized GBIX functions and obtain the Kuhn-Tucker sufficient conditions for a feasible point x^0 of (VOP) to be an efficient or properly efficient solution. We also formulate a Mond-Weir type dual for (VOP) and give various types of duality results. All these conclusions extend the corresponding results on R^n (see, e.g., [2, 4, 5, 7, 8] and the references therein) to Riemannian manifolds under the assumptions of GBIX functions and their generalization introduced in Section 2.

Let M be a Riemannian manifold and let $S \subseteq M$ be an open invex set with respect to $\eta : M \times M \rightarrow TM$. We are concerned with the following multiple objective optimization problem:

$$\begin{aligned} \text{(VOP)} \quad \min \quad & f(x) = (f_1(x), \dots, f_k(x)), \\ \text{s.t.} \quad & g(x) \leq 0, \end{aligned} \tag{59}$$

where $f : S \rightarrow R^k$ and $g : S \rightarrow R^m$ are differentiable functions. Let $D = \{x \mid x \in S, g_i(x) \leq 0, i = 1, 2, \dots, m\}$ be the set of feasible solutions for (VOP), and $I(x^0) = \{i \mid g_i(x^0) = 0\}$.

For vector inequalities we adopt the usual notions. If $x, y \in R^m$, then

$$\begin{aligned} x = y &\iff x_i = y_i, \quad i = 1, 2, \dots, m, \\ x < y &\iff x_i < y_i, \quad i = 1, 2, \dots, m. \\ x \leq y &\iff x_i \leq y_i, \quad i = 1, 2, \dots, m, \end{aligned} \tag{60}$$

$$x \leq y \iff x_i \leq y_i, \quad i = 1, 2, \dots, m, \quad x \neq y,$$

$x \not\leq y$ is the negation of $x \leq y$.

Definition 28 (see [34]). A feasible point $x^0 \in D$ is said to be an efficient solution of (VOP) if there exists no other feasible point $x \in D$ such that $f(x) \leq f(x^0)$.

Definition 29 (see [34]). The point x^0 is said to be properly efficient of (VOP) if it is efficient for (VOP) and if there exists a scalar $M > 0$ such that, for each i ,

$$\frac{f_i(x^0) - f_i(x)}{f_j(x) - f_j(x^0)} \leq M \tag{61}$$

for some j such that $f_j(x) > f_j(x^0)$, whenever x is feasible for (VOP) and $f_i(x) < f_i(x^0)$.

Theorem 30. *Let x^0 be a feasible solution to (VOP). Assume that for every feasible point x , there exist scalars $\lambda_j > 0, j = 1, 2, \dots, k$ and $\mu_i \geq 0, i \in I(x^0)$, such that*

$$\sum_{j=1}^k \lambda_j d(f_j)_{x^0} \eta(x, x^0) + \sum_{i \in I(x^0)} \mu_i d(g_i)_{x^0} \eta(x, x^0) \geq 0. \tag{62}$$

Suppose that $f_j, j = 1, 2, \dots, k$ are GBIX with respect to η and b at x^0 , and $g_i, i \in I(x^0)$ is GBIX with respect to η and \bar{b} at x^0 . If $b(x, x^0) > 0$ for any $x \in D$, then x^0 is a properly efficient solution for (VOP).

Proof. Since $f_j, j = 1, 2, \dots, k$ and $g_i, i \in I(x^0)$ are GBIX, from condition (62), we get

$$\begin{aligned} & b(x, x^0) \sum_{j=1}^k \lambda_j [f_j(x) - f_j(x^0)] \\ & \geq \sum_{j=1}^k \lambda_j d(f_j)_{x^0} \eta(x, x^0) \\ & \geq - \sum_{i \in I(x^0)} \mu_i d(g_i)_{x^0} \eta(x, x^0) \\ & \geq \bar{b}(x, x^0) \left[\sum_{i \in I(x^0)} \mu_i g_i(x^0) - \sum_{i \in I(x^0)} \mu_i g_i(x) \right] \\ & = -\bar{b}(x, x^0) \sum_{i \in I(x^0)} \mu_i g_i(x) \\ & \geq 0. \end{aligned} \tag{63}$$

Thus,

$$\begin{aligned} b(x, x^0) \sum_{j=1}^k \lambda_j f_j(x) \\ \geq b(x, x^0) \sum_{j=1}^k \lambda_j f_j(x^0), \quad \forall x \in D, \end{aligned} \quad (64)$$

which implies that

$$x^0 \text{ minimize } \sum_{j=1}^k \omega_j f_j(x), \quad \text{subject to } g(x) \leq 0, \quad (65)$$

where

$$\omega_j = \frac{\lambda_j}{b(x, x^0) \sum_{j=1}^k \lambda_j}. \quad (66)$$

Hence, from Theorem 4.11 of [35], x^0 is a properly efficient solution for (VOP). This completes the proof. \square

Remark 31. Theorem 30 is a generalization of Theorem 5.5 in [23].

Theorem 32. Let x^0 be a feasible solution for (VOP). If there exist scalars $\lambda_j \geq 0$, $j = 1, 2, \dots, k$, $\sum_{j=1}^k \lambda_j = 1$, $\mu_i \geq 0$, $i \in I(x^0)$, such that the triplet (x^0, λ_j, μ_i) satisfies (62) in Theorem 30. Assume that $\sum_{j=1}^k \lambda_j f_j$ is strictly GBIX with respect to η and b at x^0 , and g_i , $i \in I(x^0)$ is GBIX with respect to η and \bar{b} at x^0 . Then x^0 is an efficient solution for problem (VOP).

Proof. Suppose that x^0 is not an efficient solution for (VOP). Then there exists a feasible point $x \in D$ such that

$$f(x) \leq f(x^0). \quad (67)$$

Since $\sum_{j=1}^k \lambda_j f_j(x)$ is strictly GBIX, we conclude

$$\begin{aligned} b(x, x^0) \left[\sum_{j=1}^k \lambda_j f_j(x) - \sum_{j=1}^k \lambda_j f_j(x^0) \right] \leq 0 \\ \implies \sum_{j=1}^k \lambda_j d(f_j)_{x^0} \eta(x, x^0) < 0. \end{aligned} \quad (68)$$

Also, the GBIX of g_i , $i \in I(x^0)$ yields

$$\begin{aligned} \bar{b}(x, x^0) \left[\sum_{i \in I(x^0)} \mu_i g_i(x) - \sum_{i \in I(x^0)} \mu_i g_i(x^0) \right] \leq 0 \\ \implies \sum_{i \in I(x^0)} \mu_i d(g_i)_{x^0} \eta(x, x^0) \leq 0. \end{aligned} \quad (69)$$

Adding (68) and (69), we obtain a contradiction to (62). This completes the proof. \square

Remark 33. Proceeding along the same lines as in Theorem 30, it can be easily seen that x^0 becomes properly efficient to (VOP) in the above theorem, if $\lambda_j > 0$, for all $j = 1, 2, \dots, k$.

Theorem 34. Suppose that there exist a feasible point x^0 and scalars $\lambda_j > 0$, $j = 1, 2, \dots, k$, $\mu_i \geq 0$, $i \in I(x^0)$, such that (62) of Theorem 30 holds. Let $\sum_{j=1}^k \lambda_j f_j$ be GPBIX with respect to η and b at x^0 and let g_i , $i \in I(x^0)$ be GQBIX with respect to η and \bar{b} at x^0 . If $b(x, x^0) > 0$ and $\bar{b}(x, x^0) > 0$ for any $x \in D$, then x^0 is a properly efficient solution for (VOP).

Proof. Since $g_i(x) \leq g_i(x^0) = 0$, $i \in I(x^0)$, and g_i , $i \in I(x^0)$ are GQBIX functions, we obtain

$$\bar{b}(x, x^0) \sum_{i \in I(x^0)} \mu_i d(g_i)_{x^0} \eta(x, x^0) \leq 0, \quad \forall x \in D, \quad (70)$$

which along with (62) yields

$$d\left(\sum_{j=1}^k f_j\right)_{x^0} \eta(x, x^0) \geq 0. \quad (71)$$

Since $\sum_{j=1}^k \lambda_j f_j$ is GPBIX, the above inequality implies that

$$b(x, x^0) \sum_{j=1}^k \lambda_j f_j(x) \geq b(x, x^0) \sum_{j=1}^k \lambda_j f_j(x^0). \quad (72)$$

Thus, we conclude that x^0 minimize $\sum_{j=1}^k \omega_j f_j(x)$, under the constraint $g(x) \leq 0$, where

$$\omega_j = \frac{\lambda_j}{b(x, x^0) \sum_{j=1}^k \lambda_j}. \quad (73)$$

Therefore, x^0 is a properly efficient for (VOP). This completes the proof. \square

Theorem 35. Let x^0 be a feasible point for (VOP). Assume that there exist scalars $\lambda_j \geq 0$, $j = 1, 2, \dots, k$, $\sum_{j=1}^k \lambda_j = 1$, $\mu_i \geq 0$, $i \in I(x^0)$, such that (62) of Theorem 30 holds. Let $\sum_{j=1}^k \lambda_j f_j$ be strictly GPBIX with respect to η and b at x^0 , and let g_i , $i \in I(x^0)$ be GQBIX with respect to η and \bar{b} at x^0 . If $\bar{b}(x, x^0) > 0$ for any $x \in D$, then x^0 is an efficient solution for problem (VOP).

Proof. Suppose that x^0 is not efficient for (VOP). Then, there exists a feasible x of (VOP) such that

$$f(x) \leq f(x^0), \quad (74)$$

which yields

$$b(x, x^0) \sum_{j=1}^k \lambda_j f_j(x) \leq b(x, x^0) \sum_{j=1}^k \lambda_j f_j(x^0). \quad (75)$$

It follows from the strict GPBIX of $\sum_{j=1}^k \lambda_j f_j$ that

$$\sum_{j=1}^k \lambda_j d(f_j)_{x^0} \eta(x, x^0) = d\left(\sum_{j=1}^k \lambda_j f_j\right)_{x^0} \eta(x, x^0) < 0. \quad (76)$$

Also, from the GQBIX of $g_i, i \in I(x^0)$, we conclude

$$\begin{aligned} \sum_{i \in I(x^0)} \mu_i g_i(x) &\leq \sum_{i \in I(x^0)} \mu_i g_i(x^0), \\ \implies \bar{b}(x, x^0) \sum_{i \in I(x^0)} \mu_i d(g_i)_{x^0} \eta(x, x^0) &\leq 0, \quad \forall x \in D. \end{aligned} \quad (77)$$

The proof now is similar to the proof of Theorem 32. This completes the proof. \square

Remark 36. Similarly as in Theorem 34, it can be easily seen that x^0 becomes properly efficient for (VOP) in the above theorem if $\lambda_j > 0$ for all $j = 1, 2, \dots, k$.

Remark 37. In Theorems 34 and 35, the results still hold when g_i is replaced by $\mu_i g_i$ with $i \in I(x^0)$.

We now consider the following Mond-Weir vector dual of (VOP):

$$\begin{aligned} \text{(MVD)} \quad \max \quad & f(u), \\ \text{s.t.} \quad & \sum_{j=1}^k \lambda_j d(f_j)_u \eta(x, u) \\ & + \sum_{i=1}^m \mu_i d(g_i)_u \eta(x, u) = 0, \end{aligned} \quad (78)$$

$$\sum_{i=1}^m \mu_i g_i(u) \geq 0, \quad (79)$$

$$\mu_i \geq 0, \quad i = 1, 2, \dots, m, \quad (80)$$

$$\lambda_j \geq 0, \quad j = 1, 2, \dots, k, \quad \sum_{j=1}^k \lambda_j = 1, \quad (81)$$

for all $x \in D$, and f and g are differentiable functions on $S \subseteq M$.

We now prove various duality results for (VOP) and (MVD).

Theorem 38 (weak duality). *Let x and (u, λ, μ) be feasible for (VOP) and (MVD), respectively. If also either*

- (a) $\lambda_j > 0$ for all $j = 1, 2, \dots, k$ and $\sum_{j=1}^k \lambda_j f_j$ is GPBIX with respect to η and b at u , $\sum_{i=1}^m \mu_i g_i$ is GQBIX with respect to η and \bar{b} at u , and $b(x, u) > 0, \bar{b}(x, u) > 0$, or
- (b) $\sum_{j=1}^k \lambda_j f_j$ is strictly GPBIX with respect to η and b at u , $\sum_{i=1}^m \mu_i g_i$ is GQBIX with respect to η and \bar{b} at u , and $\bar{b}(x, u) > 0$,

then $f(x) \not\leq f(u)$.

Proof. We proceed by contradiction. If $f(x) \leq f(u)$, then for $\lambda_j > 0, j = 1, 2, \dots, k$ and $b > 0$, we get

$$b(x, u) \sum_{j=1}^k \lambda_j f_j(x) < b(x, u) \sum_{j=1}^k \lambda_j f_j(u), \quad (82)$$

or for $\lambda_j \geq 0, j = 1, 2, \dots, k, \sum_{i=1}^k \lambda_j = 1$, we have

$$b(x, u) \sum_{j=1}^k \lambda_j f_j(x) \leq b(x, u) \sum_{j=1}^k \lambda_j f_j(u). \quad (83)$$

Now since $\sum_{j=1}^k \lambda_j f_j$ is GPBIX or strictly GPBIX, the above two inequalities both give

$$\sum_{j=1}^k \lambda_j d(f_j)_u \eta(x, u) < 0, \quad \forall x \in D. \quad (84)$$

It follows from (79) that

$$\sum_{i=1}^m \mu_i g_i(x) \leq 0 \leq \sum_{i=1}^m \mu_i g_i(u), \quad \forall x \in D, \quad (85)$$

and the GQBIX of $\sum_{i=1}^m \mu_i g_i$ gives

$$\begin{aligned} \bar{b}(x, u) d\left(\sum_{i=1}^m \mu_i g_i\right)_u \eta(x, u) \\ = \bar{b}(x, u) \sum_{i=1}^m \mu_i d(g_i)_u \eta(x, u) \leq 0. \end{aligned} \quad (86)$$

By adding (84) and (86), we obtain a contradiction to (78). This completes the proof. \square

Theorem 39 (strong duality). *Let x^0 be an efficient solution for (VOP) at which the Kuhn-Tucker conditions are satisfied. If for all feasible solutions (u, λ, μ) of (MVD) $\sum_{j=1}^k \lambda_j f_j$ is strictly GPBIX with respect to η and b and $\sum_{i=1}^m \mu_i g_i$ is GQBIX with respect to η and \bar{b} , where $\bar{b}(x, u) > 0$ for all $x \in D$, then there exists $(x^0, \lambda^0, \mu^0) \in R^k \times R^m$ such that (x^0, λ^0, μ^0) is efficient for (MVD) and the objective function values of (VOP) and (MVD) are equal.*

Proof. The assumption in the above theorem implies that there exist scalars $\lambda_j^0 \geq 0, j = 1, 2, \dots, k, \sum_{j=1}^k \lambda_j = 1$, and $\mu_i^0 \geq 0, i = 1, 2, \dots, m$ such that the Kuhn-Tucker conditions hold:

$$\begin{aligned} \sum_{j=1}^k \lambda_j^0 d(f_j)_{x^0} \eta(x, x^0) + \sum_{i=1}^m \mu_i^0 d(g_i)_{x^0} \eta(x, x^0) = 0, \\ \sum_{i=1}^m \mu_i^0 g_i(x^0) = 0, \end{aligned} \quad (87)$$

which gives that the triplet (x^0, λ^0, μ^0) is feasible for (MVD). If (x^0, λ^0, μ^0) is not efficient, then there exists a feasible (u, λ, μ) for (MVD) such that $f(x^0) \leq f(u)$, which contradicts the weak duality. This completes the proof. \square

Theorem 40. Let x^0 be feasible for (VOP), and let (u^0, λ^0, μ^0) be feasible for (MVD) such that

$$\sum_{j=1}^k \lambda_j^0 f_j(x^0) = \sum_{j=1}^k \lambda_j^0 f_j(u^0). \tag{88}$$

Suppose that $\sum_{j=1}^k \lambda_j^0 f_j$ is strictly GPBIX at u^0 with respect to η and b , and $\sum_{i=1}^m \mu_i^0 g_i$ is GQBIX at u^0 with respect to η and \bar{b} . If $b(x^0, u^0) > 0$ and $\bar{b}(x^0, u^0) > 0$, then $x^0 = u^0$.

Proof. Let $x^0 \neq u^0$. Since $\sum_{i=1}^m \mu_i^0 g_i$ is GQBIX and (u^0, λ^0, μ^0) is feasible for (MVD), we conclude

$$\begin{aligned} \sum_{i=1}^m \mu_i^0 g_i(x^0) &\leq \sum_{i=1}^m \mu_i^0 g_i(u^0) \implies \bar{b}(x^0, u^0) \\ &\times \sum_{i=1}^m \mu_i^0 d(g_i)_{u^0} \eta(x^0, u^0) \leq 0. \end{aligned} \tag{89}$$

It follows from (78) that

$$\sum_{j=1}^k \lambda_j^0 d(f_j)_{u^0} \eta(x^0, u^0) \geq 0. \tag{90}$$

Again from the strict GPBIX of $\sum_{j=1}^k \lambda_j f_j$, we have

$$b(x^0, u^0) \sum_{j=1}^k \lambda_j^0 f_j(x^0) > b(x^0, u^0) \sum_{j=1}^k \lambda_j^0 f_j(u^0). \tag{91}$$

This is a contradiction. Therefore, $x^0 = u^0$. This completes the proof. \square

Theorem 41. Suppose that there exist a feasible x^0 for (VOP) and a feasible (u^0, λ^0, μ^0) for (MVD) such that

$$f_j(x^0) = f_j(u^0), \quad j = 1, 2, \dots, k. \tag{92}$$

If for $j = 1, 2, \dots, k$, $\lambda_j > 0$ and

$$\frac{\sum_{j=1}^k \lambda_j^0 f_j}{\sum_{i=1}^m \mu_i^0 g_i} \tag{93}$$

is GPBIX/GQBIX with respect to η and b/\bar{b} at u^0 , then x^0 is properly efficient for (VOP). Also if for each feasible (u, λ, μ) of (MVD),

$$\frac{\sum_{j=1}^k \lambda_j f_j}{\sum_{i=1}^m \mu_i g_i} \tag{94}$$

is GPBIX/GQBIX with respect to η and b/\bar{b} and $b(x^0, u^0) > 0$, $\bar{b}(x^0, u^0) > 0$, then u^0 is properly efficient of (MVD).

Proof. Suppose that x^0 is not an efficient solution for (VOP). Then there exists a feasible x for (VOP) such that

$$f(x) \leq f(x^0). \tag{95}$$

Using condition (92), we contradict the weak duality. Thus, x^0 is efficient for (VOP). If x^0 is not properly efficient for (VOP), then there exist a feasible x and an index i such that

$$f_i(x^0) - f_i(x) > M(f_j(x) - f_j(x^0)), \tag{96}$$

for all $M > 0$ and all j such that $f_j(x) > f_j(x^0)$ whenever $f_i(x) < f_i(x^0)$. Again utilizing condition (92), we obtain

$$f_i(x) - f_i(u^0) < -M(f_j(x) - f_j(u^0)) \tag{97}$$

for all $M > 0$ and all j such that

$$f_j(x) > f_j(u^0), \tag{98}$$

whenever

$$f_i(x) < f_i(u^0). \tag{99}$$

Since M can be made large, hence for $\lambda_j^0 > 0$, we get the inequality

$$\sum_{j=1}^k \lambda_j^0 (f_j(x) - f_j(u^0)) < 0, \tag{100}$$

which contradicts the weak duality. Therefore, x^0 is properly efficient for (VOP).

To prove the second half of this theorem, let us assume on the contrary that (u^0, λ^0, μ^0) is not an efficient solution of (MVD). Then there exists a feasible point (u, λ, μ) of (MVD) such that

$$f(u^0) \leq f(u). \tag{101}$$

By applying condition (92), we get a contradiction to weak duality. Hence, (u^0, λ^0, μ^0) is an efficient solution of (MVD).

If (u^0, λ^0, μ^0) is not properly efficient for (MVD), then there exist a feasible (u, λ, μ) of (MVD) and an index i such that

$$f_i(u) - f_i(u^0) > M(f_j(u^0) - f_j(u)) \tag{102}$$

for all $M > 0$ and for all j such that

$$f_j(u^0) > f_j(u), \tag{103}$$

whenever

$$f_i(u^0) < f_i(u). \tag{104}$$

Utilizing (92) again, we obtain

$$f_i(u) - f_i(x^0) > M(f_j(x^0) - f_j(u)), \tag{105}$$

for all $M > 0$ and for all j such that

$$f_j(x^0) > f_j(u), \tag{106}$$

whenever

$$f_i(x^0) < f_i(u). \quad (107)$$

Now using the same argument as in the first part of the theorem, we get

$$\sum_{j=1}^k \lambda_j^0 (f_j(x^0) - f_j(u)) < 0, \quad (108)$$

which contradicts the weak duality. This completes the proof. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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